Few-body universality: Efimov effect and its application & extension

Yusuke Nishida (Tokyo Tech)

"Quantum few- to many-body physics in ultracold atoms"

April 9-20, 2014 @ Wuhan (China)

Plan of this talk

1. Universality in physics

2. What is the Efimov effect ?

   Keywords: universality
   scale invariance
   quantum anomaly
   RG limit cycle

3. Application: Quantum magnets

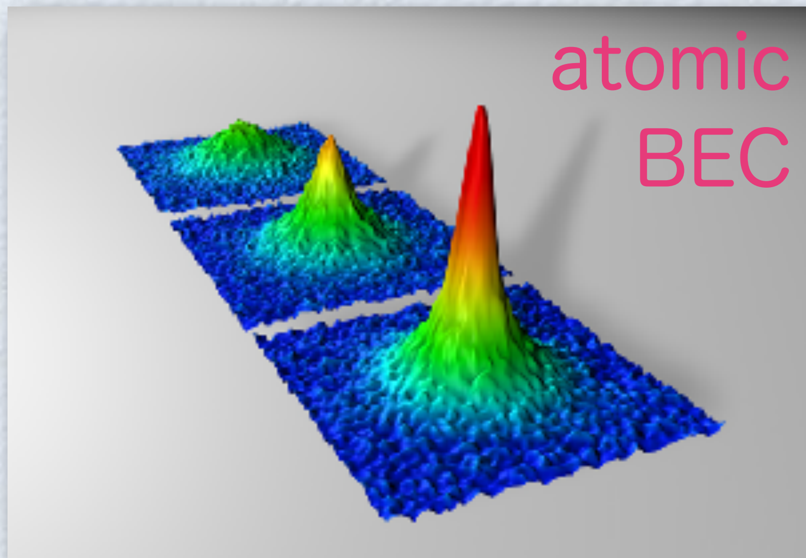
4. Extension: Super Efimov effect

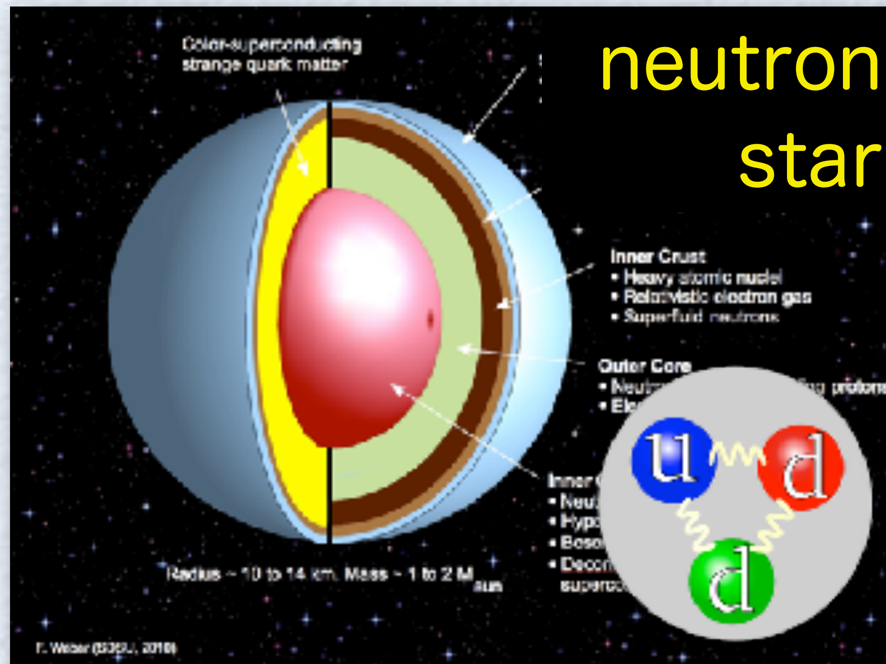
Introduction

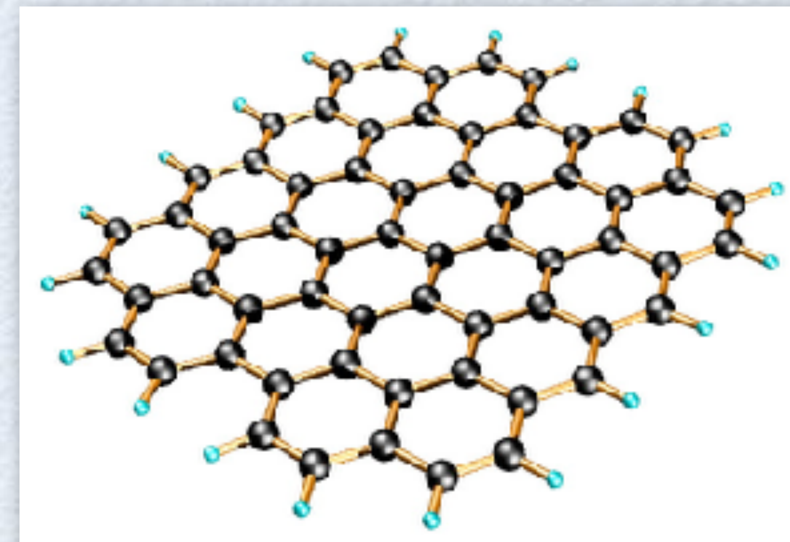
1. Universality in physics
2. What is the Efimov effect?
3. Application: Quantum magnets
4. Extension: Super Efimov effect

(ultimate) Goal of research

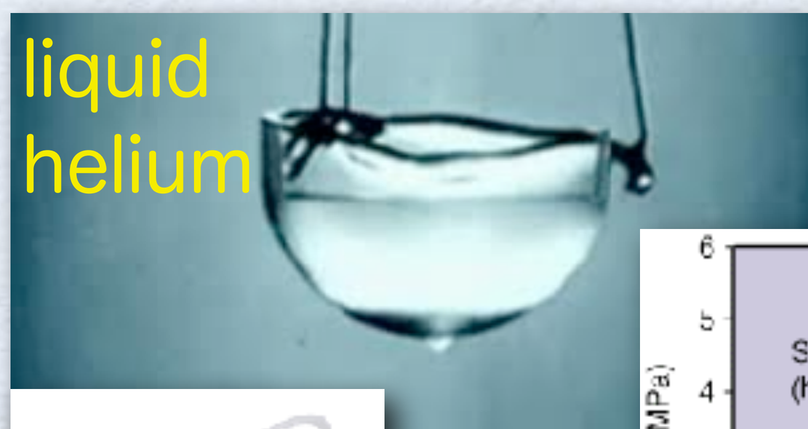
Understand physics of few and many particles governed by quantum mechanics

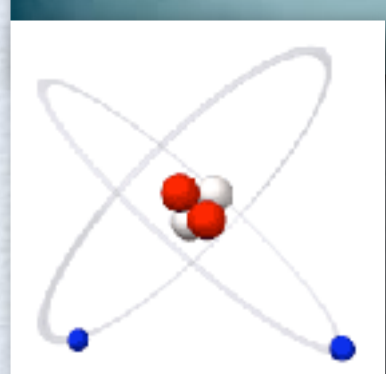


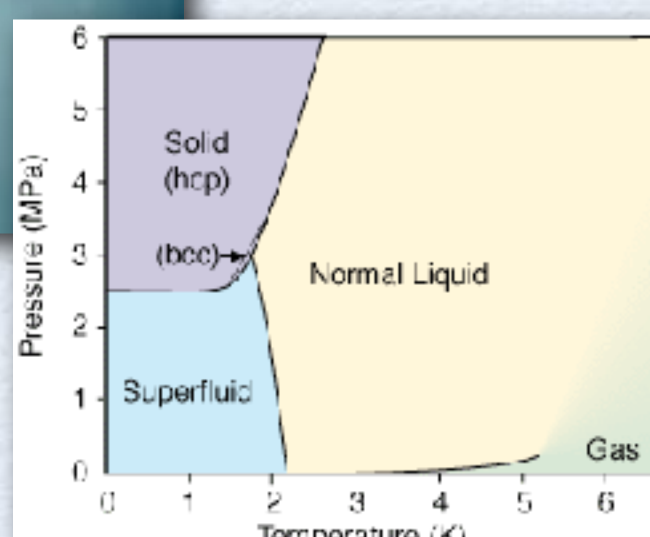




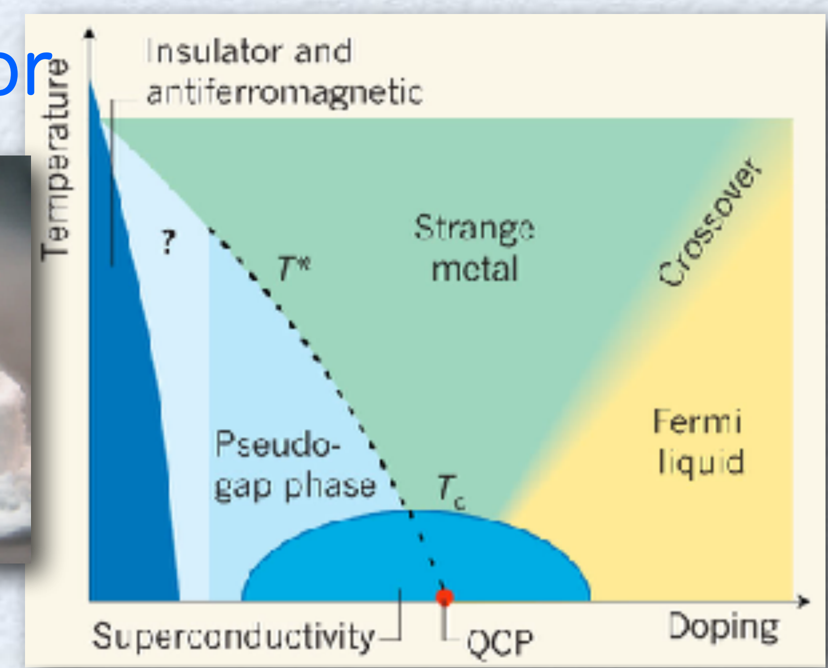
graphene







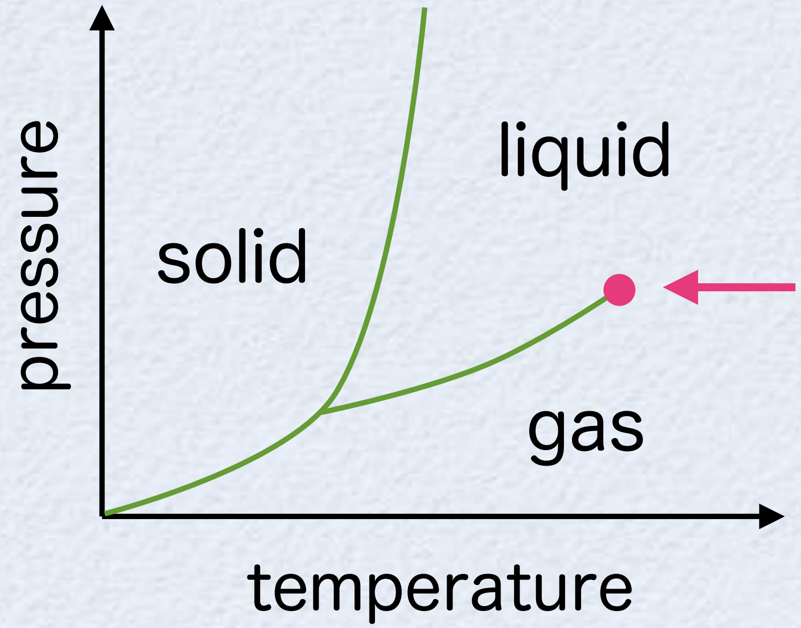
superconductor

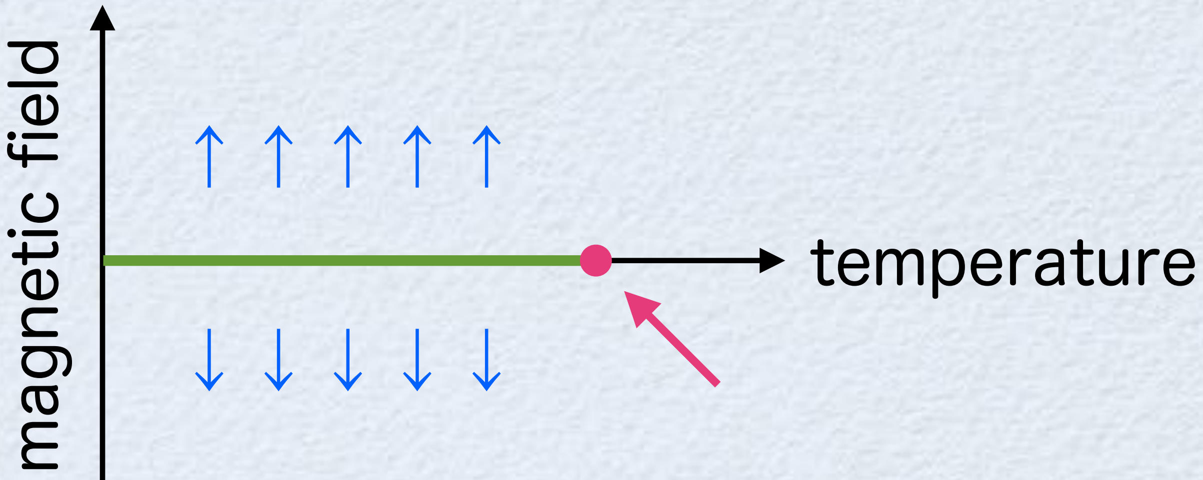


When physics is universal ?

A1. Continuous phase transitions ⇔ $\xi / r_0 \to \infty$

E.g. Water  vs. Magnet 



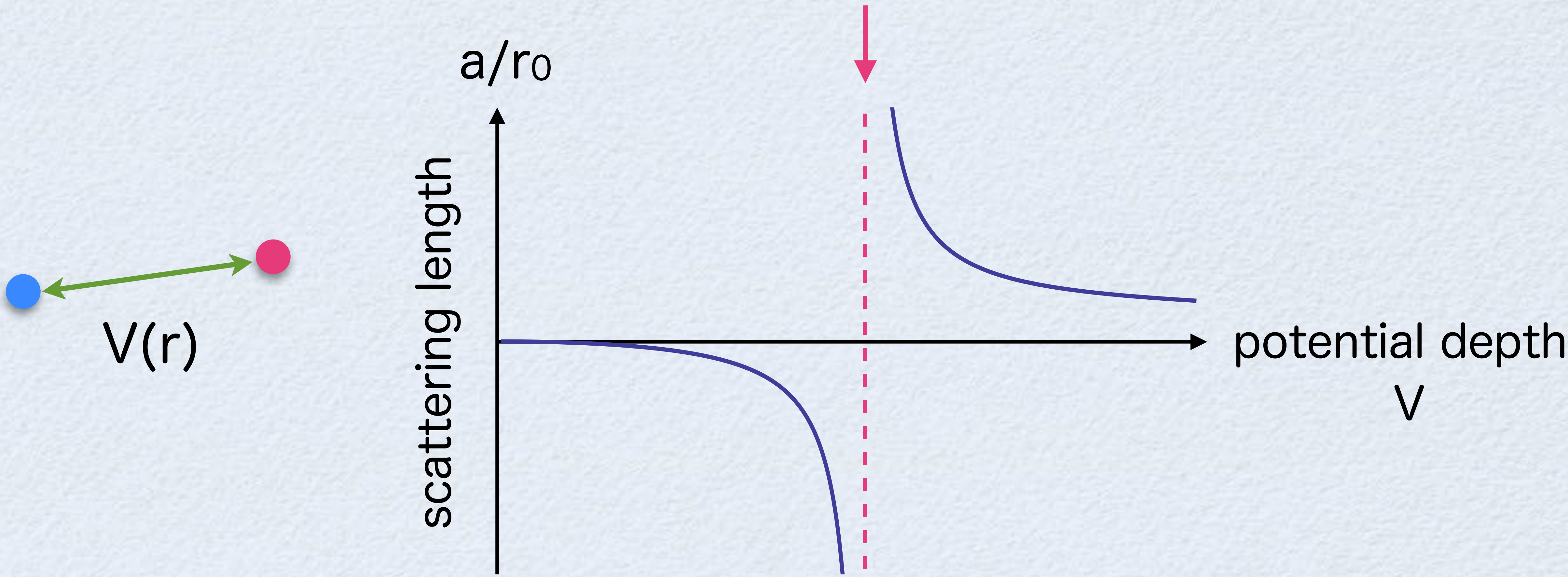


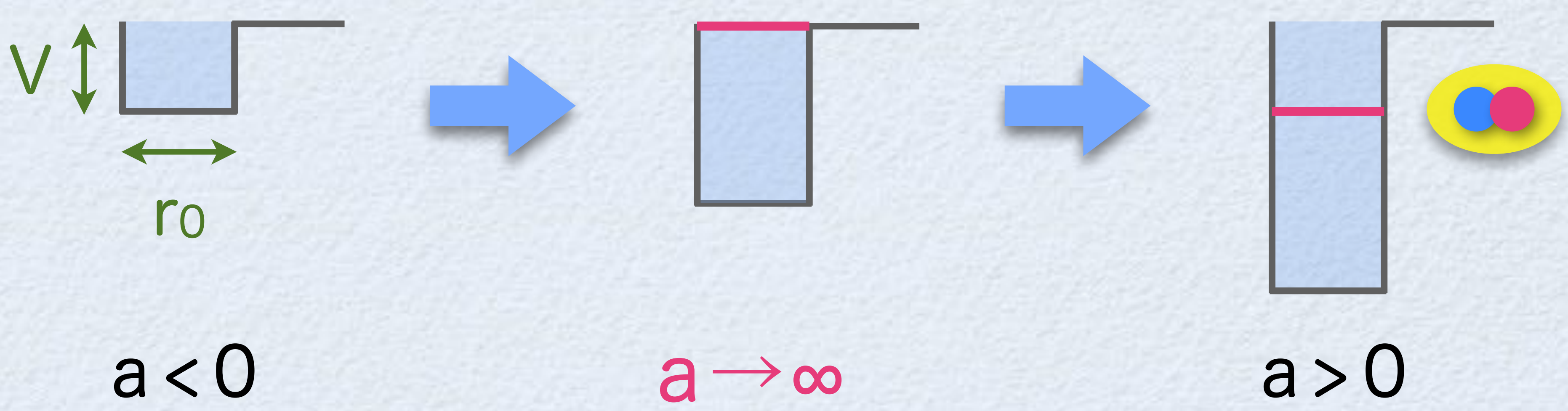
Water and magnet have the same exponent $\beta \approx 0.325$

$$\rho_{\mathrm{liq}} - \rho_{\mathrm{gas}} \sim (T_{\mathrm{c}} - T)^{\beta} \qquad M_{\uparrow} - M_{\downarrow} \sim (T_{\mathrm{c}} - T)^{\beta}$$

When physics is universal ?

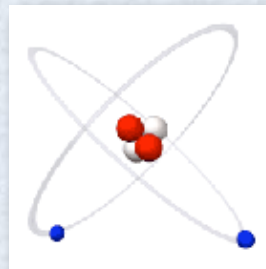
A2. Scattering resonances ⇔ $a/r_0 \to \infty$





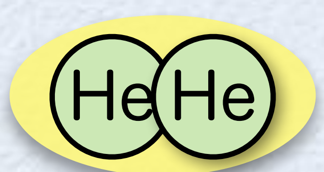
When physics is universal ?

A2. Scattering resonances $\Leftrightarrow$ $a/r_0 \to \infty$

E.g. ^{4}He atoms  vs. proton/neutron 

van der Waals force:

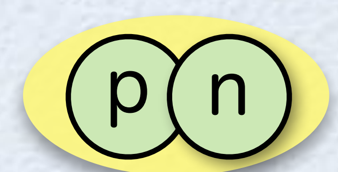
$a \approx 1\times10^{-8}$ m $\approx 20\ r_0$



$E_{binding} \approx 1.3\times10^{-3}$ K

nuclear force:

$a \approx 5\times10^{-15}$ m $\approx 4\ r_0$



$E_{binding} \approx 2.6\times10^{10}$ K

Atoms and nucleons have the same form of binding energy

$$E_{\text{binding}} \to -\frac{\hbar^2}{m\,a^2} \qquad (a/r_0 \to \infty)$$

Physics only depends on the scattering length “a”

Efimov effect

1. Universality in physics
2. **What is the Efimov effect?**
3. Application: Quantum magnets
4. Extension: Super Efimov effect

Efimov effect

Efimov (1970)

Volume 33B, number 8 PHYSICS LETTERS 21 December

ENERGY LEVELS ARISING FROM RESONANT TWO-BODY FORCES IN A THREE-BODY SYSTEM

V. EFIMOV
A.F.Ioffe Physico-Technical Institute, Leningrad, USSR

Received 20 October 1970

Resonant two-body forces are shown to give rise to a series of levels in three-particle systems. The number of such levels may be very large. Possibility of the existence of such levels in systems of three α-particles (^{12}C nucleus) and three nucleons (^{3}H) is discussed.

The range of nucleon-nucleon forces r_0 is known to be considerably smaller than the scattering lengts a. This fact is a consequence of the resonant character of nucleon-nucleon forces. Apart from this, many other forces in nuclear physics are resonant. The aim of this letter is to expose an interesting effect of resonant forces in a three-body system. Namely, for $a \gg r_0$ a series of bound levels appears. In a certain case, the number of levels may become infinite.

Let us explicitly formulate this result in the simplest case. Consider three spinless neutral

ticle bound states emerge one after the other. At $g = g_0$ (infinite scattering length) their number is infinite. As g grows on beyond g_0, levels leave into continuum one after the other (see fig. 1).

The number of levels is given by the equation

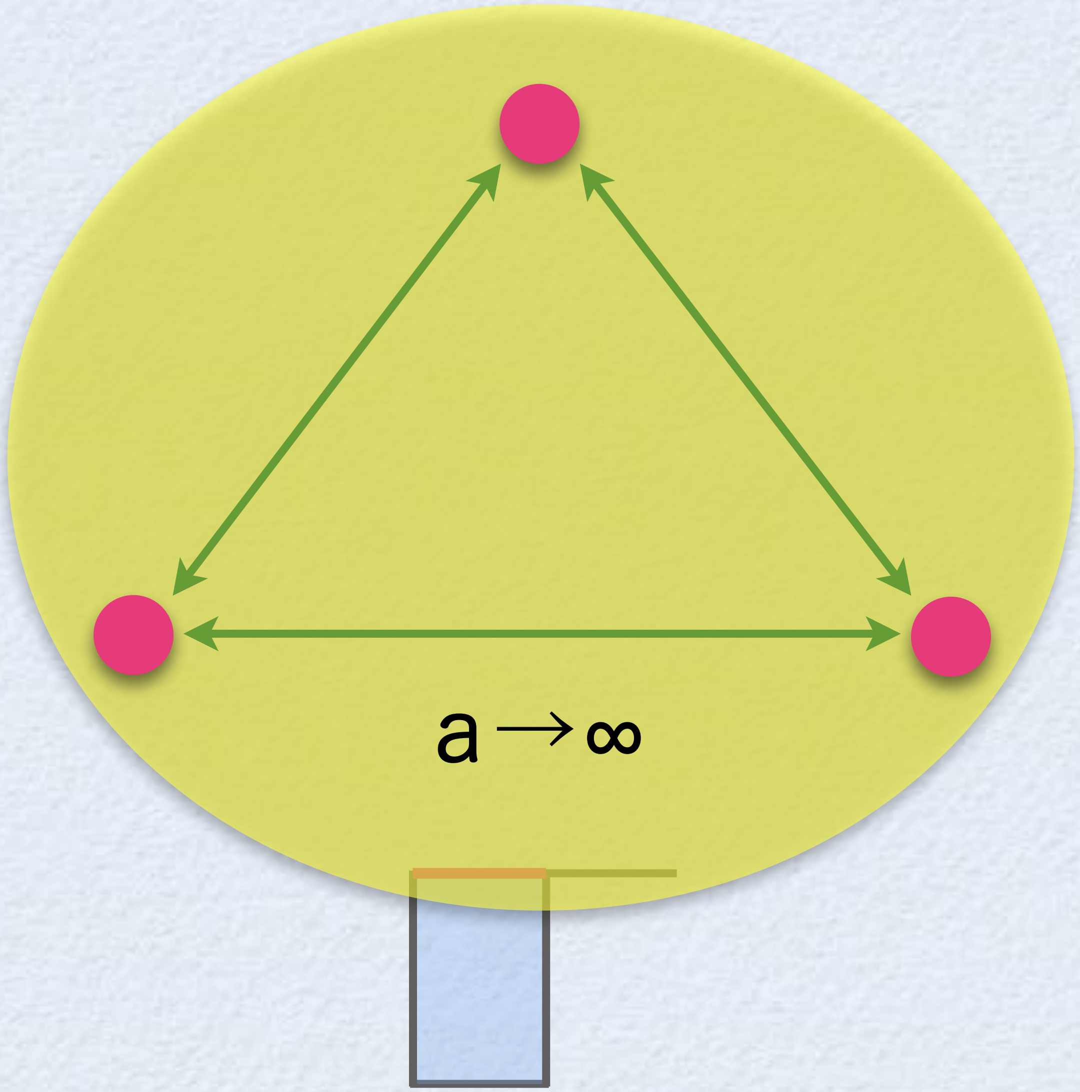
$$N \approx \frac{1}{\pi} \ln(|a|/r_0) \qquad (1)$$

All the levels are of the 0^+ kind; corresponding wave functions are symmetric; the energies $E_N \ll 1/r_0^2$ (we use $\hbar = m = 1$); the range of these bound states is much larger than r_0.

Efimov effect

When 2 bosons interact with infinite “a”,
3 bosons always form a series of bound states

Efimov (1970)



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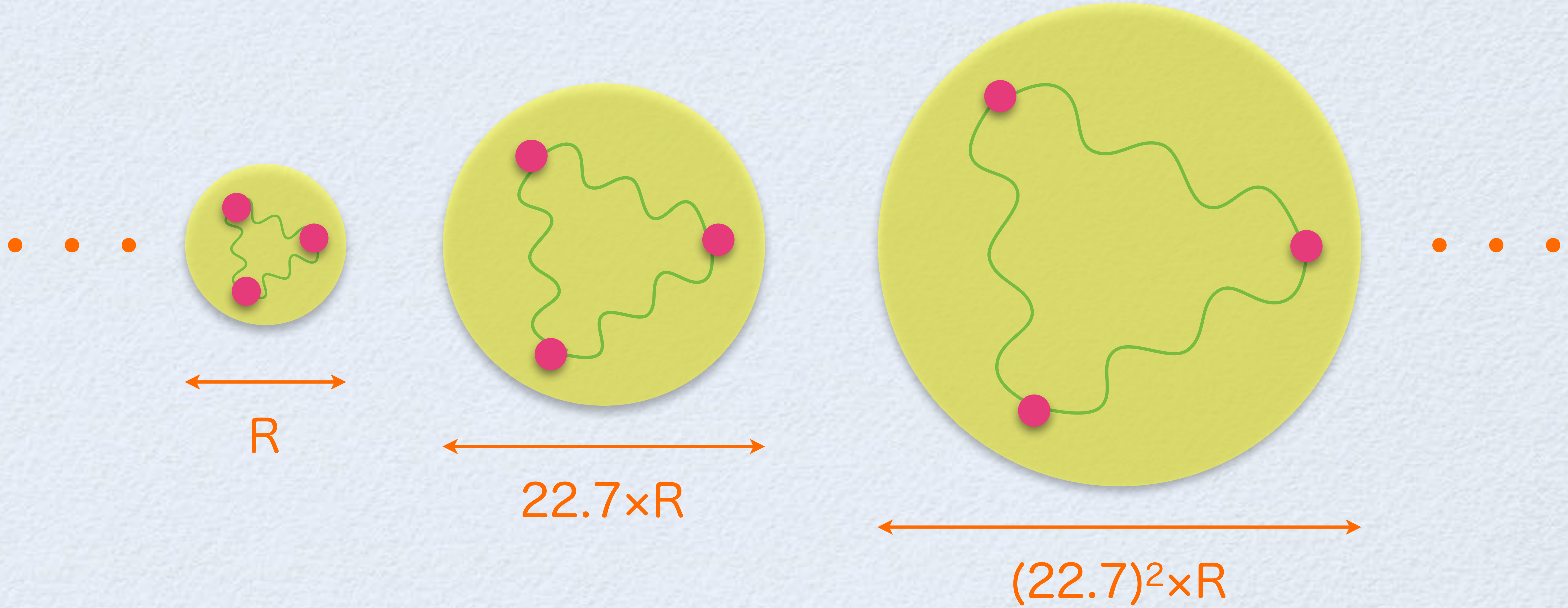

Discrete scaling symmetry

Efimov effect

When 2 bosons interact with infinite “a”,
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Efimov (1970)

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• • •

Discrete scaling symmetry

Why Efimov effect happens ?

Keywords

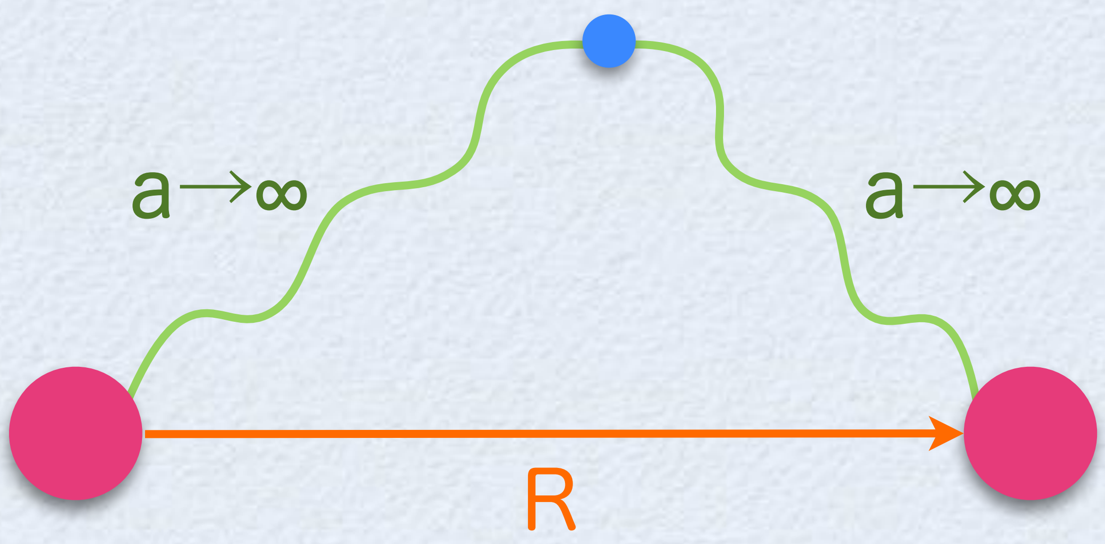
✓ Universality

- Scale invariance
- Quantum anomaly
- RG limit cycle

Why Efimov effect happens ?

Two heavy (M) and one light (m) particles

⇨ Born-Oppenheimer approximation



Binding energy of a light particle

$$E_b(R) = -\boxed{\frac{\hbar^2}{2mR^2}} \times (0.5671\ldots)^2$$

Scale invariance at a→∞

Schrödinger equation of two heavy particles :

$$\left[-\frac{\hbar^2}{M}\frac{\partial^2}{\partial \boldsymbol{R}^2} + V(R)\right]\psi(\boldsymbol{R}) = -\frac{\hbar^2\kappa^2}{M}\psi(\boldsymbol{R}) \qquad V(R) \equiv E_b(R)$$

Why Efimov effect happens ?

Schrödinger equation of two heavy particles :

$$\left[-\frac{\hbar^2}{M}\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}\right)-\frac{\hbar^2}{2mR^2}(0.5671\ldots)^2\right]\psi(R)=-\frac{\hbar^2\kappa^2}{M}\psi(R)$$

$$\psi(R)=R^{-1/2}K_{is_0}(\kappa R) \qquad s_0^2\equiv\frac{M}{2m}(0.5671\ldots)^2-\frac{1}{4}$$

$$\to R^{-1/2}\sin[s_0\ln(\kappa R)+\delta] \qquad (R\to 0)$$

ψ'/ψ has to be fixed by short-range physics

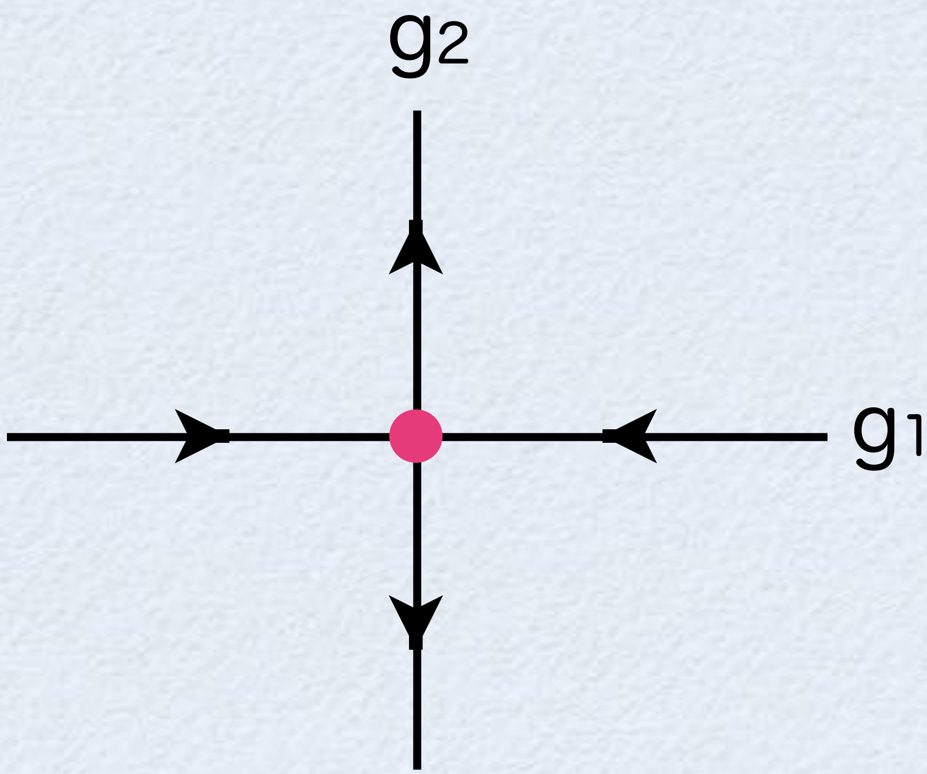
➡ If $\kappa=\kappa_*$ is a solution, $\kappa=(e^{\pi/s_0})^{-n}\kappa_*$ are solutions!

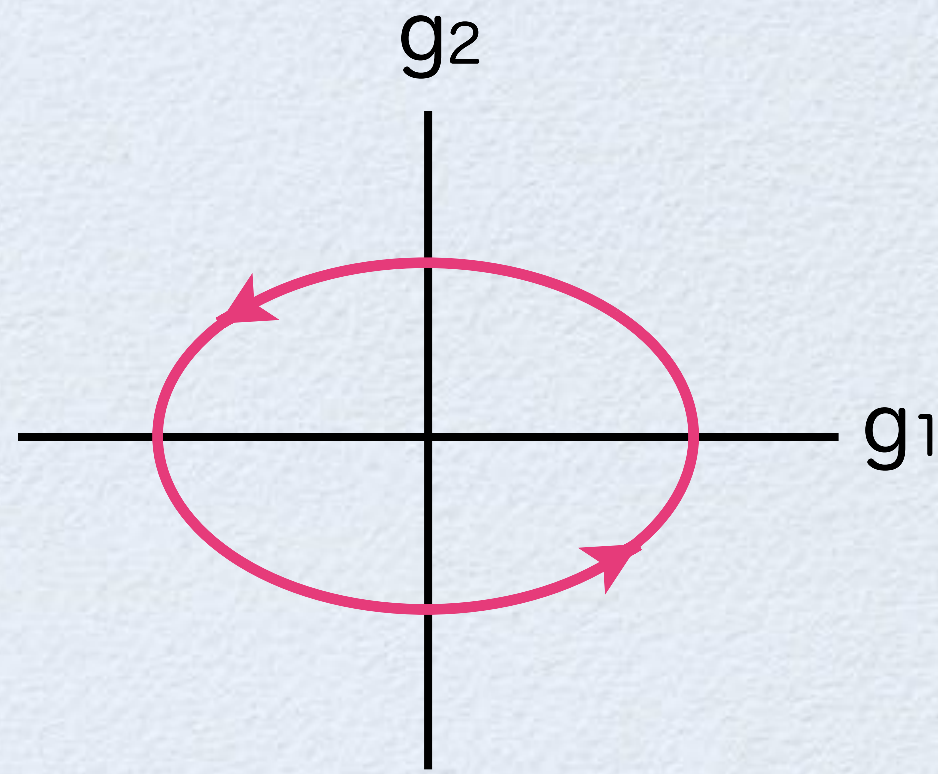
Classical scale invariance is broken by κ_*

= Quantum anomaly

Renormalization group limit cycle

Renormalization group flow diagram in coupling space





RG fixed point

⇒ Scale invariance

E.g. critical phenomena

RG limit cycle

⇒ Discrete scale invariance

E.g. E????v effect

Renormalization group limit cycle

K. Wilson (1971) considered for strong interactions

L REVIEW D VOLUME 3, NUMBER 8 15 APRIL 1971

Renormalization Group and Strong Interactions*

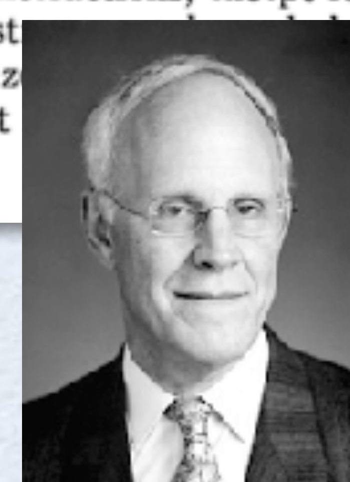
KENNETH G. WILSON
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
and
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850†

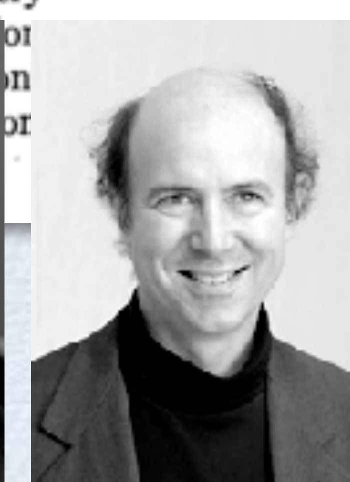
(Received 30 November 1970)

The renormalization-group method of Gell-Mann and Low is applied to field theories of strong interactions. It is assumed that renormalization-group equations exist for strong interactions which involve one or several momentum-dependent coupling constants. The further assumption that these coupling constants approach fixed values as the momentum goes to infinity is discussed in detail. However, an alternative is suggested, namely, that these coupling constants approach a limit cycle in the limit of large momenta. Some results of this paper are: (1) The e^+-e^- annihilation experiments above 1-GeV energy may distinguish a fixed point from a limit cycle or other asymptotic behavior. (2) If electrodynamics or weak interactions become strong above some large momentum Λ, then the renormalization group can be used (in principle) to determine the renormalized coupling constants of strong interactions, except for $U(3)\times U(3)$ symmetry-breaking parameters. (3) Mass terms in the Lagrangian of st… must break a symmetry of the combined interactions with z… weak interactions can be understood assuming only that … interactions.

QCD is asymptotic free

(2004 Nobel prize)





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Efimov effect (1970) is its rare manifestation!

Effective field theory

PHYSICAL REVIEW LETTERS

VOLUME 82 | 18 JANUARY 1999 | NUMBER 3

Renormalization of the Three-Body System with Short-Range Interactions

P. F. Bedaque,[1,*] H.-W. Hammer,[2,†] and U. van Kolck[3,4,‡]

[1]*Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195*
[2]*TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada V6T 2A3*
[3]*Kellogg Radiation Laboratory, 106-38, California Institute of Technology, Pasadena, California 91125*
[4]*Department of Physics, University of Washington, Seattle, Washington 98195*
(Received 9 September 1998)

We discuss renormalization of the nonrelativistic three-body problem with short-range forces. The problem becomes nonperturbative at momenta of the order of the inverse of the two-body scattering length, and an infinite number of graphs must be summed. This summation leads to a cutoff dependence that does not appear in any order in perturbation theory. We argue that this cutoff dependence can be absorbed in a single three-body counterterm and compute the running of the three-body force with the cutoff. We comment on the relevance of this result for the effective field theory program in nuclear and molecular physics. [S0031-9007(98)08276-3]

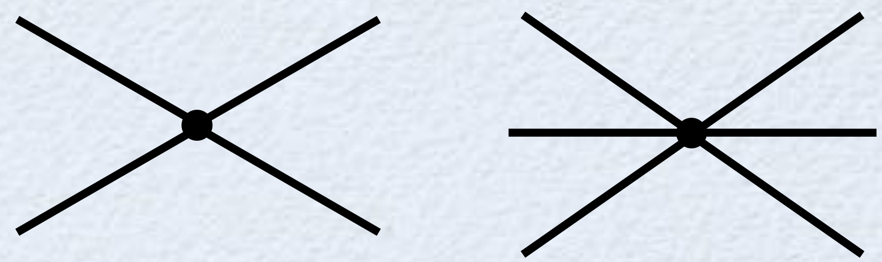
PACS numbers: 03.65.Nk, 11.80.Jy, 21.45.+v, 34.20.Gj

Systems composed of particles with momenta k much

dence can be absorbed in the coefficients of the leading-

Effective field theory

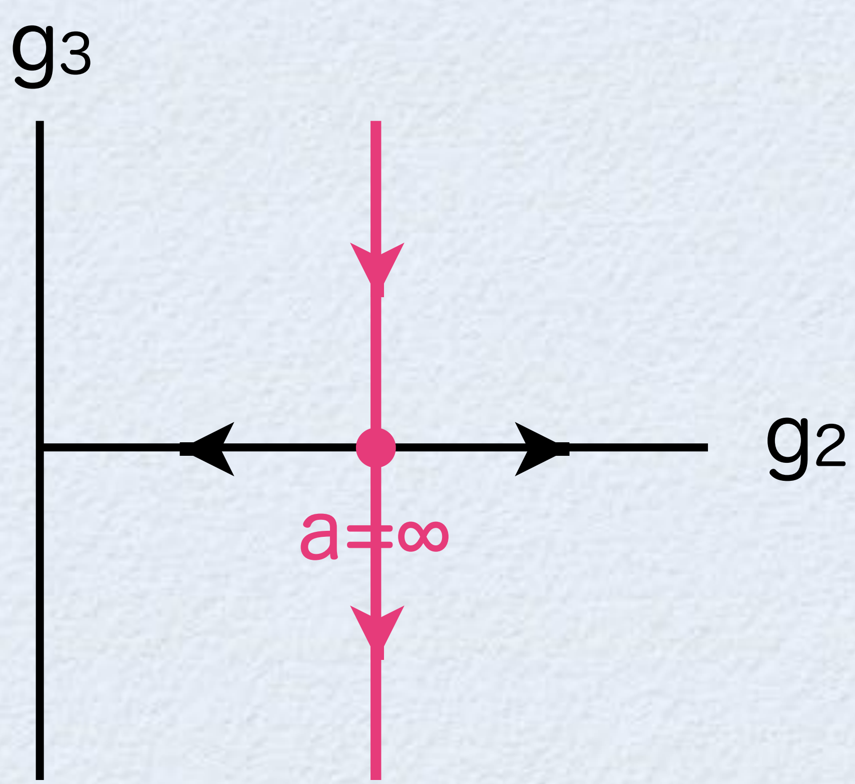
$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi + g_2 (\psi^\dagger \psi)^2 + g_3 (\psi^\dagger \psi)^3$$



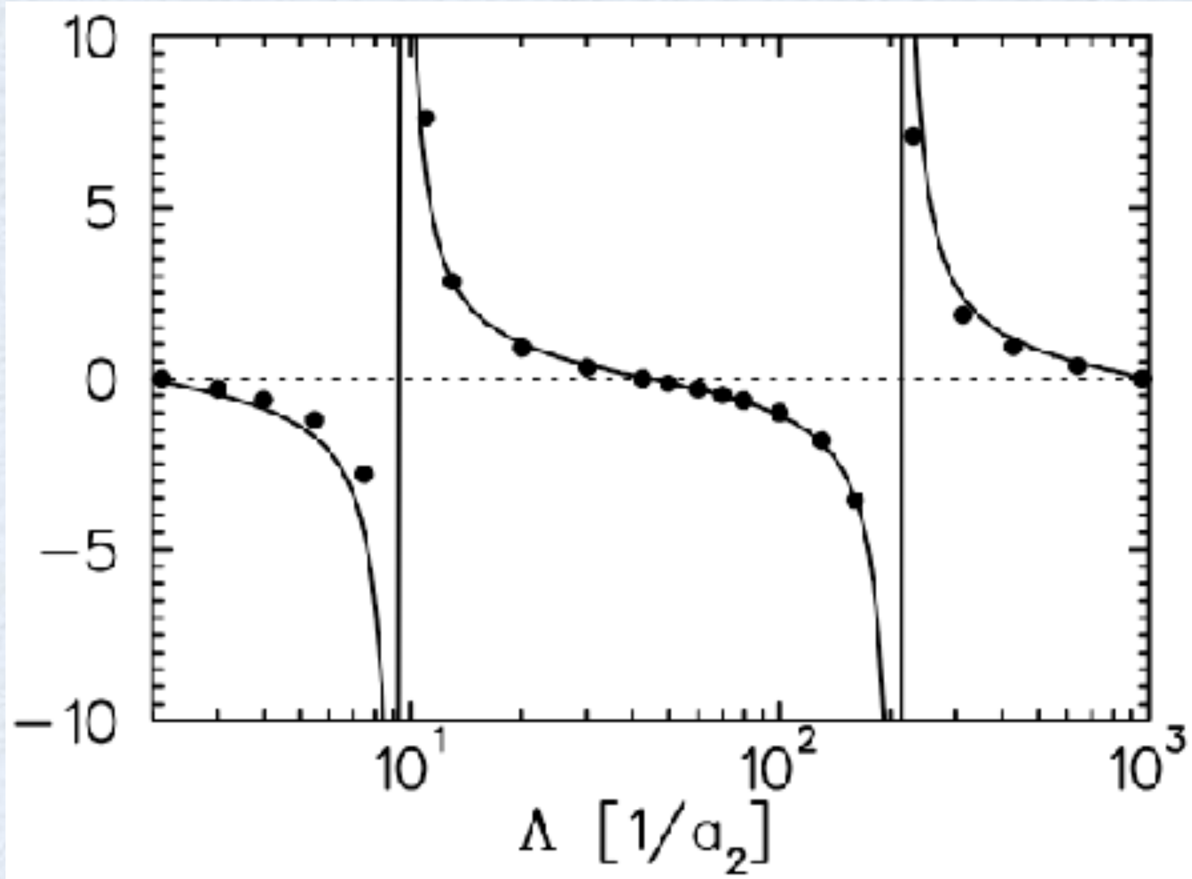
g_2 has a fixed point corresponding to a=∞

What is flow of g_3?

$$g_3(\Lambda) = -\frac{\sin[s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0)]}{\sin[s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0)]}$$



RG limit cycle



Effective field theory

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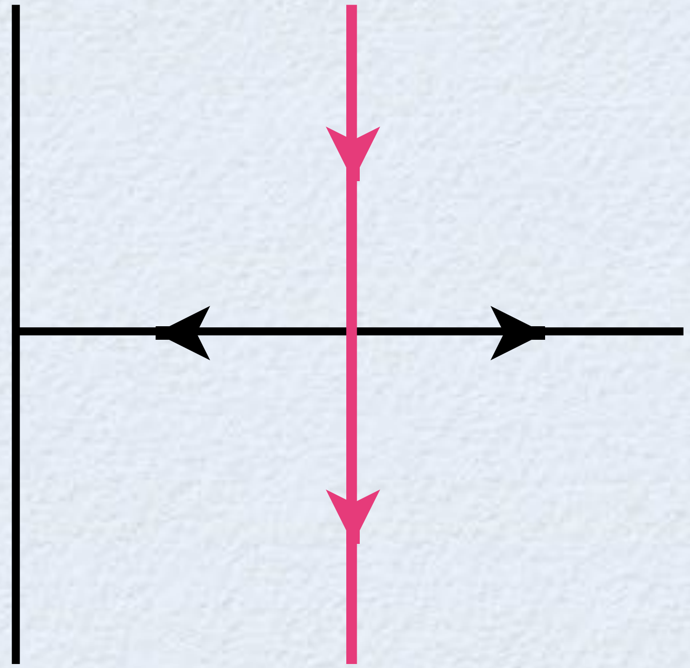
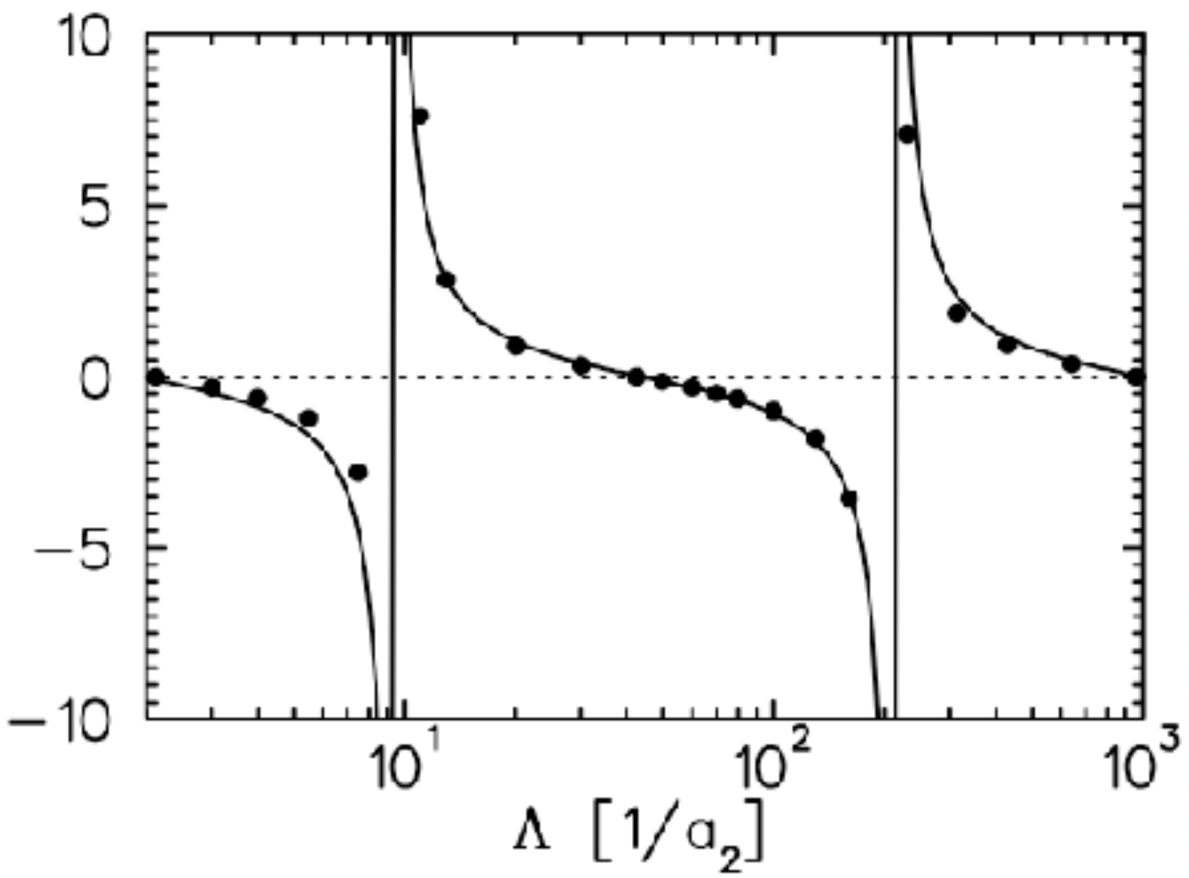
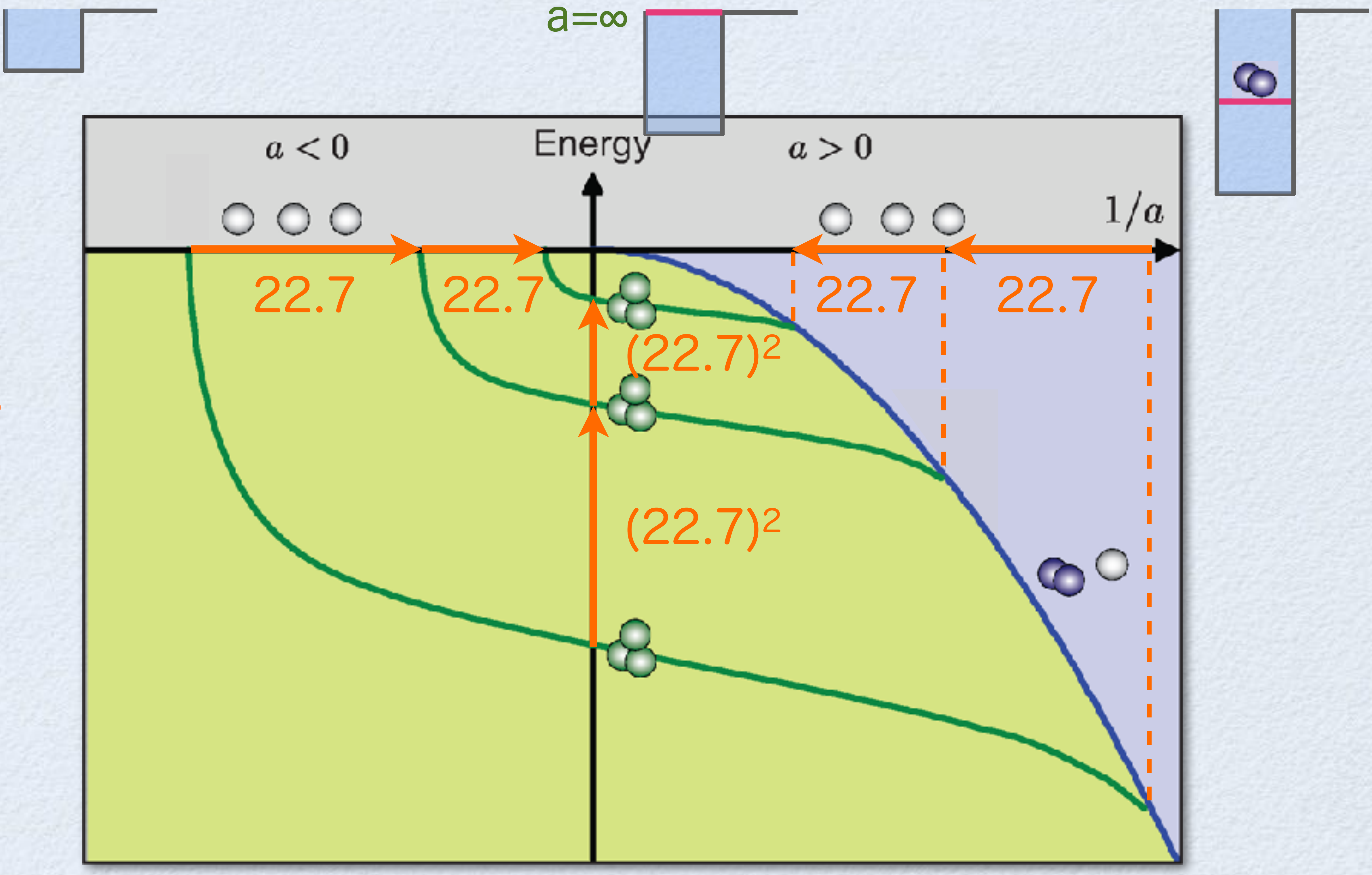




Efimov effect at a≠∞



Ferlaino & Grimm, Physics (2010)

Discrete scaling symmetry

Why 22.7 ?

Just a numerical number given by

22.6943825953666951928602171369...

log(22.6943825953666951928602171369...)

= 3.122211743110421968073091732438...

= π / 1.00623782510278148906406681234...

= π / s_0

$$\frac{2\pi}{s_0}\frac{\sinh(\frac{\pi}{6}s_0)}{\cosh(\frac{\pi}{2}s_0)}=\frac{\sqrt{3}\pi}{4}$$

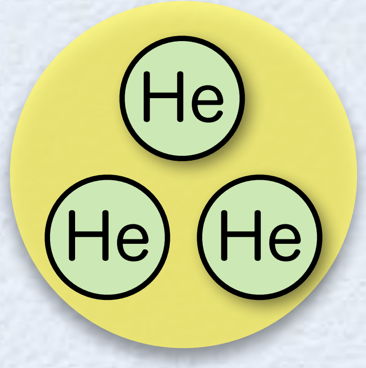
22.7 = exp(π / 1.006...)

Where Efimov effect appears ?

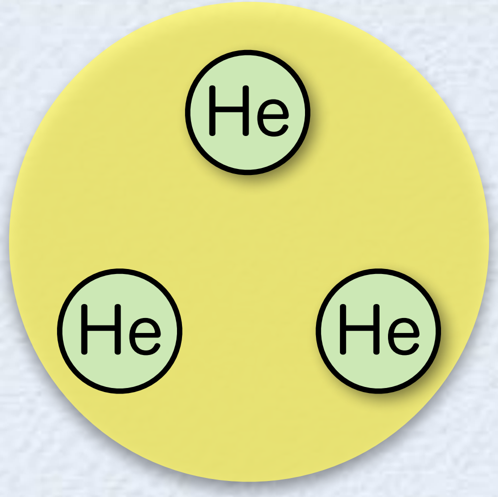
× Originally, Efimov considered
^{3}H nucleus (≈3n) and ^{12}C nucleus (≈3α)

△ ^{4}He atoms ($a \approx 1\times10^{-8}$ m $\approx 20 r_0$) ?

2 trimer states were predicted
and observed in 1994 and 2015



E_b = 125.8 mK

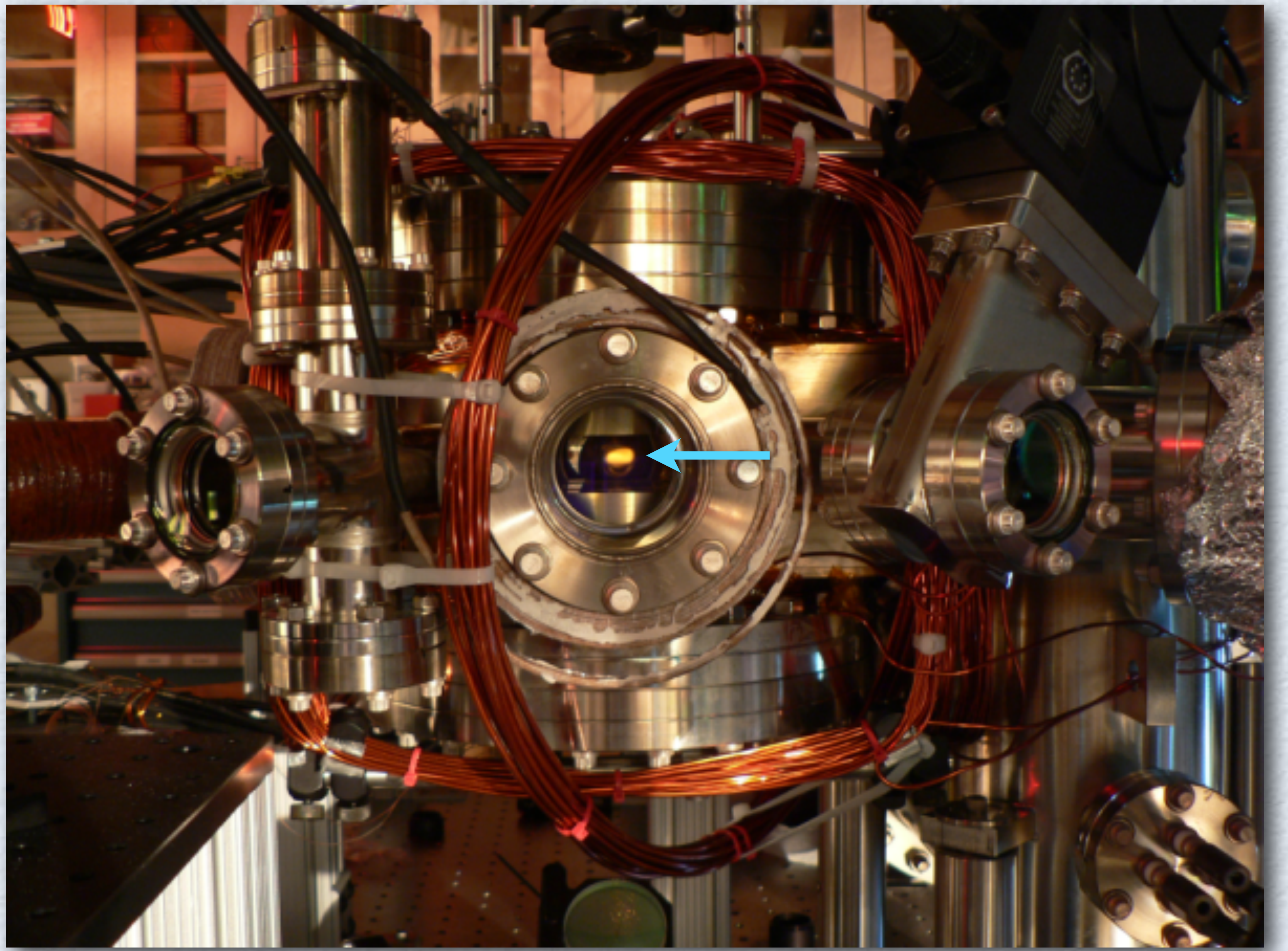


E_b = 2.28 mK

→ Ultracold atoms !

Ultracold atom experiments

Ultracold atoms are ideal to study universal quantum physics because of the ability to design and control systems at will



Ultracold atom experiments

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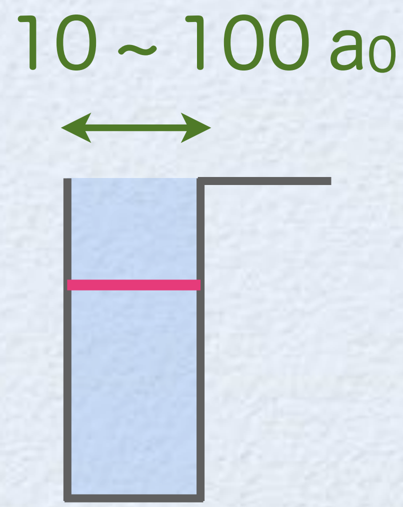
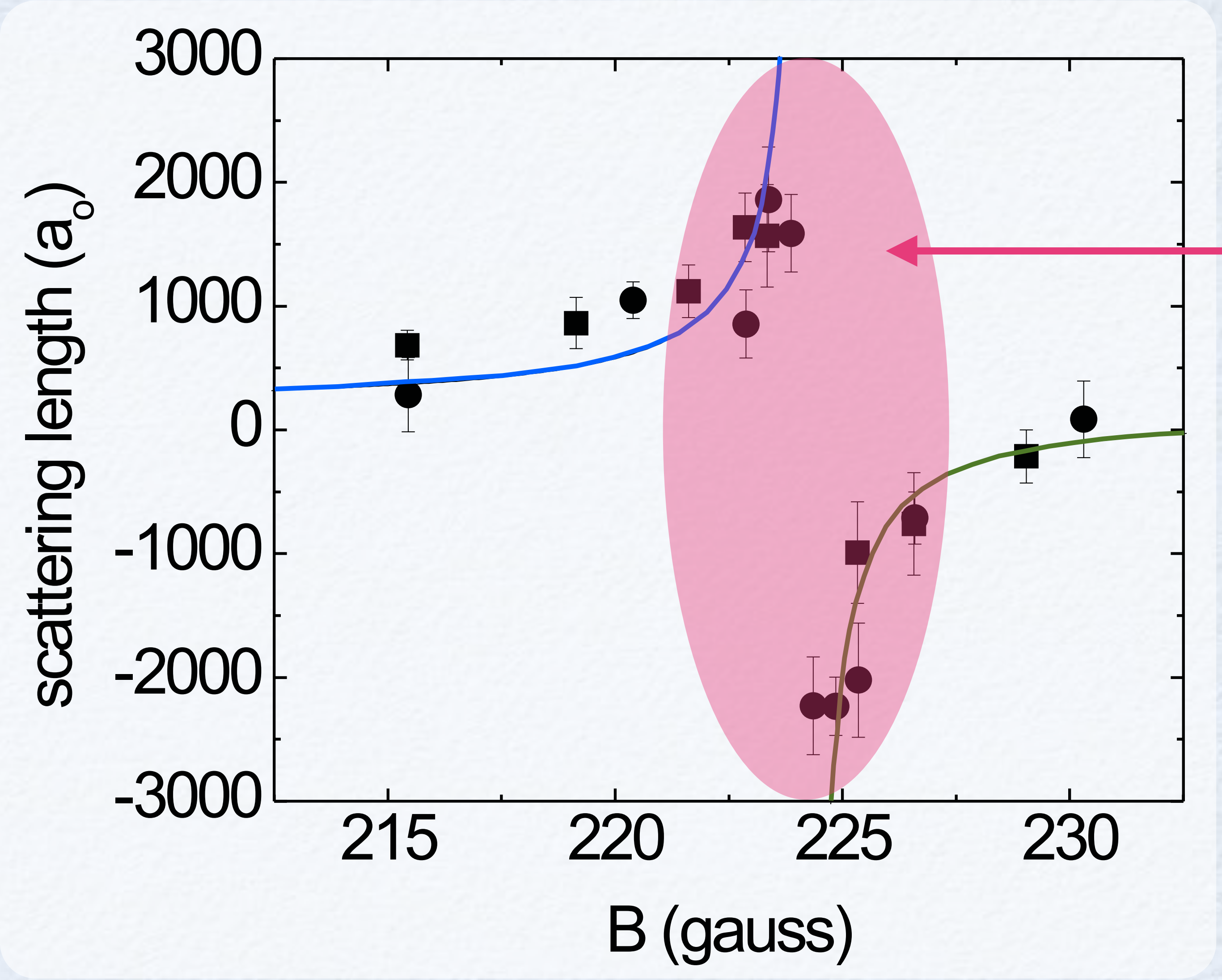
✓ Interaction strength by Feshbach resonances





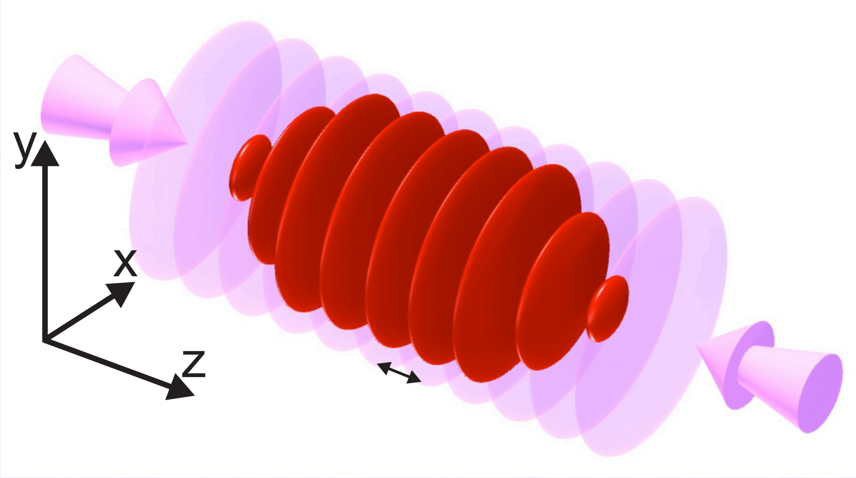
C.A. Regal & D.S. Jin
PRL90 (2003)

Ultracold atom experiments

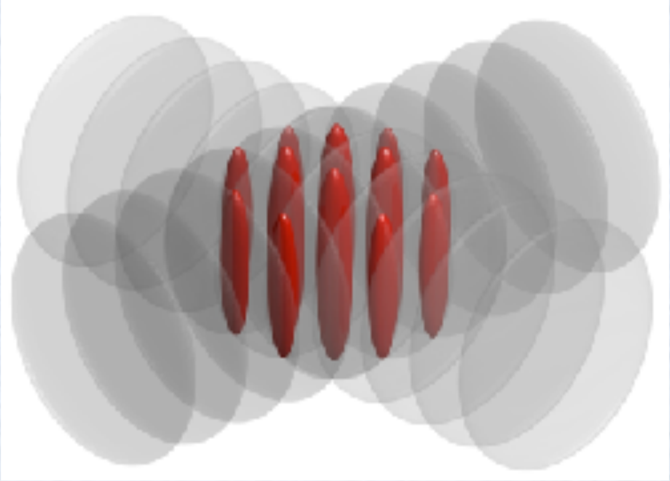
Ultracold atoms are ideal to study universal quantum physics because of the ability to design and control systems at will

- ✓ Interaction strength by Feshbach resonances
- ✓ Spatial dimensions by strong optical lattices

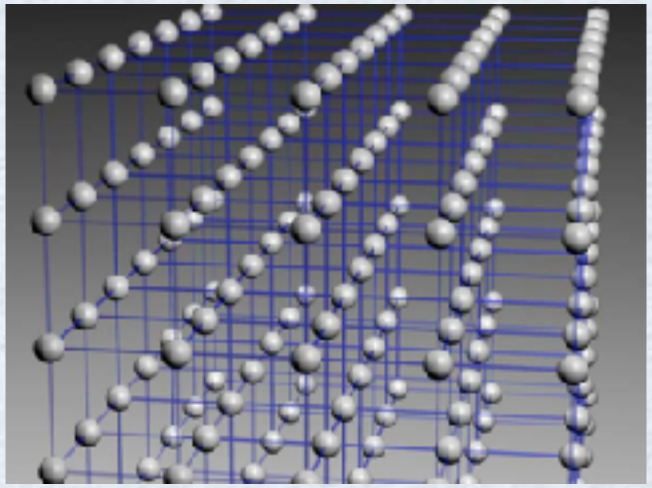
2D

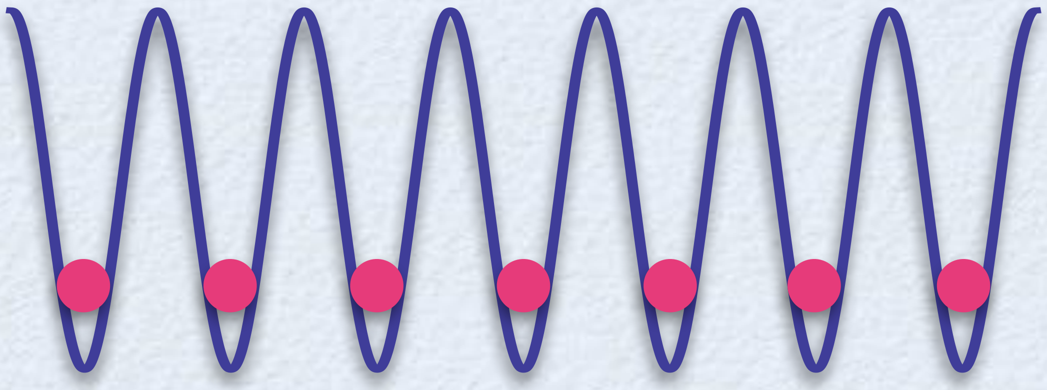


1D



0D





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Ultracold atoms are ideal to study universal quantum physics because of the ability to design and control systems at will

✓ Interaction strength by Feshbach resonances

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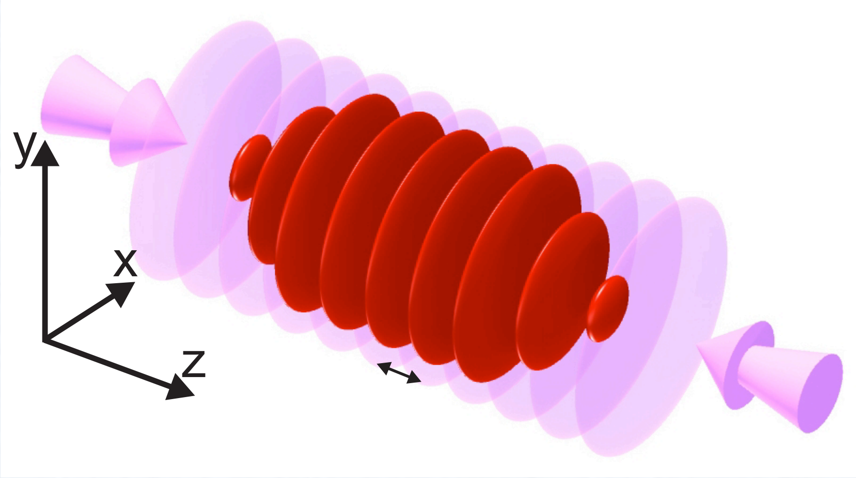
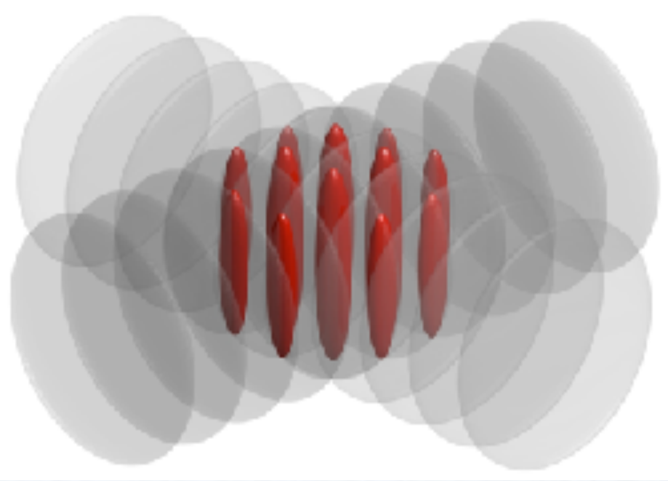
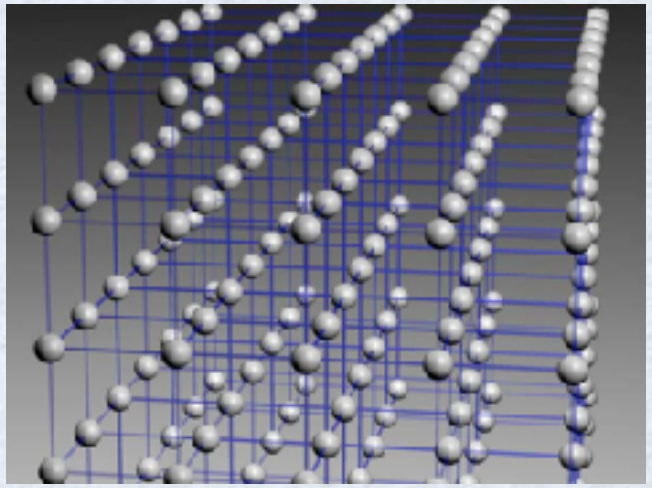


1D



0D

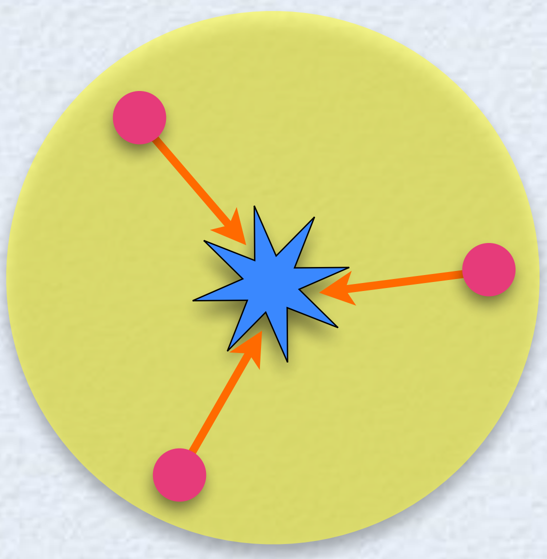


✓ Quantum statistics of particles

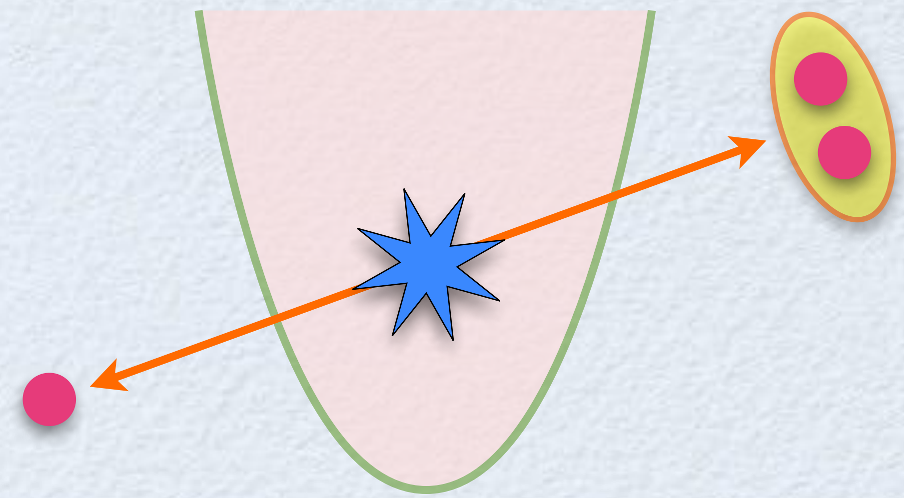
- Bosonic atoms (^{7}Li, ^{23}Na, ^{39}K, ^{41}K, ^{87}Rb, ^{133}Cs, ...)
- Fermionic atoms (^{6}Li, ^{40}K, ...)

Ultracold atom experiments

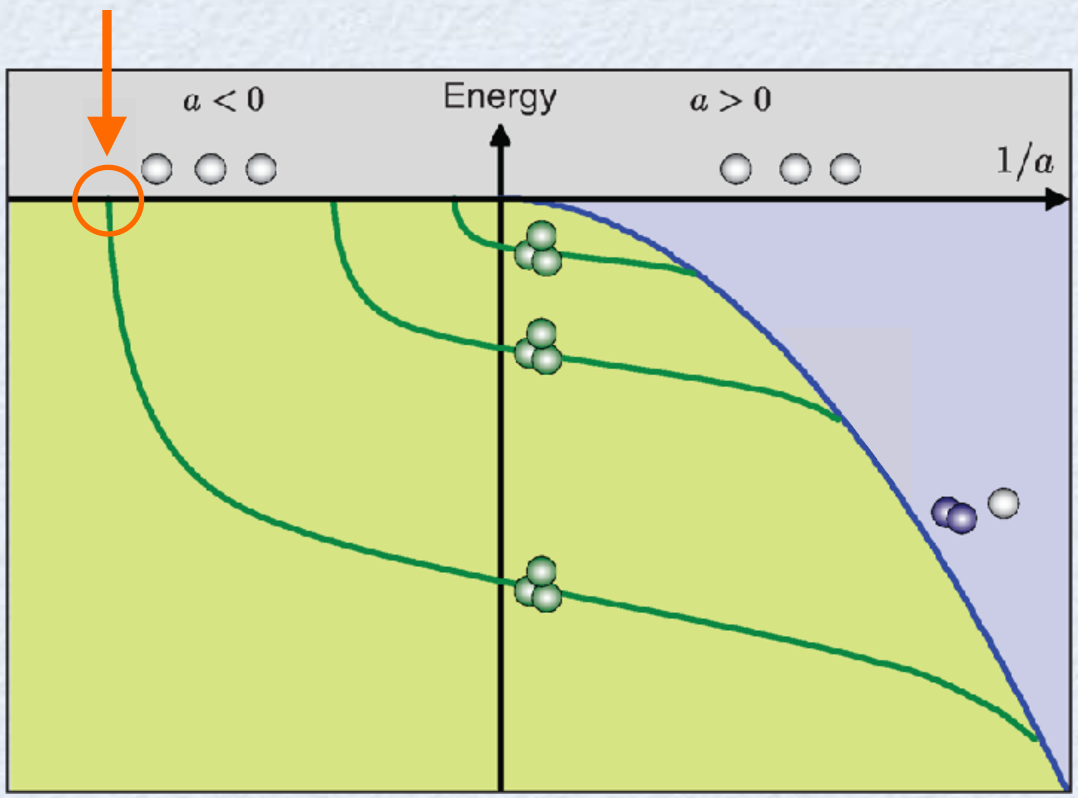
First experiment by Innsbruck group for ^{133}Cs (2006)

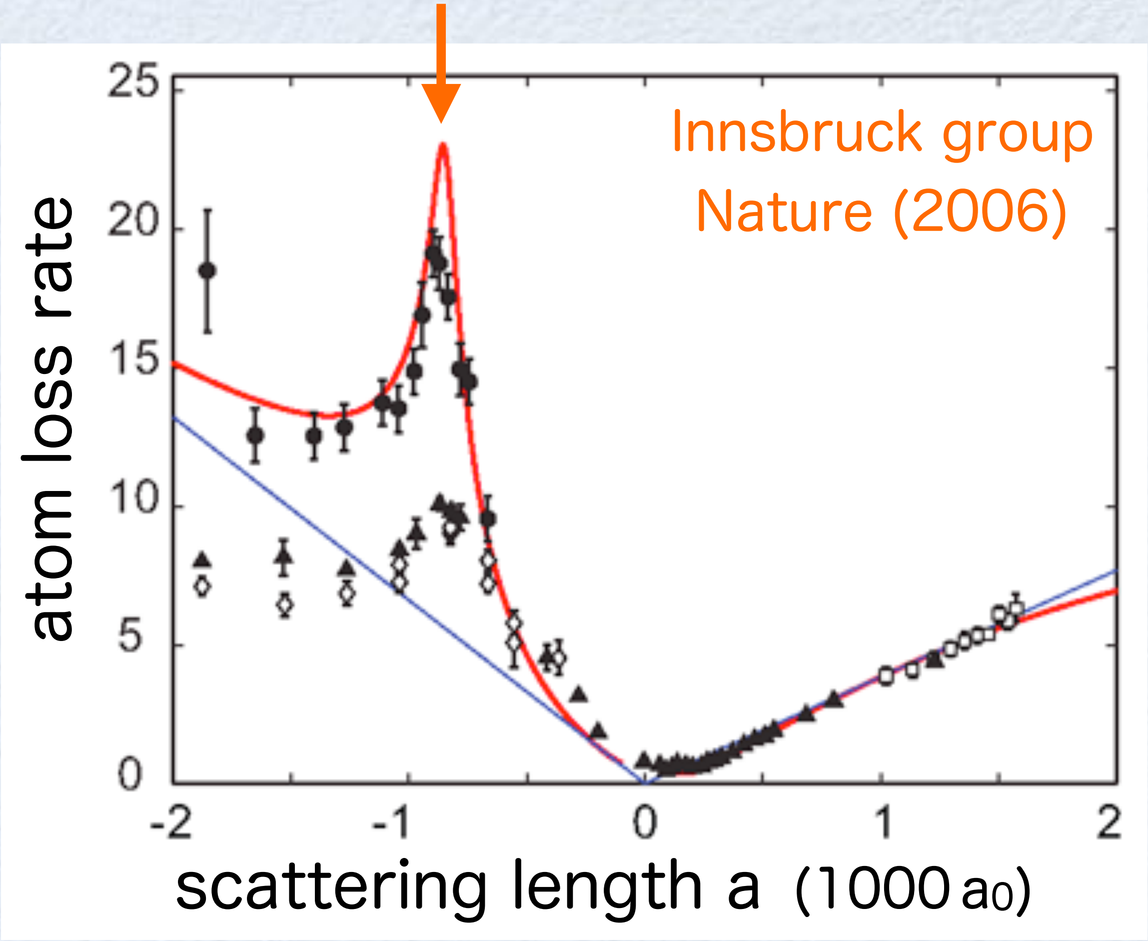


Trimer is unstable → atom loss

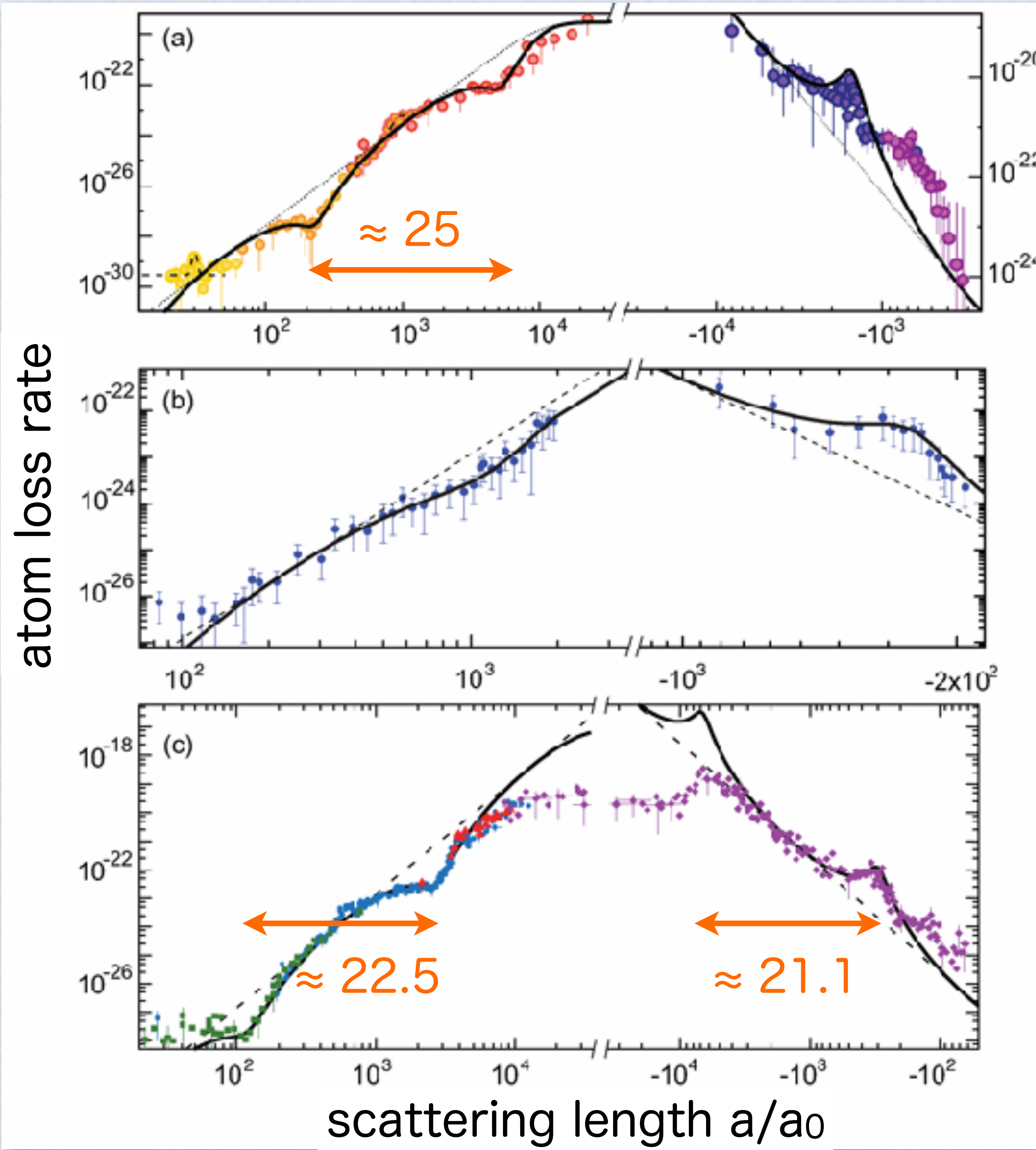


signature of trimer formation





Ultracold atom experiments



Florence group
for ^{39}K (2009)

Bar-Ilan University
for ^{7}Li (2009)

Rice University
for ^{7}Li (2009)

Discrete scaling
& Universality !

Application

1. Universality in physics
2. What is the Efimov effect?
3. **Application: Quantum magnets**
4. Extension: Super Efimov effect

Magnons

nature physics

ARTICLES

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Efimov effect in quantum magnets

Yusuke Nishida*, Yasuyuki Kato and Cristian D. Batista

Physics is said to be universal when it emerges regardless of the underlying microscopic details. A prominent example is the Efimov effect, which predicts the emergence of an infinite tower of three-body bound states obeying discrete scale invariance when the particles interact resonantly. Because of its universality and peculiarity, the Efimov effect has been the subject of extensive research in chemical, atomic, nuclear and particle physics for decades. Here we employ an anisotropic Heisenberg model to show that collective excitations in quantum magnets (magnons) also exhibit the Efimov effect. We locate anisotropy-induced two-magnon resonances, compute binding energies of three magnons and find that they fit into the universal scaling law. We propose several approaches to experimentally realize the Efimov effect in quantum magnets, where the emergent Efimov states of magnons can be observed with commonly used spectroscopic measurements. Our study thus opens up new avenues for universal few-body physics in condensed matter systems.

Sometimes we observe that completely different systems exhibit the same physics. Such physics is said to be universal and its most famous example is the critical phenomena[1]. In the vicinity of second-order phase transitions where the correlation length diverges, microscopic details become unimportant and the critical phenomena are characterized by only a few ingredients; dimensionality, interaction range and symmetry of the order parameter. Accordingly, fluids and magnets exhibit the same critical exponents. The universality in critical phenomena has been one of the central themes in condensed matter physics.

Similarly, we can also observe universal physics in the vicinity of scattering resonances where the *s*-wave scattering length diverges. Here low-energy physics is characterized solely by the *s*-wave scattering length and does not depend on other microscopic details.

emergent Efimov states of magnons. Our study thus opens up new avenues for universal few-body physics in condensed matter systems. Also, in addition to the Bose–Einstein condensation of magnons[24], the Efimov effect provides a novel connection between atomic and magnetic systems.

Anisotropic Heisenberg model

To demonstrate the Efimov effect in quantum magnets, we consider an anisotropic Heisenberg model on a simple cubic lattice:

$$H = -\frac{1}{2}\sum_{\mathbf{r}}\sum_{\hat{\mathbf{e}}}(J\, S^+_{\mathbf{r}} S^-_{\mathbf{r}+\hat{\mathbf{e}}} + J_z S^z_{\mathbf{r}} S^z_{\mathbf{r}+\hat{\mathbf{e}}}) - D\sum_{\mathbf{r}}(S^z_{\mathbf{r}})^2 - B\sum_{\mathbf{r}} S^z_{\mathbf{r}} \quad (2)$$

where $\sum_{\hat{\mathbf{e}}}$ is a sum over six unit vectors; $\sum_{\hat{\mathbf{e}}=\pm\hat{\mathbf{x}},\pm\hat{\mathbf{y}},\pm\hat{\mathbf{z}}}$. Two types

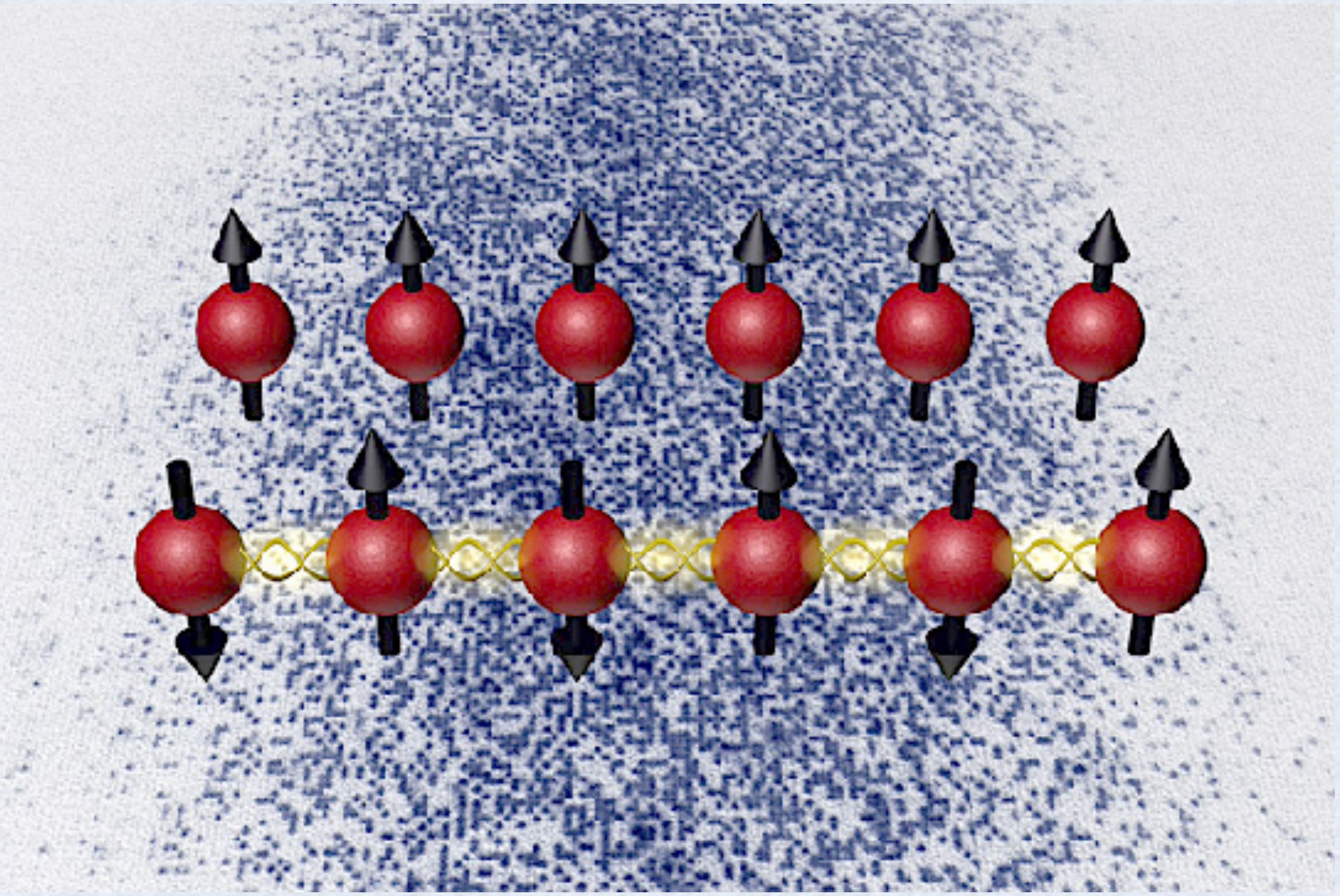
Quantum magnet

Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\left[\sum_{\hat{e}}(JS_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z) + D(S_r^z)^2 - BS_r^z\right]$$

exchange anisotropy

single-ion anisotropy



Quantum magnet

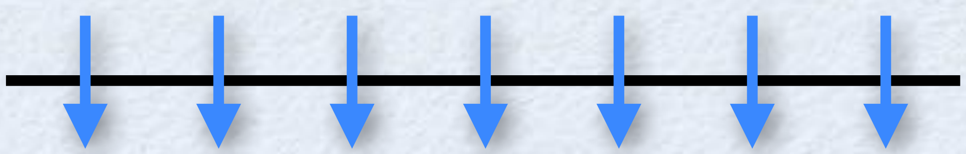
Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\left[\sum_{\hat{e}}\left(J S_r^{+} S_{r+\hat{e}}^{-} + J_z S_r^{z} S_{r+\hat{e}}^{z}\right) + D\left(S_r^{z}\right)^2 - B S_r^{z}\right]$$

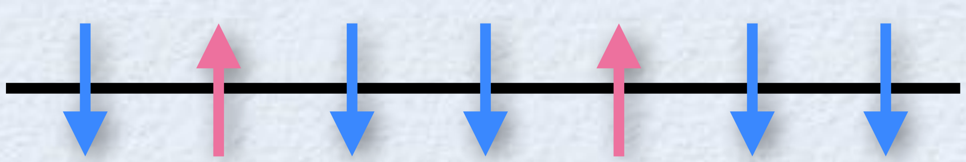
exchange anisotropy (for J and J_z)

single-ion anisotropy (for D)

Spin-boson correspondence

 ⇔

fully polarized state (B→∞) — No boson = vacuum

 ⇔ 

N spin-flips — N bosons = magnons

Quantum magnet

Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_r \left[\sum_{\hat{e}} (J S_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z) + D (S_r^z)^2 - B S_r^z \right]$$

xy-exchange coupling

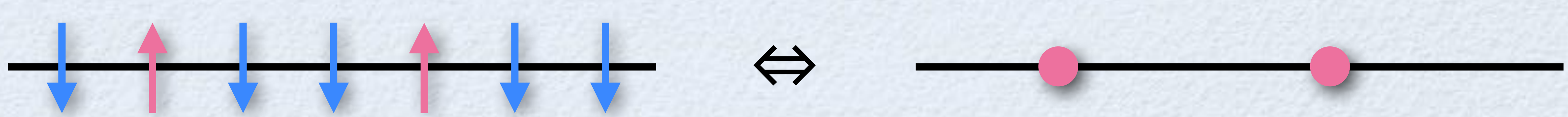
⇔ hopping

single-ion anisotropy

⇔ on-site attraction

z-exchange coupling

⇔ neighbor attraction



N spin-flips ⇔ N bosons = magnons

Quantum magnet

Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\left[\sum_{\hat{e}}(JS_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z) + D(S_r^z)^2 - BS_r^z\right]$$

xy-exchange coupling
⇔ hopping

single-ion anisotropy
⇔ on-site attraction

z-exchange coupling
⇔ neighbor attraction

Tune these couplings to induce
scattering resonance between two magnons
⇒ Three magnons show the Efimov effect

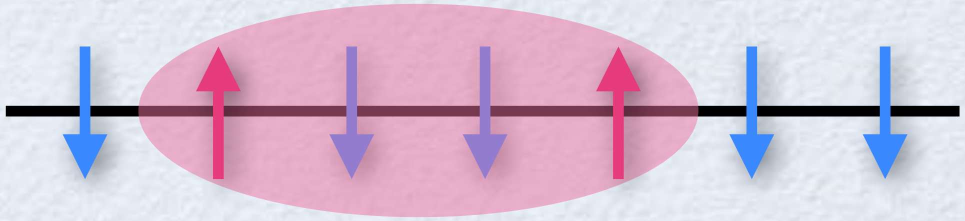
Two-magnon resonance

Scattering length between two magnons

$$\frac{a_s}{a} = \frac{\frac{3}{2\pi}\left[1 - \frac{D}{3J} - \frac{J_z}{J}\left(1 - \frac{D}{6SJ}\right)\right]}{2S - 1 + \frac{J_z}{J}\left(1 - \frac{D}{6SJ}\right) + 1.52\left[1 - \frac{D}{3J} - \frac{J_z}{J}\left(1 - \frac{D}{6SJ}\right)\right]}$$

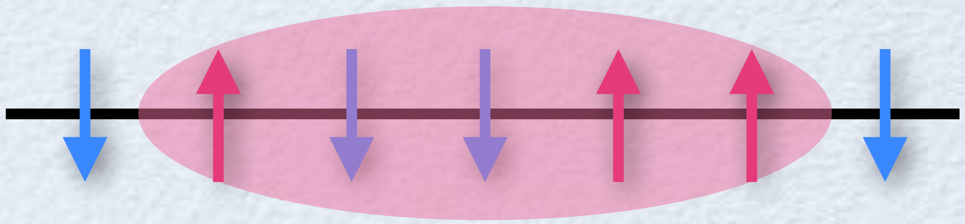
Two-magnon resonance ($a_s \to \infty$)

- $J_z/J = 2.94$ (spin-1/2)
- $J_z/J = 4.87$ (spin-1, D=0)
- $D/J = 4.77$ (spin-1, ferro $J_z=J>0$)
- $D/J = 5.13$ (spin-1, antiferro $J_z=J<0$)
- …



Three-magnon spectrum

At the resonance, three magnons form bound states with binding energies E_n



- **Spin-1/2**

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-2.09×10^{-1}	—
1	-4.15×10^{-4}	22.4
2	-8.08×10^{-7}	22.7

- **Spin-1, D=0**

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-5.16×10^{-1}	—
1	-1.02×10^{-3}	22.4
2	-2.00×10^{-6}	22.7

- **Spin-1, J_z=J>0**

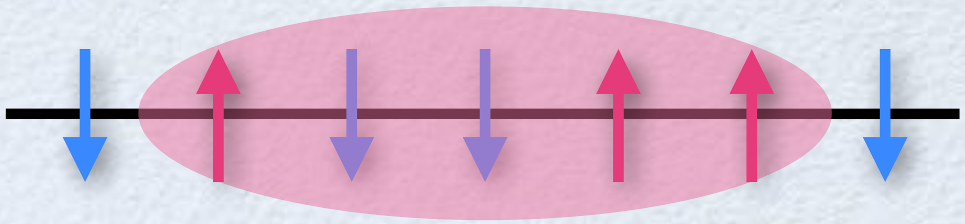
n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-5.50×10^{-2}	—
1	-1.16×10^{-4}	21.8

- **Spin-1, J_z=J<0**

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-4.36×10^{-3}	—
1	-8.88×10^{-6}	22.2

Three-magnon spectrum

At the resonance, three magnons form bound states with binding energies E_n



- Spin-1/2

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
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- Spin-1, D=0

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-5.16×10^{-1}	—
1	-1.02×10^{-3}	22.4
2	-2.00×10^{-6}	22.7

Universal scaling law by ~ 22.7 confirms they are Efimov states !

Lecture 2

How to solve the two-magnon problem ?

1. Schrödinger equation

2. wave function

3. scattering length

Extension

1. Universality in physics
2. What is the Efimov effect?
3. Application: Quantum magnets
4. **Extension: Super Efimov effect**

Few-body universality

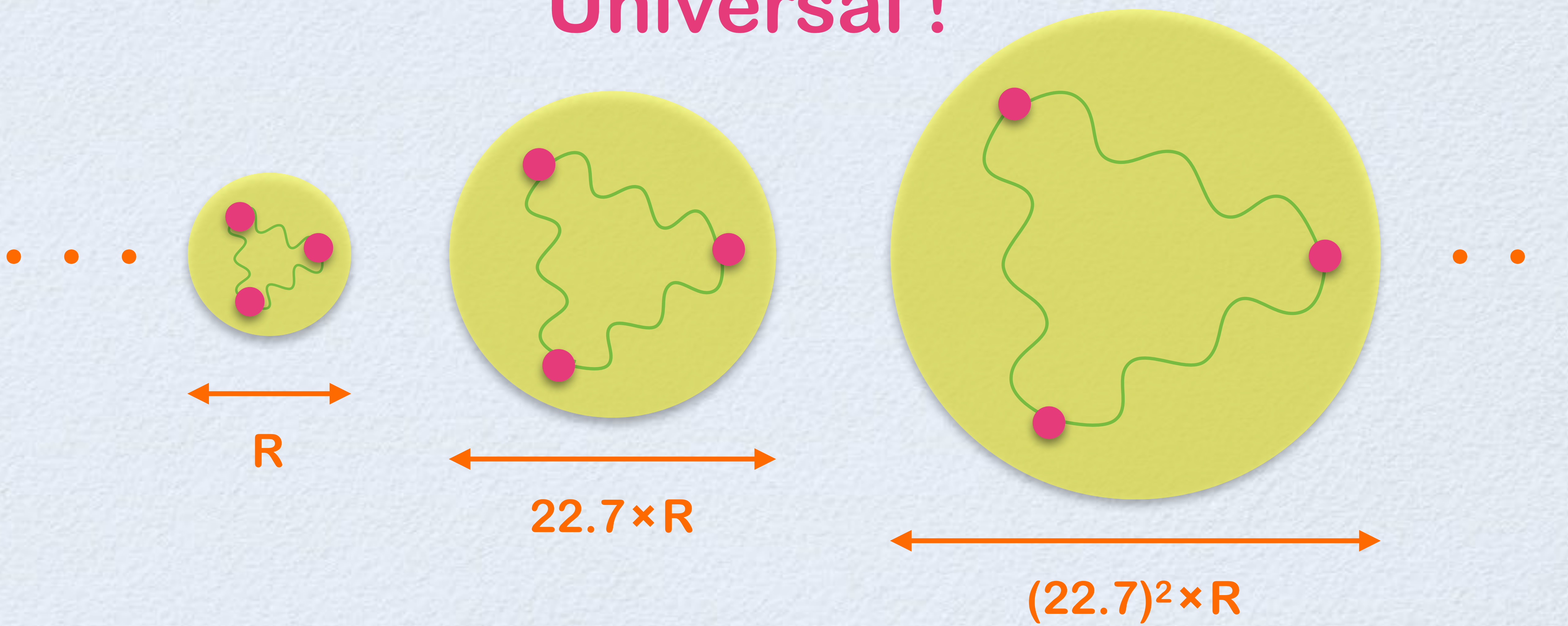
Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Universal !



Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Efimov effect in other systems ?

No, only in 3D with s-wave resonance

	s-wave	p-wave	d-wave
3D	O	x	x
2D	x	x	x
1D	x	x	

Y.N. & S.Tan, Few-Body Syst. (2011)

Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Different universality in other systems ?

Yes, super Efimov effect in 2D with p-wave !

	s-wave	p-wave	d-wave
3D	O	x	x
2D	x	!x!	x
1D	x	x	

Y.N. & S.Tan,
Few-Body Syst.
(2011)

Model confirmation

Spinless fermions with a separable potential

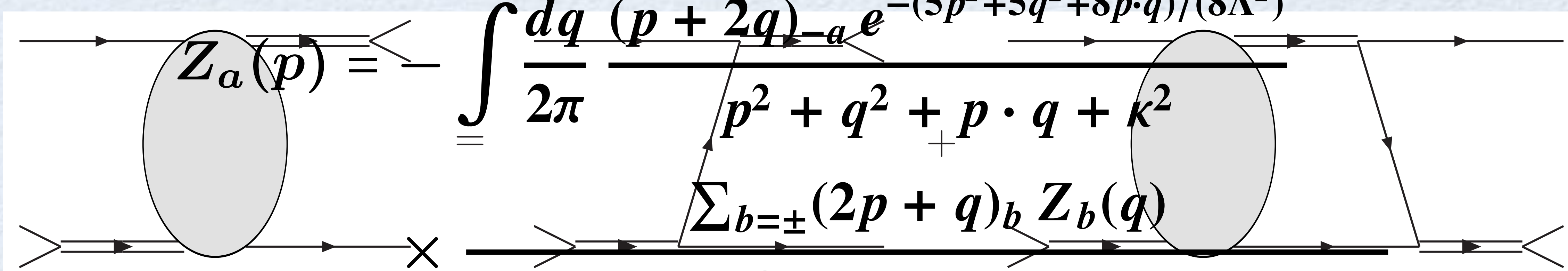
$$H = \int \frac{dk}{(2\pi)^2} \frac{k^2}{2m} \psi_k^\dagger \psi_k - v_0 \sum_{a=\pm} \int \frac{dk\,dp\,dq}{(2\pi)^6} \psi^\dagger_{\frac{k}{2}+p} \chi_a(p) \psi^\dagger_{\frac{k}{2}-p} \times \psi_{\frac{k}{2}-q} \chi_{-a}(q) \psi_{\frac{k}{2}+q}$$

v_0: resonance (a→∞)

$$\chi_\pm(p) = (p_x \pm i p_y)\, e^{-p^2/(2\Lambda^2)}$$

3-body binding energies $\lambda_n = \ln\ln (mE_n/\Lambda^2)^{-1/2}$

⇒ solve STM equation numerically



$$Z_a(p) = -\int \frac{dq}{2\pi} \frac{(p+2q)_{-a}\, e^{-(5p^2+5q^2+8p\cdot q)/(8\Lambda^2)}}{p^2+q^2+p\cdot q+\kappa^2} \times \frac{\sum_{b=\pm}(2p+q)_b\, Z_b(q)}{(\frac{3}{4}q^2+\kappa^2)\, e^{(\frac{3}{4}q^2+\kappa^2)/\Lambda^2}\, \mathrm{E}_1[(\frac{3}{4}q^2+\kappa^2)/\Lambda^2]}$$

Model confirmation

$$H = \int \frac{dk}{(2\pi)^2} \frac{k^2}{2m} \psi_k^\dagger \psi_k$$

$$- v_0 \sum_{a=\pm} \int \frac{dk\,dp\,dq}{(2\pi)^6} \underbrace{\psi^\dagger_{\frac{k}{2}+p} \chi_a(p) \psi^\dagger_{\frac{k}{2}-p}}_{} \times \underbrace{\psi_{\frac{k}{2}-q} \chi_{-a}(q) \psi_{\frac{k}{2}+q}}_{}$$

v_0: resonance (a→∞)

Spinless fermions with a separable potential

$$\chi_\pm(p) = (p_x \pm i p_y)\, e^{-p^2/(2\Lambda^2)}$$

3-body binding energies $\lambda_n = \ln\ln(mE_n/\Lambda^2)^{-1/2}$

n	λ_n	$\lambda_n - \lambda_{n-1}$
0	0.5632	—
1	2.770	2.207
2	5.078	2.308
3	7.430	2.352
4	9.785	2.355
5	12.141	2.356
∞	—	2.35619 ← $3\pi/4$

Super Efimov effect !

⇒ **doubly exponential scaling** $mE_n/\Lambda^2 \propto e^{-2e^{3\pi n/4+\theta}}$

Efimov vs super Efimov

Efimov effect

- 3 bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$E_n \sim e^{-2\pi n}$$

Super Efimov effect

- 3 fermions
- 2 dimensions
- p-wave resonance

"doubly" exponential

$$E_n \sim e^{-2e^{3\pi n/4}}$$

PRL 110, 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

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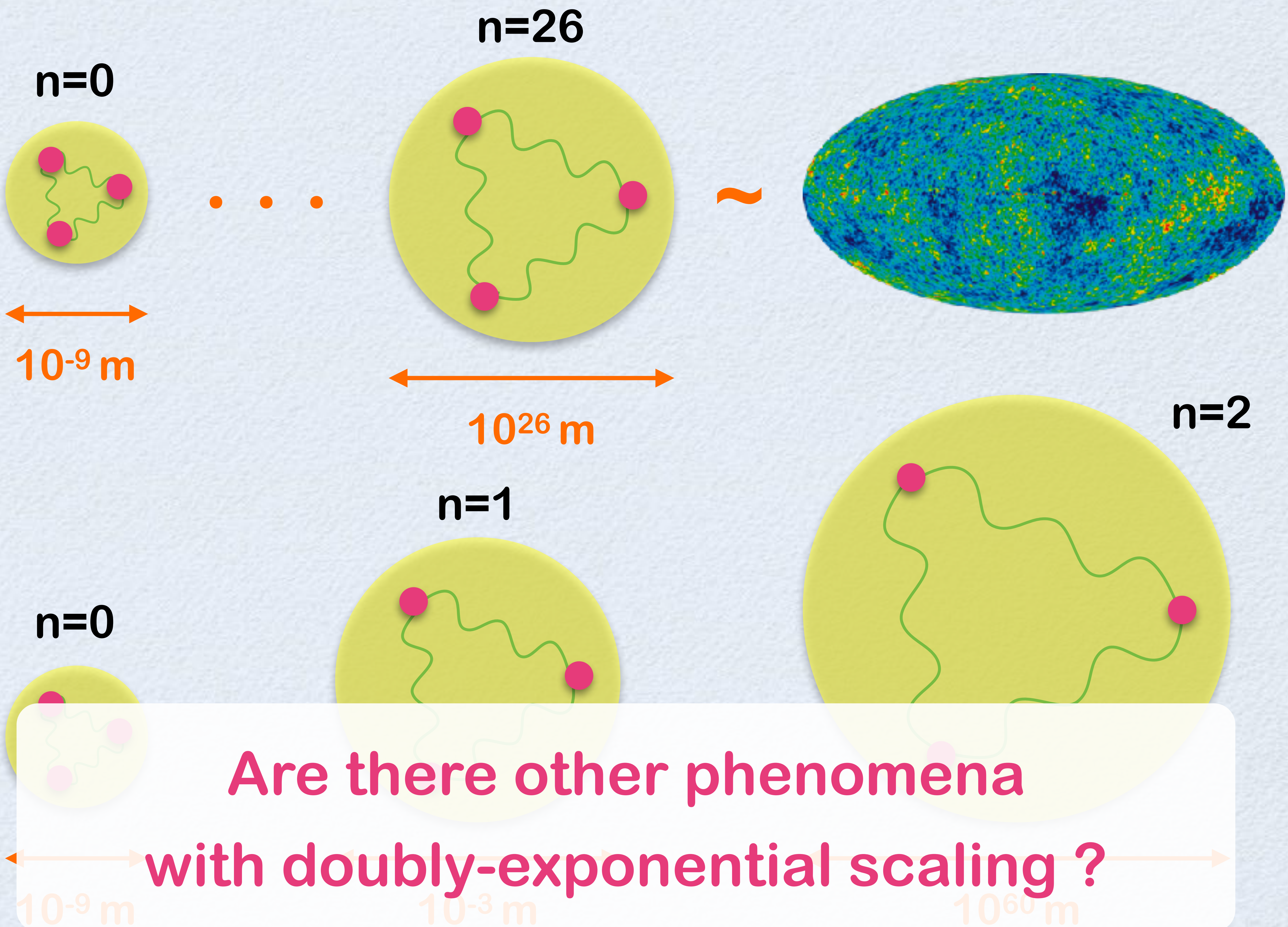
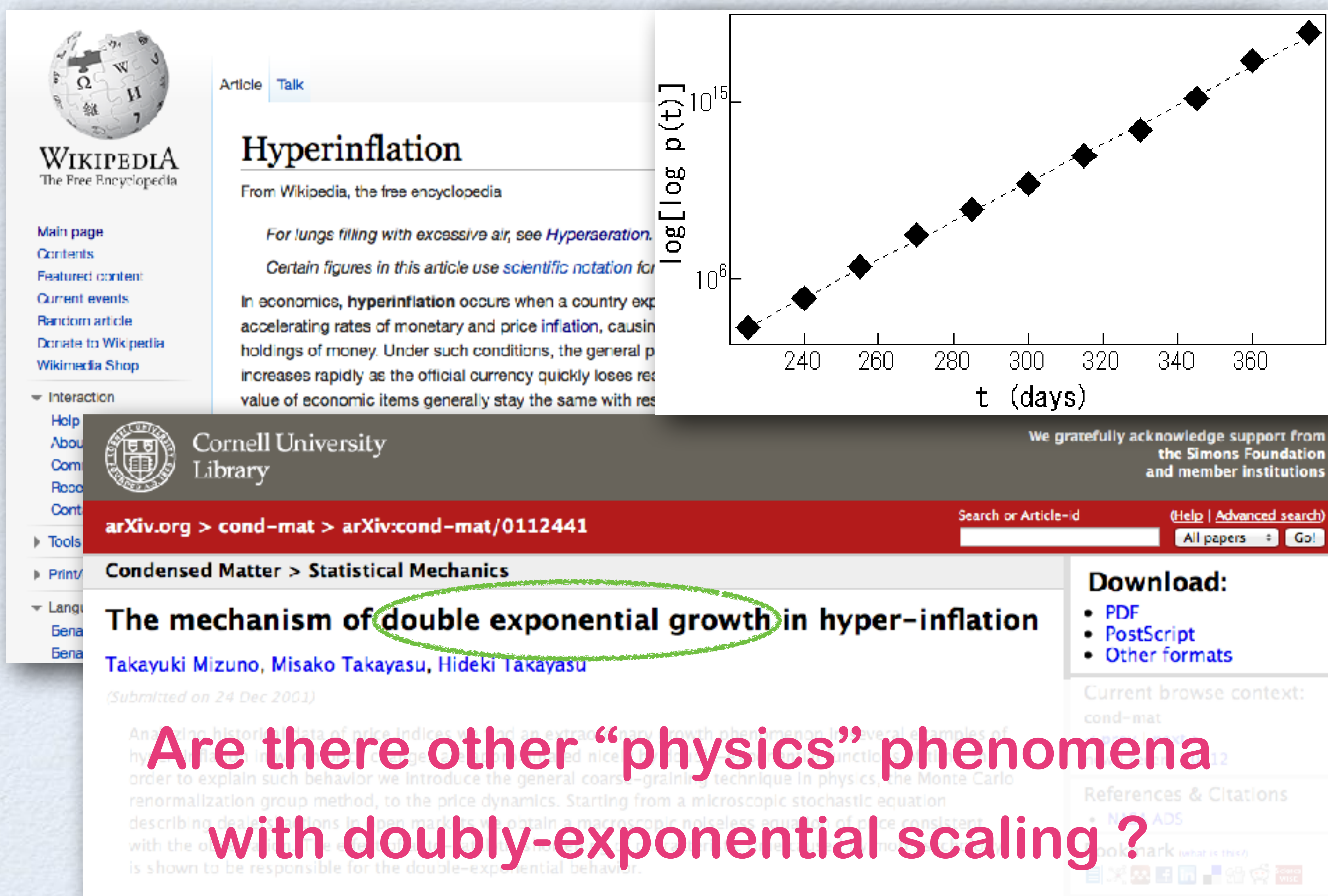
Efimov vs super Efimov



Are there other phenomena with doubly-exponential scaling ?

Efimov vs super Efimov



Are there other "physics" phenomena with doubly-exponential scaling ?

Lecture 3

How to derive the super Efimov effect ?

1. effective field theory

2. renormalization group equations

3. limit cycle

Summary

Efimov effect: universality, discrete scale invariance, RG limit cycle

nuclear physics	atomic physics	condensed matter
prediction (1970)	realization (2006)	proposal (2013)

✓ **Application: Quantum magnets**

Y.N, Y.K, C.D.B, Nature Physics 9, 93-97 (2013)

✓ **Extension: Super Efimov effect**

Y.N, S.M, D.T.S, Phys Rev Lett 110, 235301 (2013)