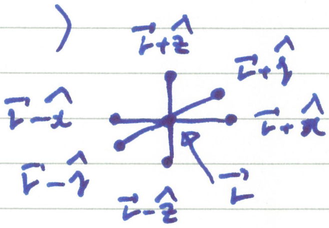
☆ Application to magnons

✓ Schrödinger equation

<u>Anisotropic Heisenberg model</u> on a cubic lattice

XXZ model

$$H = -\frac{1}{2} \sum_{\vec{r}} \sum_{\hat{e} = \pm\hat{x}, \pm\hat{y}, \pm\hat{z}} \left(J S^x_{\vec{r}} S^x_{\vec{r}+\hat{e}} + J S^y_{\vec{r}} S^y_{\vec{r}+\hat{e}} + J_z S^z_{\vec{r}} S^z_{\vec{r}+\hat{e}} \right)$$



$$[S^i_{\vec{r}}, S^j_{\vec{r}'}] = i\epsilon_{ijk} S^k_{\vec{r}} \quad (\vec{r} = \vec{r}')$$
$$= 0 \quad (\vec{r} \neq \vec{r}')$$

$$S^{\pm}_{\vec{r}} = S^x_{\vec{r}} \pm i S^y_{\vec{r}}$$

$$\Longrightarrow \begin{cases} [S^+_{\vec{r}}, S^-_{\vec{r}'}] = 2 S^z_{\vec{r}} \delta_{\vec{r},\vec{r}'} \\ [S^z_{\vec{r}}, S^{\pm}_{\vec{r}'}] = \pm S^{\pm}_{\vec{r}} \delta_{\vec{r},\vec{r}'} \end{cases}$$

↑

raising / lowering operators of spins

⟺ creation / annihilation operators of magnons

$$\left(\begin{array}{l} [\hat{b}^\dagger_i, \hat{b}_j] = -\delta_{ij} \\ [\underbrace{\hat{n}_i}_{\hat{b}^\dagger_i \hat{b}_i}, b^{(\dagger)}_j] = \pm b^{(\dagger)}_i \delta_{ij} \end{array} \right)$$

$$\Rightarrow H = -\frac{1}{2}\sum_{\vec{r}}\sum_{\hat{e}}\Big(J\,\underbrace{S^{+}_{\vec{r}}S^{-}_{\vec{r}+\hat{e}}}_{\text{hopping of magnons}} + J_z\,\underbrace{S^{z}_{\vec{r}}S^{z}_{\vec{r}+\hat{e}}}_{\text{nearest-neighbor interaction}}\Big)$$

- "vacuum" $\equiv$ fully polarized state

$$|0\rangle \equiv |\downarrow\downarrow\cdots\downarrow\rangle \Rightarrow \begin{cases} S^{z}_{\vec{r}}|0\rangle = -S|0\rangle \\ S^{-}_{\vec{r}}|0\rangle = 0 \quad \text{for } \forall\vec{r} \end{cases}$$

- 2-magnon state

$$|\vec{r}_1,\vec{r}_2\rangle \equiv S^{+}_{\vec{r}_1}S^{+}_{\vec{r}_2}|0\rangle = |\vec{r}_2,\vec{r}_1\rangle$$

bosons

- energy eigenstate $H|E\rangle = E|E\rangle$

The wave function $\Psi(\vec{r}_1,\vec{r}_2) \equiv \langle\vec{r}_1,\vec{r}_2|E\rangle$ obeys

$$E\,\Psi(\vec{r}_1,\vec{r}_2) = \langle\vec{r}_1,\vec{r}_2|H|E\rangle = \cdots$$

$= -SJ\sum_{\hat{e}}\Psi(\vec{r}_1+\hat{e},\vec{r}_2)$ ← hopping

$\quad -SJ\sum_{\hat{e}}\Psi(\vec{r}_1,\vec{r}_2+\hat{e})$

$\quad +J\sum_{\hat{e}}\delta_{\vec{r}_1,\vec{r}_2}\Psi(\vec{r}_1+\hat{e},\vec{r}_1)$ ← (*)

$\quad -J_z\sum_{\hat{e}}\delta_{\vec{r}_1,\vec{r}_2+\hat{e}}\Psi(\vec{r}_1,\vec{r}_2)$ ← nearest-neighbor interaction

$\quad +\underbrace{(\text{const.})}_{(-\frac{1}{2}S^2J_z+2SJ_z)\times 6}\,\Psi(\vec{r}_1,\vec{r}_2)$ ← constant energy shift → 0

(*) "hardcore" for $s=\frac{1}{2}$

set $\vec{r}_1 = \vec{r}_2$

$$E\,\Psi(\vec{r}_1,\vec{r}_1) = \underbrace{(-2SJ+J)}_{=0 \text{ for } s=\frac{1}{2}} \sum_{\hat{e}} \Psi(\vec{r}_1+\hat{e},\vec{r}_1)$$

$$\Longrightarrow \Psi(\vec{r}_1,\vec{r}_1) = 0 \Longleftrightarrow (S^+_{\vec{r}_1})^2 = 0$$

$$E\,\Psi(\vec{r}_1,\vec{r}_2)$$

$$= \Big[\underbrace{-SJ \sum_{\hat{e}} (\nabla^1_{\hat{e}} + \nabla^2_{\hat{e}})}_{\equiv H_0} + \underbrace{J \sum_{\hat{e}} \delta_{\vec{r}_1,\vec{r}_2} \nabla^1_{\hat{e}} - J_z \sum_{\hat{e}} \delta_{\vec{r}_1,\vec{r}_2+\hat{e}}}_{V} \Big] \Psi(\vec{r}_1,\vec{r}_2)$$

displacement operator

$$\nabla^1_{\hat{e}} \Psi(\vec{r}_1,\vec{r}_2) \equiv \Psi(\vec{r}_1+\hat{e},\vec{r}_2)$$

✓ Solution

Schrödinger eq. $\Rightarrow$ Lippmann-Schwinger eq.

$$(H_0 + V)\,\Psi(\vec{r}_1, \vec{r}_2) = E\,\Psi(\vec{r}_1, \vec{r}_2)$$

$$\Rightarrow \quad \underset{f(\vec{r}_1 - \vec{r}_2)}{\Psi(\vec{r}_1, \vec{r}_2)} = \underbrace{\Phi(\vec{r}_1, \vec{r}_2)}_{\substack{\text{solution for} \\ H_0 \Phi = E \Phi}} + \frac{1}{E - H_0} V \underset{f(\vec{r}_1 - \vec{r}_2)}{\Psi(\vec{r}_1, \vec{r}_2)}$$

translational symmetry

$$\Rightarrow \quad \Psi(\vec{r}_1, \vec{r}_2) = e^{i\vec{K}\cdot\vec{R}} f_{\vec{K}}(\vec{r})$$

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

We consider zero center-of-mass momentum

$$\vec{K} = \vec{0}$$

• Fourier transformation

$$\Rightarrow \quad \tilde{f}(\vec{k}) = \sum_{\vec{r}} e^{-i\vec{k}\cdot\vec{r}} f(\vec{r})$$

$$= \tilde{\phi}(\vec{k}) + \frac{1}{E + 2SJ \sum_{\hat{e}} \cos(\vec{k}\cdot\hat{e})}$$

$$\times \underbrace{\sum_{\vec{r}} e^{-i\vec{k}\cdot\vec{r}} \left[J \sum_{\hat{e}} \delta_{\vec{r},\vec{0}} - J_z \sum_{\hat{e}} \delta_{\vec{r},\hat{e}} \right]}_{\sum_{\hat{e}} [J - J_z \cos(\vec{k}\cdot\hat{e})]} f(\hat{e})$$

- Inverse Fourier transformation

$$\Longrightarrow f(\vec{r}) = \int_{-\pi}^{\pi} \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \mathcal{F}(\vec{k})$$

$$e^{i\vec{k}\cdot\vec{r}} \rightsquigarrow \cos(\vec{k}\cdot\vec{r})$$

$$= \phi(\vec{r}) + \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{E + 2SJ \sum_{\hat{e}} \cos(\vec{k}\cdot\hat{e})}$$

$$\times \sum_{\hat{e}} [J - J_z \cos(\vec{k}\cdot\hat{e})] \underbrace{f(\hat{e})}_{\text{unknown!}}$$

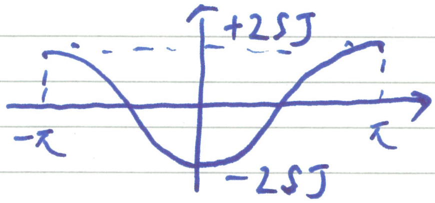
set $\vec{r} = \hat{e}$ $\Rightarrow$ equations to determine $f(\hat{e})$ //

When $f(\pm\hat{x}) = f(\pm\hat{y}) = f(\pm\hat{z})$

$$\Longrightarrow f(\hat{e}) = \frac{\phi(\hat{e})}{1 - \int \frac{d\vec{k}}{(2\pi)^3} \frac{\cos(\vec{k}\cdot\hat{e})}{E + 2SJ \sum_{\hat{e}} \cos(\vec{k}\cdot\hat{e})} \sum_{\hat{e}} [J - J_z \cos(\vec{k}\cdot\hat{e})]}$$

✓ Scattering length

dispersion of single magnon $E(\vec{k}) = -SJ\sum_{\hat{e}}\cos(\vec{k}\cdot\hat{e})$



k_x, k_y, k_z $\qquad E_{min} = -6SJ$

∴ two magnon threshold $\underline{E = -12SJ}$ $\leftrightarrow$ $k=0$

$\Rightarrow f(\vec{r}) = const. = C$

When $r_0 \ll |\vec{r}| \ll k^{-1} \to \infty$

$$\Rightarrow f(\vec{r}) \propto \frac{1}{|\vec{r}|} - \underset{\sim\sim}{\frac{1}{a}}$$

$$f(\vec{r})\Big|_{k=0} = C + \int\frac{d\vec{k}}{(2\pi)^3}\frac{\cos(\vec{k}\cdot\vec{r})}{-12SJ + 2SJ\sum_{\hat{e}}\cos(\vec{k}\cdot\hat{e})} \times \sum_{\hat{e}}\left[J - J_z\cos(\vec{k}\cdot\hat{e})\right] f(\hat{e})$$

When $|\vec{r}| \to \infty$, $|\vec{k}| \sim 0$ is dominant

$$\to C + \underbrace{\int\frac{d\vec{k}}{(2\pi)^3}\frac{\cos(\vec{k}\cdot\vec{r})}{-2SJ(\vec{k})^2}}_{-\frac{1}{2SJ}\times\frac{1}{4\pi|\vec{r}|}\ (|\vec{r}|\to 0)}(J-J_z)\sum_{\hat{e}} f(\hat{e})$$

$$= C\left[1 - \frac{1}{|\vec{r}|}\underbrace{\frac{J-J_z}{8\pi SJ}\frac{\sum_{\hat{e}} f(\hat{e})}{C}}_{\equiv a}\right]$$

magnon-magnon scattering length

$$a = \frac{J-J_z}{8\pi S J} \cdot \frac{\sum_{\hat{e}} f(\hat{e})}{c}$$

$$= \frac{3}{2\pi} \cdot \frac{J-J_z}{2SJ + (J-J_z)(3W-1)}$$

Watson's triple integral

$$W \equiv \int_{-\pi}^{\pi} \frac{d\vec{k}}{(2\pi)^3} \frac{2}{\sum_{\hat{e}} [1-\cos(\vec{k}\cdot\hat{e})]}$$

$$= \frac{\sqrt{6}}{96\pi^3} \Gamma(\tfrac{1}{24}) \Gamma(\tfrac{5}{24}) \Gamma(\tfrac{7}{24}) \Gamma(\tfrac{11}{24})$$

$$\approx 0.505462$$

When $J = J_z$ (isotropic exchange int.) $\Rightarrow a = 0$

$a = \infty$ is achieved when

$$\frac{J_z}{J} = \frac{2S + 3W - 1}{3W - 1} \approx 2.93654 \quad (S = \tfrac{1}{2})$$

$$\approx 4.87307 \quad (S = 1)$$

2-magnon resonance ($a = \infty$)

$\Downarrow$ universality

3-magnon Efimov effect !!!