☆ Super Efimov effect

2D
3-fermions
③ p-wave resonance

$\Longrightarrow$ An infinite tower of bound states with
$E_n \sim e^{-2e^{3\pi n/4}}$ $(n \gg 1)$

low-energy EFT
+ RG analysis

2D Schrödinger eq for $l = \pm 1$

$$\left[-\frac{\nabla^2}{2\mu} + V(r)\right] f(\vec{r}) = E f(\vec{r})$$

$$\searrow f_l(\vec{r}) = e^{il\theta} \varphi(r)$$

$$\Longrightarrow \left[-\partial_r^2 - \frac{1}{r}\partial_r + \frac{l^2}{r^2} + 2\mu V(r)\right]\varphi(r) = \underbrace{2\mu E}_{k^2}\varphi(r)$$

When $\underbrace{r_0 < r}_{V(r)\to 0} \ll \underbrace{k^{-1}}_{E \to 0}$

$$\Longrightarrow \left[-\partial_r^2 - \frac{1}{r}\partial_r + \frac{l^2}{r^2}\right]\varphi(r) = 0$$

$$\therefore \ \varphi(r) \sim r^{|l|},\ r^{-|l|}$$

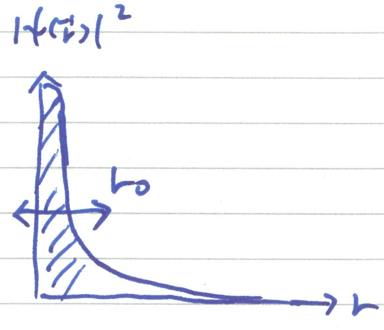
$$l = \pm 1 \Longrightarrow \varphi(r) \propto \left(\frac{1}{r} - \frac{r}{A}\right)$$

$\nwarrow$ scattering "area"

normalization integral

$$\int d\vec{r}\, |\psi(\vec{r})|^2$$

$$= 2\pi \int dr\, r\, \underbrace{|\psi(r)|^2}_{\frac{1}{r^2}\ (r\to 0)}$$



$$= 2\pi \int_{r_0} \frac{dr}{r} \sim \ln \frac{1}{r_0}$$ ← log-divergence when $r_0 \to 0$

$\Rightarrow$ 2 fermions behave as a point-like $\underbrace{\text{dimer}}_{\ell = \pm 1}$

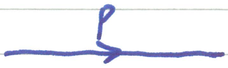
☑ Low-energy EFT

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m}\right)\psi + \phi_a^\dagger \left(i\partial_t + \frac{\nabla^2}{4m} - \varepsilon_0\right)\phi_a$$
$$+ \frac{g}{m}\phi_a^\dagger \psi (-i\nabla_a)\psi + \frac{g}{m}\psi^\dagger(-i\nabla_{-a})\psi^\dagger \phi_a$$

sum over $a = \pm$ $\quad \nabla_\pm \equiv \nabla_x \pm i \nabla_y$

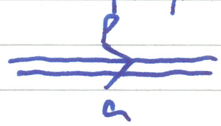
Feynman rules

- fermion propagator



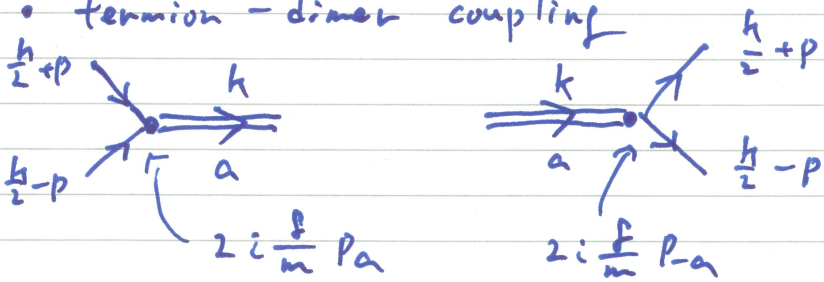
$$iG(p) = \frac{i}{p_0 - \frac{\mathbf{p}^2}{2m} + i0^+}$$

- dimer propagator with $\ell = a$ $(= \pm 1)$



$$iD_a(p) = \frac{i}{p_0 - \frac{\mathbf{p}^2}{4m} + i0^+}$$

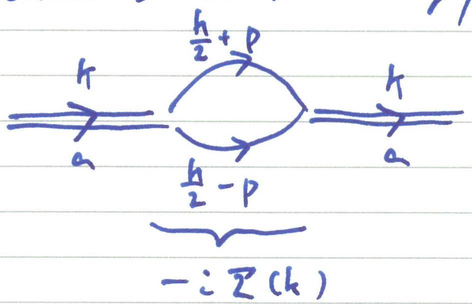
- fermion – dimer coupling

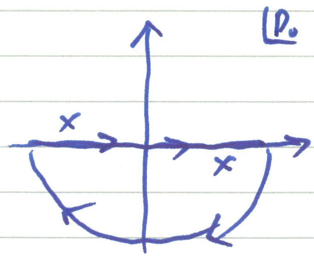


- Wilsonian renormalization group analysis

study the change of $\mathcal{L}_\Lambda$ when modes in $e^{-s}\Lambda < |\vec{p}| < \Lambda$ are integrated out.

- dimer's self-energy





$$\Sigma(k) = \frac{i}{2}\left(2i\frac{g}{m}\right)^2 \int_p \vec{p}^{\,2}\, iG\left(\frac{k}{2}+p\right) iG\left(\frac{k}{2}-p\right)$$

$$= 2i\frac{g^2}{m^2}\int\frac{dp_0}{2\pi}\int\frac{d\vec{p}}{(2\pi)^2}\,\vec{p}^{\,2}\,\frac{1}{\frac{k_0}{2}+p_0-\frac{(\frac{\vec{k}}{2}+\vec{p})^2}{2m}+i0^+}$$

$$\times\frac{1}{\frac{k_0}{2}-p_0-\frac{(\frac{\vec{k}}{2}-\vec{p})^2}{2m}+i0^+}$$

$$= 2\frac{g^2}{m^2}\int\frac{dp\cdot p}{2\pi}\,\frac{\vec{p}^{\,2}}{k_0-\frac{\vec{k}^2}{4m}-\frac{\vec{p}^{\,2}}{m}+i0^+}$$

$$= 2\frac{g^2}{m^2}\int\frac{dp\cdot p}{2\pi}\left[\underbrace{-m}_{\substack{\text{quadratically divergent}\\ \text{but cancelled by chosing } \varepsilon_0=\frac{g^2\Lambda^2}{2\pi m}}} + \underbrace{m\,\frac{k_0-\frac{\vec{k}^2}{4m}}{k_0-\frac{\vec{k}^2}{4m}-\frac{\vec{p}^2}{m}+i0^+}}\right]$$

$$\longrightarrow\ 2\frac{g^2}{m^2}\int_{e^{-s}\Lambda}^{\Lambda}\frac{dp\cdot p}{2\pi}\;\frac{k_0-\frac{\vec{k}^2}{4m}}{-\frac{\vec{p}^2}{m}}$$

$$= -\frac{g^2}{\pi}\left(k_0-\frac{\vec{k}^2}{4m}\right)\underbrace{\ln\frac{\Lambda}{e^{-s}\Lambda}}_{s}$$

$$\therefore\ \mathcal{L}_\Lambda \to \mathcal{L}_{e^{-s}\Lambda} = \psi^\dagger\left(i\partial_t+\frac{\nabla^2}{2m}\right)\psi$$

$$+\ \phi_a^\dagger\underbrace{\left(i\partial_t+\frac{\nabla^2}{4m}-\varepsilon_0-\Sigma\right)}_{\left(i\partial_t+\frac{\nabla^2}{4m}\right)\underbrace{\left(1-\frac{g^2}{\pi}s\right)}_{=\frac{1}{Z}}}\phi_a$$

$$+\ \frac{g}{m}\,\phi_a^\dagger\,\psi(-i\nabla_a)\psi\ +\ h.c.$$

rescaling of $\phi_a \to \sqrt{Z}\,\phi_a$

$$\Longrightarrow\ \mathcal{L}_{e^{-s}\Lambda} = \psi^\dagger\left(i\partial_t+\frac{\nabla^2}{2m}\right)\psi + \phi_a^\dagger\left(i\partial_t+\frac{\nabla^2}{4m}\right)\phi_a$$

$$+\ \frac{\sqrt{Z}\,g}{m}\,\phi_a^\dagger\,\psi(-i\nabla_a)\psi\ +\ h.c.$$

$\therefore$ We find the coupling changes as

$$g_\Lambda \to \underbrace{g_{e^{-s}\Lambda}}_{g(s)} = \sqrt{Z}\, g_\Lambda = \left(1 - \frac{g^2}{2\pi} s\right) g$$

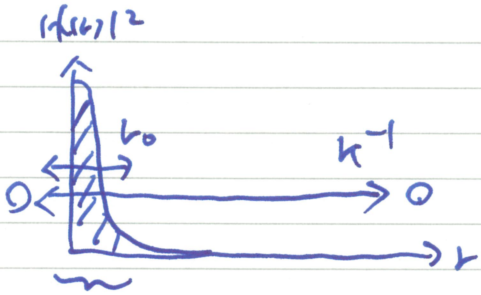
$$\frac{dg}{ds} = -\frac{g^3}{2\pi} \qquad \text{" RG equation "}$$

$$\Rightarrow \quad g^2(s) = \frac{1}{\frac{1}{g_0^2} + \frac{s}{\pi}} \longrightarrow \frac{\pi}{s} \qquad (s \to \infty)$$

fermion - dimer coupling decreases

by lowering the momentum scale

$\Longleftrightarrow$ increasing the length scale



effective dimer state

$$e^{-s}\Lambda = k$$

$\uparrow$

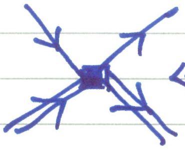
momentum scale of our interest.

$$s = \ln \frac{\Lambda}{k}$$

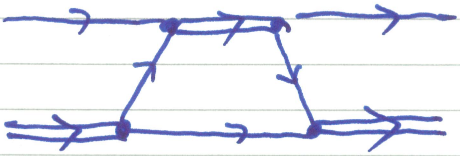
☆ 3-body problem

$\Longleftrightarrow$ scattering of fermion and dimer

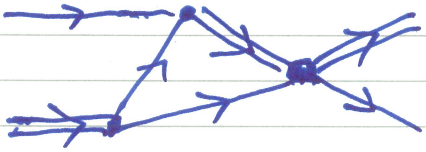
$$\mathcal{L}_{3\text{-body}} = \frac{v_3}{m}\psi^\dagger \phi_a^\dagger \phi_a \psi$$



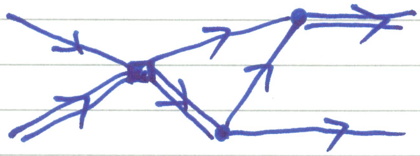
$\leftarrow$: $\frac{v_3}{m}$ is renormalized by



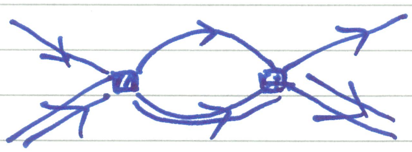
$\sim g^4 \ln \frac{\Lambda}{e^{s}\Lambda}$



$\sim g^2 v_3 \ln \frac{\Lambda}{e^{-s}\Lambda}$



$\sim g^2 v_3 \ln \frac{\Lambda}{e^{-s}\Lambda}$



$\sim v_3^2 \ln \frac{\Lambda}{e^{-s}\Lambda}$

$\therefore$ under $\Lambda \to e^{-s}\Lambda$, v_3 changes as

$$v_3 \to \underbrace{Z}_{1-\frac{g^2}{\pi}s} v_3 + \frac{16 g^4}{3\pi} s - \frac{8 g^2 v_3}{3\pi} s + \frac{2 v_3^2}{3\pi} s \equiv v_3(s)$$

$\therefore$ RG equation for u_3

$$\frac{du_3}{ds} = \frac{16}{3\pi} g^4 - \frac{11}{3\pi} g^2 u_3 + \frac{2}{3\pi} u_3^2$$

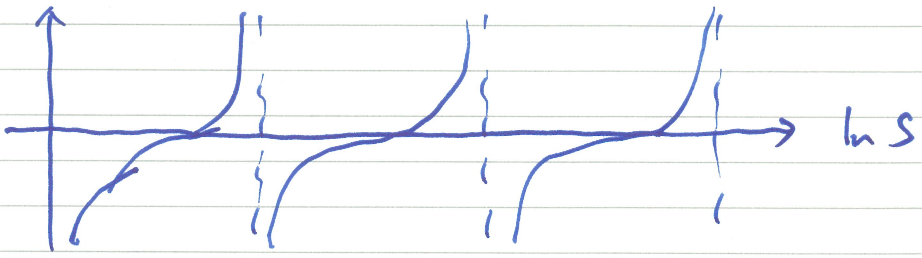
$$\xrightarrow[s\to\infty]{} \frac{16\pi}{3s^2} - \frac{11}{3s} u_3 + \frac{2}{3\pi} u_3^2 \quad \left(g^2 \to \frac{\pi}{s}\right)$$

$$\underbrace{s\,\frac{d(su_3)}{ds}}_{\frac{d(su_3)}{d(\ln s)}} = s^2 \frac{du_3}{ds} + s u_3$$

$$= \frac{16\pi}{3} - \frac{8}{3}(su_3) + \frac{2}{3\pi}(su_3)^2$$

$$= \frac{2}{3\pi}\left[2\pi - (su_3)\right]^2 + \frac{8\pi}{3}$$

$$s u_3 \xrightarrow[s\to\infty]{} 2\pi \left\{ 1 - \cot\left[\frac{4}{3}(\ln s - \theta)\right]\right\}$$



$s u_3$ diverges when $\frac{4}{3}(\ln s - \theta) = \pi n$

indicates bound state!

$$\underbrace{\ln s}_{\sim \ln\left(\ln \frac{\Lambda}{k}\right)} = \frac{3\pi n}{4} + \theta$$

$\ln\left(\ln \frac{\Lambda}{k}\right) \leftarrow$ characteristic momentum scale

$$k_n \sim \Lambda\, e^{-e^{3\pi n/4 + \theta}}$$

$$\Longleftrightarrow \underbrace{E_n}_{} \sim \frac{k_n^2}{m} \sim \frac{\Lambda^2}{m} e^{-e^{3\pi n/4 + \theta}}$$

infinite tower of 3-body binding energies

$\Longrightarrow$ Super Efimov effect !!!