Understanding quantum separation and togetherness in 1D ultracold atoms

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Outline

- Lecture I. Quantum liquid and quantum criticality in one dimension
  - Lieb-Liniger Bose gas
  - Spin-1/2 Heisenberg chain
- Lecture II. Spin charge separation in one dimension
  - Two component Fermi gas
  - Two component Bose gas

”Man is born free and everywhere he is in chains—-Jean-Jacques Rousseau”
Giamarchi, Quantum physics in one dimension, Oxford Scientific Publication

Bethe's Hypothesis (1931) for spin-1/2 Heisenberg chain

$$\text{Heisenberg spin chain}: \quad H = \frac{J}{2}\sum_{i=1}^{L}(\sigma_i \cdot \sigma_{i+1} + 1)$$

$$\text{Eigenstate}: \quad \Psi = \sum_{1 \le x_1 \cdots x_N \le L} a(x_1, \ldots, x_N)|x_1 x_2 \cdots x_N\rangle$$

$$\text{Bethe's hypothesis}: \quad a(x_1, \ldots, x_N) = \sum_{\{P\}} A_{\{P\}} e^{\mathrm{i}(k_{P_1}x_1 + \ldots + k_{P_N}x_N)}$$

In general, N! plane waves are N-fold products of individual exponential phase factors $e^{\mathrm{i}k_i x_j}$, where the N distinct wave numbers, k_i, are permuted among the N distinct coordinates, x_j. Each of the N! plane waves have an amplitude coefficient in each of regions.

- Problem: find eigenvalues E and eigenvectors $\Psi(x_1, \ldots, M)$

$$H\Psi(x_1, \ldots, M) = E\Psi(x_1, \ldots, M)$$

- H is is $2^L \times 2^L$ matrix
- DMRG numerical diagonalization(only finite L)
- coordinate Bethe ansatz
- Quantum Inverse Scattering Method/algebraic Beth ansatz
- Fractioanl excitations/ magnon and spinon
- Magnetism
- Luttinger liquid
- quantum criticality
- entanglement
- thermodynamics in and out of equilibrium
- quantum dynamics
- quantum information
- $\cdots$

$$\text{Pauli matrix}: \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Tensor product of two matrices

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \otimes \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{12}y_{11} & x_{12}y_{12} \\ x_{11}y_{21} & x_{11}y_{22} & x_{12}y_{21} & x_{12}y_{22} \\ x_{21}y_{11} & x_{21}y_{12} & x_{22}y_{11} & x_{22}y_{12} \\ x_{21}y_{21} & x_{21}y_{22} & x_{22}y_{21} & x_{22}y_{22} \end{pmatrix}$$

Tensor product of two vectors

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix}$$

Permutation matrix:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{P}_{12}\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Hamiltonian operator:

$$H_{jj+1} = \frac{1}{2}\left(\sigma_j \sigma_{j+1} + 1\right) = \mathcal{P}_{jj+1}$$

Basic notations:

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \sigma_1^x = \sigma^x \otimes id_2, \; \sigma_2^x = id_1 \otimes \sigma^x$$

$$H_{12}|--\rangle = |--\rangle, \; H_{12}|++\rangle = |++\rangle$$
$$H_{12}|+-\rangle = |-+\rangle, \; H_{12}|-+\rangle = |+-\rangle$$

$$\text{Eigenstate}: \quad H\Psi = \sum_{j=1}^{L-1} (H_{j,j+1} + H_{L,1})\Psi$$

$$\text{N} = 0: \quad \Psi = | - \; - \cdots - \cdots - \; - \rangle, \quad E = L$$

$$\text{N} = 1: \quad \Psi = \sum_{x=1}^{L} a(x) |\psi(x)\rangle = \sum_{x=1}^{L} a(x) | - \; - \cdots + \cdots - \; - \rangle.$$

$$H \sum_{x=1}^{L} a(x) |\psi(x)\rangle \quad = \quad \sum_{x=1}^{L} \left((L-2)a(x) + a(x-1) + a(x+1) \right) |\psi(x)\rangle$$

$$\text{Spin wave}: \qquad a(x) = e^{ikx}, \; E = (L - 2 + 2\cos k)$$

$$a(L+1) = a(1), \; a(0) = a(L)$$

$$\text{Bethe ansatz Eq.}: \qquad e^{ikL} = 1$$

- Eigenstate for $N = 2$:

$$\Psi = \sum_{1 \le x_1 < x_2}^{L} a(x_1, x_2)|\psi(x_1 x_2)\rangle = \sum_{1 \le x_1 < x_2}^{L} a(x_1, x_2)\sigma_{x_1}^{\dagger}\sigma_{x_2}^{\dagger}|- - \cdots - -\rangle$$

- $x_2 > x_1 + 1$

$$E \sum_{1 \le x_1 < x_2}^{L} a(x_1, x_2)|\psi(x_1 x_2)\rangle = \sum_{1 \le x_1 < x_2}^{L} \{(L-4)a(x_1, x_2) + a(x_1 - 1, x_2) + a(x_1 + 1, x_2) + a(x_1 + 1, x_2) + a(x_1, x_2 - 1) + a(x_1, x_1 + 1)\} |\psi(x_1 x_2)\rangle$$

- $x_2 = x_1 + 1$

$$E \sum_{1 \le x_1 < x_2}^{L} a(x_1, x_2)|\psi(x_1 x_2)\rangle = \sum_{1 \le x_1 < x_2}^{L} \{(L-2)a(x_1, x_1 + 1) + a(x_1 - 1, x_1 + 1) + a(x_1, x_1 + 2)\} |\psi(x_1 x_2)\rangle$$

$$\text{compatible condition}: \quad 2a(x_1, x_1+1) = a(x_1, x_1) + a(x_1+1, x_1+1)$$

$$\text{wave function}: \quad a(x_1, x_2) = A_{12} e^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + A_{21} e^{\mathrm{i}(k_2 x_1 + k_1 x_2)}$$

$$\text{energy}: \quad E = L - 4 + 2\cos k_1 + 2\cos k_2$$

$$\text{two} - \text{body scatteringmatrix}: \quad S_{12}(k_1 - k_2) = \frac{A_{12}}{A_{21}} = -\frac{1 - 2e^{\mathrm{i}k_1} + e^{\mathrm{i}(k_1+k_2)}}{1 - 2e^{\mathrm{i}k_2} + e^{\mathrm{i}(k_1+k_2)}}$$

$$\text{periodic boundary condition}: \quad a(x_1 = 0, x_2) = a(x_2, x_1 = L)$$

$$\text{Bethe ansatz Eqs.}: \quad e^{\mathrm{i}k_1 L} = \frac{A_{12}}{A_{21}}, \quad e^{\mathrm{i}k_2 L} = \frac{A_{21}}{A_{12}}$$

general case : $a(x_1, \dots, x_M) = \sum_P A_{P_1,\dots P_M} e^{ik_{P_1} x_1 \dots k_M x_M}$

Beth ansatz : $e^{ik_{p_1} L} = \frac{A_{P_1 \dots P_M}}{A_{P_2 \dots P_M P_1}}, \; A_{P_1 \dots P_M} = \epsilon_P \prod_{i \le i \le j \le M} S_{P_j, P_i}$

parameterization : $S_{ij} = 1 - 2e^{ik_j} + e^{i(k_j + k_i)}, \; e^{ik_j} = \frac{\lambda_j - i/2}{\lambda_j + i/2}$

Bethe ansatz Eqs. : $\left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^L = -\prod_{\ell=1}^{M} \frac{\lambda_j - \lambda_\ell - i}{\lambda_j - \lambda_\ell + i}, \qquad j = 1, \dots M$

energy : $E = L - \sum_{j=1}^{M} (2\cos k_j - 2) = L - \sum_{j=1}^{M} \frac{1}{\lambda_j^2 + 1/4}$

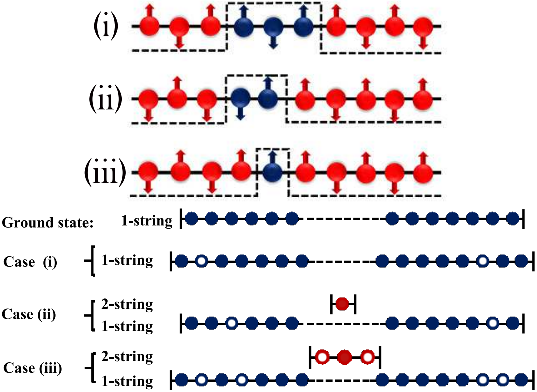
Yang & Yang, Phys. Rev. **150**, 321 (1966)
Yang & Yang, Phys. Rev. **150**, 327 (1966)
Yang & Yang, Phys. Rev. **151**, 258 (1966)

ground state :
$$\theta_1(\lambda_j) = \frac{2\pi I_j}{L} + \frac{1}{L}\sum_{\ell=1}^{M}\theta_2(\lambda_j-\lambda_\ell),\ \theta_n(x) = \pi - 2arctan(2x/n)$$
$$z(\lambda) = \frac{1}{2\pi}\left(\theta_1(\lambda) - \frac{1}{L}\sum_{\ell=1}^{M}\theta_2(\lambda-\lambda_\ell)\right),\ \sigma(\lambda) = \frac{dz(\lambda)}{d\lambda}$$
$$\sigma(\lambda) = a_1(\lambda) - \int_{-\Lambda}^{\Lambda}\sigma(\mu)a_2(\lambda-\mu)du,\ a_n(\lambda) = \frac{1}{2\pi}\frac{n}{\lambda^2+n^2/4}$$
$$\frac{E}{N} = -2\pi J\int a_1(\lambda)\sigma(\lambda)d\lambda + J/2$$

distribution :
$$\tilde{\sigma}(\omega) = \frac{\tilde{a}_1(\omega)}{1+\tilde{a}_2(\omega)} = \frac{1}{2\cosh\frac{\omega}{2}},\ \tilde{a}_n(\omega) = e^{-\frac{n}{2}|\omega|}$$

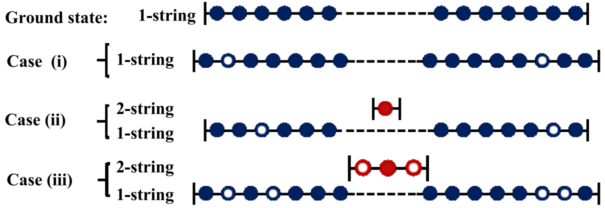
magnon distribution :
$$\sigma(\lambda) = \frac{1}{2\cosh(\pi\lambda)}$$

magnon density, energy :
$$M/N = \int\sigma(\lambda)d\lambda,\ E = (1-2\ln 2)J$$



$$\text{Spin strings :} \quad \lambda_{\alpha}^{n,a} = \lambda_{\alpha}^{n} + \frac{1}{2}\mathrm{i}(n+1-2a) + O(e^{-\delta L}), \quad a = 1, \cdots, n$$

- (i) $M^z = 1$ and 2 spinons
- (ii) $M^z = 0$, $\nu_2 = 1$ and 2 spinons
- (iii) $M^z = 1$, $\nu_2 = 1$ and 4 spinons



- (i) $M^z = 1$ and 2 spinons

$$
\begin{aligned}
\sigma(\lambda) &= a_1(\lambda) - \int_{-\Lambda}^{\Lambda} \sigma(\mu) a_2(\lambda-\mu) du \\
\sigma(\lambda) + \sigma_h(\lambda) &= \frac{dZ(\lambda)}{d\lambda} = a_1(\lambda) - \int a_2(\lambda-\mu)\sigma(\mu) d\mu \\
\sigma_h(\lambda) &= \frac{1}{L}\left(\delta(\lambda-\lambda_1^h) + \delta(\lambda-\lambda_2^h)\right) \\
\delta\sigma(\lambda) &= -\sigma_h(\lambda) - \int a_2(\lambda-\mu)\delta\sigma(\mu) d\mu \\
\delta\tilde{\sigma}(\omega) &= -\frac{1}{L}\frac{e^{i\lambda_1^h\omega} + e^{i\lambda_2^h\omega}}{1+e^{-|\omega|}}
\end{aligned}
$$

$$\text{dressed energy}: \quad \epsilon(\lambda) = \epsilon_0(\lambda) - \int a_2(\lambda-\mu)\epsilon(\mu)d\mu$$

$$\epsilon_0(\lambda) = 2\pi a_1(\lambda), \quad \epsilon(\lambda) = \frac{1}{2\cosh(\pi\lambda)}$$

$$\text{spinons}: \quad \delta E = -2\pi L \int \delta\sigma(\lambda) a_1(\lambda) d\lambda = -L \int \delta\tilde{\sigma}(\omega)\tilde{a}_1(\omega)d\omega$$

$$= \epsilon(\lambda_1^h) + \epsilon(\lambda_2^h)$$

$$\text{spin triplet}: \quad S = -L\int \delta\sigma(\lambda)d\lambda = 1$$

The BA general equations :

$$\sigma_n^h(\lambda) = a_n(\lambda) - \sum_{m=1}^{\infty} \int A_{mn}(\lambda - \mu)\sigma_m(\mu)d\mu$$

$$A_{mn}(\lambda) = A_{m+n}(\lambda) + 2\theta_{m+n-2}(\lambda) + \cdots + 2\theta_{|m-n|+2}(\lambda$$

$$+A_{|m-n|}(\lambda), \quad a_0(\lambda) \equiv \delta(\lambda)$$

energy :

$$E(\lambda_1, \ldots, \lambda_m) = -2\pi J \sum_{j=1}^{\infty} a_n(\lambda)\sigma_n(\lambda)d\lambda + \frac{1}{2}L(J - h)$$

2 − string excitation $\lambda_1^h, \ \lambda_2^h, \ \lambda_s \pm \mathrm{i}/2$

$$\sigma_1^h(\lambda) = \frac{1}{L}\left(\delta(\lambda-\lambda_1^h) + \delta(\lambda-\lambda_2^h)\right), \quad \sigma_2(\lambda) = \frac{1}{L}\delta(\lambda-\lambda_s)$$

length − 2 string :

$$\sigma_1(\lambda) + \sigma_1^h(\lambda) = a_1(\lambda) - \int a_2(\lambda-\mu)\sigma_1(\mu)d\mu - \int \left(a_1(\lambda-\mu) + a_3(\lambda-\mu)\right)\sigma_2(\mu)d\mu$$

$$\delta\tilde{\sigma}_1(\omega) = -\frac{1}{L}\frac{e^{\mathrm{i}\lambda_1^h\omega} + e^{\mathrm{i}\lambda_2^h\omega}}{1+e^{-|\omega|}} - \frac{1}{L}\frac{e^{-\frac{1}{2}|\omega|} + e^{-\frac{3}{2}|\omega|}}{1+e^{-|\omega|}}e^{\mathrm{i}\lambda_s\omega}$$

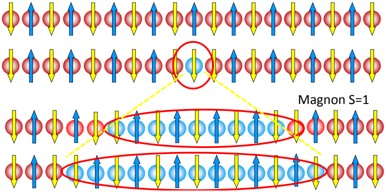
excitation energy

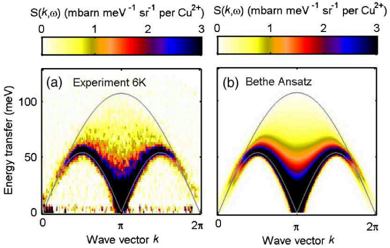
$$\delta E = -N\int \delta\tilde{\sigma}_1(\omega)\tilde{a}_1(\omega)d\omega - 2\pi a_2(\lambda_s) = \epsilon(\lambda_1^h) + \epsilon(\lambda_2^h),$$

see the definition of dressed energy

total magnetization

$$S = \frac{1}{2}\left(L - 2L\int\sigma_1(\lambda)d\lambda - 4\int\sigma_2(\lambda)d\lambda\right) = 0$$



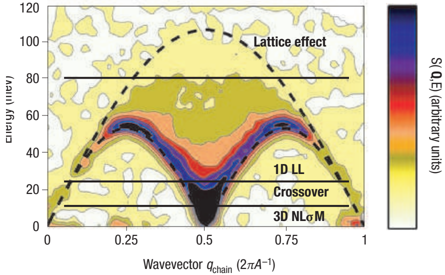


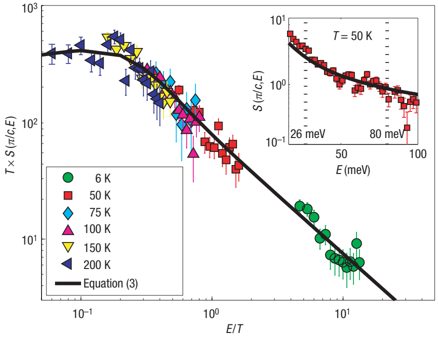
Two-spinon continuum with $k = p_1 + p_2$ and $\omega = \epsilon(p_1) + \epsilon(p_2)$: $\frac{\pi}{2}J|sin\, k| \leq \omega \leq \pi J|\sin\frac{k}{2}|$. Many neutron scattering experiments present novel measurements of such fractional excitations through measuring the dynamic structure factor:

$$S^{a\bar{a}}(q,\omega) = \frac{1}{N}\sum_{j,j'=1}^{N}\int_{-\infty}^{\infty} dt e^{-iq(j-j')+i\omega t} < S_j^a(t) S_{j'}^{\bar{a}}(0) >, \quad a = z, -, +$$

$$S(q,E) = \frac{A_M \Theta(E - E_l(q))\Theta(E_u(q) - E)}{\sqrt{E^2 - E_L^2(q)}}$$

Faddeev & Takhtajian, Phys. Lett. A 85, 375 (1981); Klauser, et al. J. Stat. Mech, 2012/P03012
Lake et al. PRL 111, 137205 (2013); Stone et al, PRL 91, 037205 (2003)
He, Jiang, Yu, Lin, Guan, PRB 96, 220401(R) (2017)





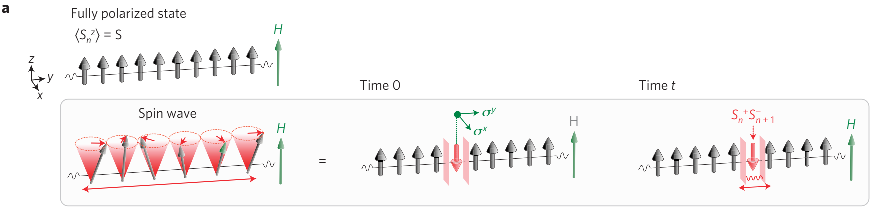
Left Panel: Inelastic neutron scattering data for weakly coupled spin chain $KCuF_3$ in 3D

1D Luttinger liquid: $S(\pi, E) = \frac{e^{E/T}}{e^{E/T}-1} \frac{A}{T} \mathrm{Im} \left[\rho \left(\frac{E}{4\pi T} \right)^2 \right]$.

Right Panel: Phase diagram–different physics regimes at finite temperatures

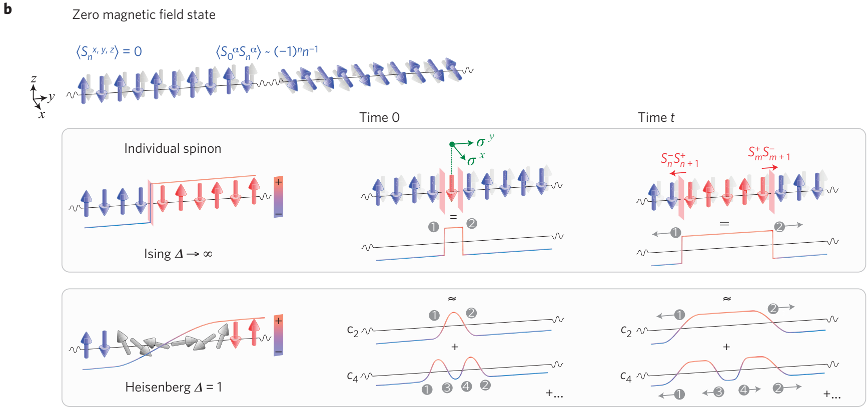
$$H = J \sum_{n,r} S_{n,r} \bullet S_{n+1,r} + J_r \sum_{n,r,\delta} S_{n,r} \bullet S_{n,r+\delta}$$

Schulz, PRB, **34**, 6372 (1986)
Lake *et al*, Nature Materials **4**, 329 (2005)



$$H = J \sum_{n=1}^{L} \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z \right)$$

Fractional excitations in copper sulphate $CuSO_4 \cdot 5D_2O$. a) Single spin flip causes a spin wave propagation in fully polarized state. Mourigal et. al. Nat. Phys. 2013.



$$H = J \sum_{n=1}^{L} \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z \right)$$

Fractional excitations in copper sulphate $CuSO_4 \cdot 5D_2O$. b) $h = 0$ case, local spin flip decomposes into two spinons for a large Δ, however, $\Delta = 1$ it decomposes into a rapid converge series of 2, 4, and higher of spions. Mourigal et. al. Nat. Phys. 2013.

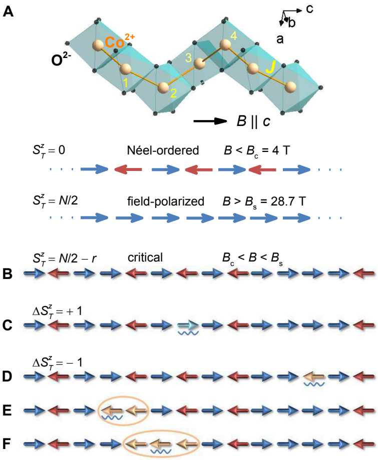
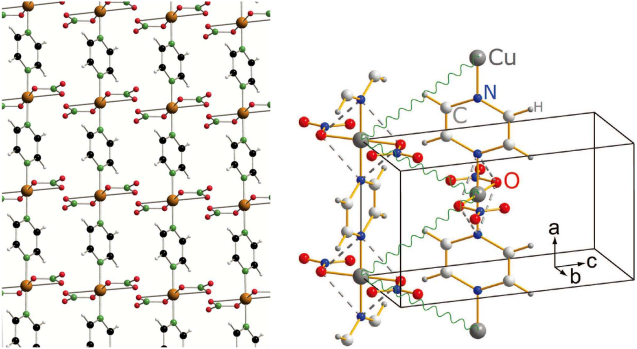


Fig. 1. Quantum spin chain in $SrCo_2V_2O_8$ and characteristic magnetic excitations of one dimension in the critical regime: psinons-(anti)psinons and strings.

Schematic spin string of spin-1/2 XXZ model $SrCo_2V_2O_8$

Wang, et al. Nature 554, 219 (2018)

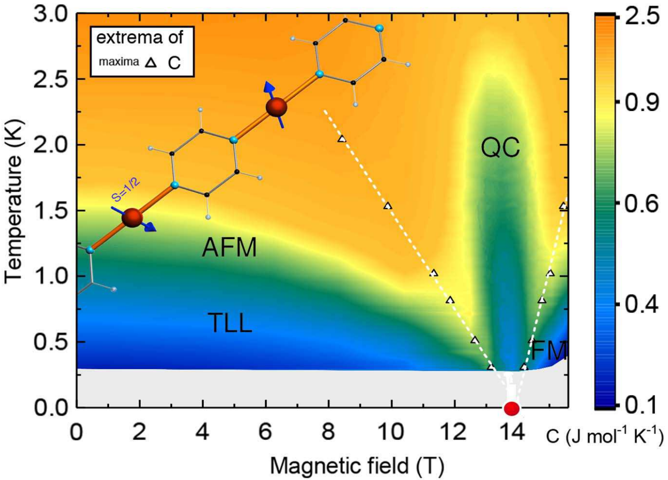


Schematic structure of CuPzN

$$\mathcal{H} = 2J \sum_{j=1}^{N} \overrightarrow{S}_j \cdot \overrightarrow{S}_{j+1} - g\mu_B H \sum_{j=1}^{N} \overrightarrow{S}_j^{\,z}$$

Cooper pyrazine dinitrate (CuPzN) $Cu(C_4H_4N_2)(NO_3)_2$: yellow sphere cu^{2+} linked by the pyrazine rings to form a 1D spin-1/2 chain with intrachain coupling $2J \approx 10.8$K (a-direction). The wave lines indicates the interchain contacts ($J' \approx 0.046$K).

PRB, 59, 1008 (1999); PRL 114, 037202 (2015).



$Cu(C_4H_4N_2)(NO_3)_2$

- Luttinger liquid
- Quantum criticality
- Gapped phase

Density of energy :
$$\frac{E}{L} = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\varepsilon_n(\lambda)\sigma_n(\lambda)d\lambda + \frac{1}{2}(1-h)$$
$$\varepsilon_n(\lambda) = -2\pi a_n(\lambda) + nH$$

degeneracy :
$$dW = \prod_{n=1}^{\infty}\frac{\left[L(\sigma_n(\lambda)+\sigma_n^h(\lambda))d\lambda\right]!}{[\sigma_n(\lambda)d\lambda]!\,[\sigma_n(\lambda)d\lambda]!}$$

$$dS(\lambda) = \ln dW(\lambda) = L\sum_{n=1}^{\infty}\left\{\left(\sigma_n(\lambda)+\sigma_n^h(\lambda)\right)\ln\left(\sigma_n(\lambda)+\sigma_n^h(\lambda)\right)\right.$$
$$\left. -\sigma_n(\lambda)\ln\sigma_n(\lambda) - \sigma_n^h(\lambda)\ln\sigma_n^h(\lambda)\right\} d\lambda$$

free energy per length :
$$f = -\frac{E}{L} - T\frac{S}{L} - \frac{1}{2}(1-h)$$

a thermal equilibrium state :
$$\frac{\delta f}{\delta\sigma_n(\lambda)} = 0,$$

a BA equations :
$$\delta\sigma_n^h(\lambda) = -\delta\sigma_n(\lambda) - \sum_{n=1}^{\infty}T_{nm}\star\delta\sigma_n(\lambda)$$

TBAequations : $$\ln(1+\eta_n) = \frac{\varepsilon_n}{T} + \sum_m A_{m,n} * \ln\left(1+\eta_m^{-1}\right)$$

$$\varepsilon_n = -2\pi a_n + nH, \quad n = 1, \ldots, \infty$$

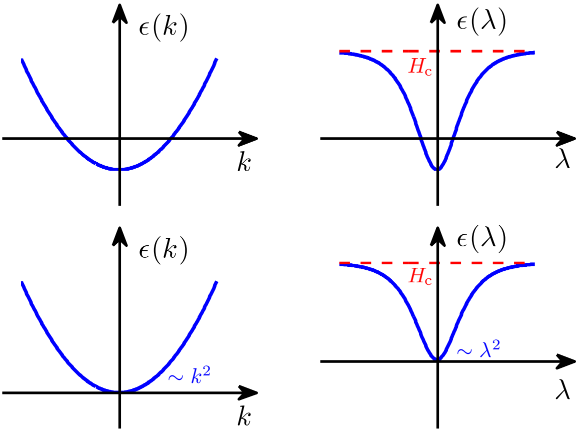
free energy : $$f = -T\sum_n \int a_n(\lambda)\ln\left(1+\eta_m^{-1}(\lambda)\right)d\lambda$$

$$\varepsilon_n^+ = \varepsilon_n - \sum_m A_{m,n} * \varepsilon_n^-, \quad \varepsilon^\pm = \pm T\ln(1+e^{\pm\varepsilon_n/T})$$

$$A_{mn}(\lambda) = A_{m+n}(\lambda) + 2\theta_{m+n-2}(\lambda) + \cdots + 2\theta_{|m-n|+2}(\lambda) + A_{|m-n|}(\lambda), \quad a_0(\lambda) \equiv \delta(\lambda)$$

Takahashi, Thermodynamics of One-Dimensional Solvable Models (Cambridge: Cambridge University Press) 1999

He, Jiang, Yu, Lin, Guan, Phys. Rev. B 96, 220401(R) (2017)



- Quantum Liquid

- Quantum criticality

Guan, Batchelor and Lee, Rev. Mod. Phys. 85, 1633 (2013)

$$\varepsilon_1^{(0)}(\lambda) = -2\pi J a_1(\lambda) + H - \int_{-Q}^{Q} a_2(\lambda-\mu)\,\varepsilon_1^{(0)}(\mu)\,d\mu$$

$$\varepsilon_1(\lambda) = -2\pi J a_1(\lambda) + H + T\int a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu$$

$$\begin{aligned}
\varepsilon_1(\lambda) &= -2\pi J a_1(\lambda) + H + T\int a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \\
&= -2\pi J a_1(\lambda) + H + T\left(\int_{-\infty}^{-Q}+\int_{Q}^{\infty}\right) a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \\
&\quad + T\int_{-Q}^{Q} a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \\
&= -2\pi J a_1(\lambda) + H + T\left(\int_{-\infty}^{-Q}+\int_{Q}^{\infty}\right) a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \\
&\quad + T\int_{-Q}^{Q} a_2(\lambda-\mu)\ln\left(1+e^{\frac{\varepsilon_1(\mu)}{T}}\right)d\mu - \int_{-Q}^{Q} a_2(\lambda-\mu)\,\varepsilon_1(\mu)\,d\mu \\
&= -2\pi J a_1(\lambda) + H - \int_{-Q}^{Q} a_2(\lambda-\mu)\,\varepsilon_1(\mu)\,d\mu \\
&\quad + T\int_{-\infty}^{\infty} a_2(\lambda-\mu)\ln\left(1+e^{\frac{-|\varepsilon_1(\mu)|}{T}}\right)d\mu
\end{aligned}$$

$$\begin{aligned}\varepsilon_1(\lambda) &= \varepsilon_1^{(0)}(\lambda)+\eta(\lambda)\\ &= -2\pi J a_1(\lambda)+H-\int_{-Q}^{Q} a_2(\lambda-\mu)\,\varepsilon_1^{(0)}(\mu)\,d\mu+\eta(\lambda)\\ &= -2\pi J a_1(\lambda)+H-\int_{-Q}^{Q} a_2(\lambda-\mu)\left(\varepsilon_1(\mu)-\eta(\mu)\right)d\mu+\eta(\lambda)\end{aligned}$$

$$\begin{aligned}\eta(\lambda) &= T\int_{-\infty}^{\infty} a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\left|\varepsilon_1(\mu)\right|}{T}}\right)d\mu-\int_{-Q}^{Q} a_2(\lambda-\mu)\,\eta(\mu)\,d\mu\\ &= I-\int_{-Q}^{Q} a_2(\lambda-\mu)\,\eta(\mu)\,d\mu\end{aligned}$$

$$\varepsilon_1(\lambda)=t(\lambda-Q)+O\left((\lambda-Q)^2\right)\qquad t=\frac{d\varepsilon}{d\lambda}\Big|_{\lambda=Q}$$

He, Jiang, Yu, Lin, Guan, Phys. Rev. B 96, 220401(R) (2017)

$$
\begin{aligned}
\varepsilon_1(\lambda) &= t(\lambda - Q) + O\left((\lambda-Q)^2\right) \qquad t = \frac{d\varepsilon}{d\lambda}\Big|_{\lambda=Q} \\
I &= T\int_{-\infty}^{+\infty} a_2(\lambda-\mu)\ln\left(1+e^{\frac{-|\varepsilon_1(\mu)|}{T}}\right)d\mu \\
&= T\left[\int_{-Q-s}^{-Q+s} a_2(\lambda-\mu)\ln\left(1+e^{\frac{-t|\mu+Q|}{T}}\right)d\mu\right. \\
&\quad \left. + \int_{Q-s}^{Q+s} a_2(\lambda-\mu)\ln\left(1+e^{\frac{-t|\mu-Q|}{T}}\right)d\mu\right] \\
&= \frac{T^2}{t}\left[\int_{-\frac{ts}{T}}^{\frac{ts}{T}} a_2(\lambda+Q)\ln\left(1+e^{\frac{-t|\mu+Q|}{T}}\right)d\left(\frac{t(\mu+Q)}{T}\right)\right. \\
&\quad \left. + \int_{-\frac{ts}{T}}^{\frac{ts}{T}} a_2(\lambda-Q)\ln\left(1+e^{\frac{-t|\mu-Q|}{T}}\right)d\left(\frac{t(\mu-Q)}{T}\right)\right] \\
&= \frac{T^2}{t}\left[\int_{-\infty}^{\infty} a_2(\lambda+Q)\ln\left(1+e^{-y}\right)dy + \int_{-\infty}^{\infty} a_2(\lambda-Q)\ln\left(1+e^{-y}\right)dy\right] \\
&= \frac{\pi^2 T^2}{6t}\left[a_2(\lambda+Q)+a_2(\lambda-Q)\right]
\end{aligned}
$$

$$\eta(\lambda) = \frac{\pi^2 T^2}{6t}\left[a_2(\lambda+Q) + a_2(\lambda-Q)\right] - \int_{-Q}^{Q} a_2(\lambda-\mu)\,\eta(\mu)\,d\mu$$

$$f_0(0,H) = \int_{-Q}^{Q} a_1(\lambda)\,\varepsilon_1^{(0)}(\lambda)\,d\lambda, \quad f(T,H) = -T\int_{-\infty}^{\infty} a_1(\lambda)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda$$

$$\begin{aligned}
f - f_0 &= -T\int_{-\infty}^{\infty} a_1(\lambda)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda - \int_{-Q}^{Q} a_1(\lambda)\,\varepsilon_1^{(0)}(\lambda)\,d\lambda \\
&= -T\left[\left(\int_{-\infty}^{-Q} + \int_{Q}^{\infty}\right) a_1(\lambda)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda\right. \\
&\quad \left. + \int_{-Q}^{Q} a_1(\lambda)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda\right] - \int_{-Q}^{Q} a_1(\lambda)\,\varepsilon_1^{(0)}(\lambda)\,d\lambda \\
&= -T\left(\int_{-\infty}^{-Q} + \int_{Q}^{\infty}\right) a_1(\lambda)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda - T\int_{-Q}^{Q} a_1(\lambda)\ln\left(1+e^{\frac{\varepsilon_1(\lambda)}{T}}\right)d\lambda \\
&\quad + \int_{-Q}^{Q} a_1(\lambda)\,\varepsilon_1(\lambda)\,d\lambda - \int_{-Q}^{Q} a_1(\lambda)\,\varepsilon_1^{(0)}(\lambda)\,d\lambda \\
&= -T\int_{-\infty}^{\infty} a_1(\lambda)\ln\left(1+e^{\frac{-|\varepsilon_1(\lambda)|}{T}}\right)d\lambda + \int_{-Q}^{Q} a_1(\lambda)\,\eta(\lambda)\,d\lambda \\
&= -\frac{\pi^2 T^2}{3t} a_1(Q) + \int_{-Q}^{Q} a_1(\lambda)\,\eta(\lambda)\,d\lambda
\end{aligned}$$

Finally, we obtain

$$f = f_0 - \frac{\pi^2 T^2}{3t} a_1(Q) + \int_{-Q}^{Q} a_1(\lambda)\,\eta(\lambda)\,d\lambda$$

For ground state, the density of inverse spin in λ sea is

$$\rho_0(\lambda) = a_1(\lambda) - \int_{-Q}^{Q} a_2(\lambda-\mu)\,\rho_0(\mu)\,d\mu$$

There is a general formula

$$f = f_0 - a_2 * f$$

$$g = g_0 - a_2 * g$$

then

$$\int_{-Q}^{Q} f(\lambda)\,g_0(\lambda)\,d\lambda = \int_{-Q}^{Q} g(\lambda)\,f_0(\lambda)\,d\lambda$$

using this formula with equation (36) and (40), we have

$$\int_{-Q}^{Q} \frac{\pi^2 T^2}{6t}\left[(a_2(\lambda+Q) + a_2(\lambda-Q))\right]\rho_0(\lambda)\,d\lambda = \int_{-Q}^{Q} a_1(\lambda)\,\eta(\lambda)\,d(\lambda)$$

since

$$\rho_0(Q) = a_1(Q) - \int_{-Q}^{Q} a_2(Q-\mu)\,\rho_0(\mu)\,d\mu$$

$$\rho_0(-Q) = a_1(-Q) - \int_{-Q}^{Q} a_2(-Q-\mu)\,\rho_0(\mu)\,d\mu$$

Summing this two formula on both side,then

$$\int_{-Q}^{Q} \left[(a_2(\lambda+Q) + a_2(\lambda-Q))\right]\rho_0(\lambda)\,d\lambda = 2a_1(Q) - 2\rho_0(Q)$$

so

$$\int_{-Q}^{Q} a_1(\lambda)\,\eta(\lambda)\,d(\lambda) = \frac{\pi^2 T^2}{6t}\left[2a_1(Q) - 2\rho_0(Q)\right]$$

together with the formula of the free energy per site (40), we know

$$\begin{aligned} f &= f_0 - \frac{\pi^2 T^2}{3t} a_1(Q) + \int_{-Q}^{Q} a_1(\lambda)\,\eta(\lambda)\,d\lambda \\ &= f_0 - \frac{\pi^2 T^2}{3t} a_1(Q) + \frac{\pi^2 T^2}{6t}\left[2a_1(Q) - 2\rho_0(Q)\right] \\ &= f_0 - \frac{\pi^2 T^2}{3t}\rho_0(Q) \end{aligned}$$

we define Fermi velocity

$$v_s = \frac{1}{2\pi}\frac{d\varepsilon_1(\lambda)/d\lambda}{\rho_0(\lambda)}|_{\lambda=Q} = \frac{1}{2\pi}\frac{t}{\rho_0(Q)} \tag{48}$$

Finally, the free energy per site is

$$f = f_0 - \frac{\pi T^2}{6v_s} \tag{49}$$

Since f_0 is the free energy per site at zero temperature, it is not a function of T , it follows that the specific heat at LL region is given by

$$C_v = -T\frac{\partial^2 f}{\partial^2 T} = \frac{\pi T}{3v_s} \propto T \tag{50}$$

from

$$\frac{C_v}{T} \propto T^\alpha \Rightarrow \alpha = 1 \tag{51}$$

with

$$\alpha = 1 - (d+z)/z \tag{52}$$

so at one dimension, $d = 1$, the critical exponent at LL region reads

$$z = 1 \tag{53}$$

simultaneously

$$\frac{C_v}{T} = \frac{\pi}{3v_s} \tag{54}$$

near the critical point, the effective contribution to dressed energy (55) comes from small negative part of λ, so the effective λ can be regard as a infinitesimal number, then we can expand above equation with λ too.

$$a_1(\lambda) = \frac{1}{2\pi}\frac{1}{\lambda^2+\frac{1}{4}} \approx \frac{2}{\pi}\left(1-4\lambda^2+\cdots\right) \tag{57}$$

$$a_2(\lambda-\mu) = \frac{1}{2\pi}\frac{2}{1+(\lambda-\mu)^2} \approx \frac{1}{\pi}\left[1-(\lambda-\mu)^2+\cdots\right] \tag{58}$$

then we can get

$$f = -T\int a_1(\lambda)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda \approx -\frac{2T}{\pi}\int\left(1-4\lambda^2\right)\ln\left(1+e^{\frac{-\varepsilon_1(\lambda)}{T}}\right)d\lambda \tag{59}$$

$$\begin{aligned}\varepsilon_1(\lambda) &= -2\pi J a_1(\lambda) + H + T\int a_2(\lambda-\mu)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \\ &\approx -2\pi J\cdot\frac{2}{\pi}\left(1-4\lambda^2\right) + H + \frac{T}{\pi}\int\left(1-(\lambda-\mu)^2\right)\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \\ &= 16\pi J\lambda^2 - 4J + H + \frac{T}{\pi}\left(1-\lambda^2\right)\int\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu - \frac{T}{\pi}\int\mu^2\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu\end{aligned} \tag{60}$$

we define

$$b_1 = T\int\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \tag{61}$$

$$b_2 = T\int\mu^2\ln\left(1+e^{\frac{-\varepsilon_1(\mu)}{T}}\right)d\mu \tag{62}$$

then dressed energy equation (55) becomes

$$\varepsilon_1(\lambda) = \left(16J-\frac{b_1}{\pi}\right)\lambda^2 - 4J + H + \frac{b_1}{\pi} - \frac{b_2}{\pi} \tag{63}$$

After some calculation, we get

$$b_1 = -\frac{\sqrt{\pi}T^{\frac{3}{2}}}{\left(16J-\frac{b_1}{\pi}\right)^{\frac{1}{2}}}\mathrm{Li}_{\frac{3}{2}}\left(-e^{\frac{A_0}{T}}\right) \tag{64}$$

$$b_2 = -\frac{1}{2}\frac{\sqrt{\pi}T^{\frac{5}{2}}}{\left(16J-\frac{b_1}{\pi}\right)^{\frac{3}{2}}}\mathrm{Li}_{\frac{5}{2}}\left(-e^{\frac{A_0}{T}}\right) \tag{65}$$

where

$$A_0 = 4J - H - \frac{b_1}{\pi} + \frac{b_2}{\pi} \tag{66}$$

- Free Energy from Bethe ansatz

$$f = E_0 - \pi T^2/(6v_s) + O(T^3), \qquad v_s = \frac{1}{2\pi}\frac{\mathrm{d}\varepsilon_1(\lambda)/\mathrm{d}\lambda}{\rho_0(\lambda)}|_{\lambda=Q}$$

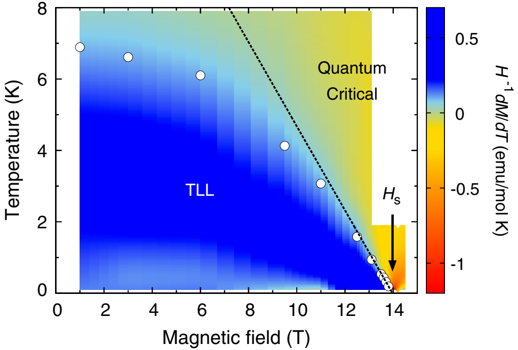
- For spin-1/2 Heisenberg chain, the effective Hamiltonian

$$H = \frac{\hbar}{2\pi}\int dx\left[\frac{v_s K_s}{\hbar^2}(\pi\Pi(x))^2 + \frac{v_s}{K_s}(\nabla\phi(x))^2\right]$$

- The canonical momentum Π conjugate to the phase ϕ obeying the standard Bose commutation relations $[\phi(x), \Pi(y)] = \mathrm{i}\delta(x-y)$. Collective excitations are analogous to sound waves.
- The correlation functions

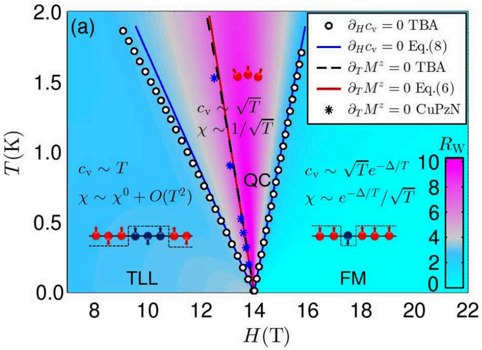
$$\begin{aligned}\langle s^z(x)s^z(0)\rangle &= \frac{1}{x^2} + (-1)^x A_z\left(\frac{1}{x}\right)^{2K}\\ \langle s^-(x)s^+(0)\rangle &= (-1)^x A_x\left(\frac{1}{x}\right)^{1/(2K)} + B_x\left(\frac{1}{x}\right)^{1/(2K)+2k}\end{aligned}$$

Giamarchi, T. *Quantum Physics in one dimension* (Oxford University Press, Oxford, 2004)



$$\mathcal{H} = 2J \sum_{j=1}^{N} \vec{S}_j \cdot \vec{S}_{j+1} - g\mu_B H \sum_{j=1}^{N} \vec{S}_j^{\,z}$$

- How to determine Lutttinger liquid phase boundary?
- How to precisely determine quantum scalings?
- How can the interacting spins form free fermion criticality?



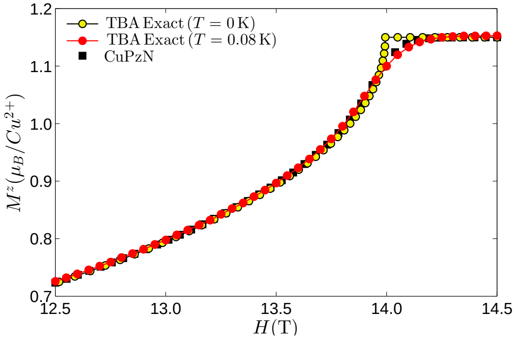
Wilson ratio of spin-$1/2$ chain CuPzN with $2J = 10.81$K

$$f \approx -\frac{2b_1}{\pi} + \frac{8b_2}{\pi}, \quad A = 4J - H - \frac{b_1}{\pi} + \frac{b_2}{\pi}$$

$$b_1 = -\frac{\sqrt{\pi}T^{\frac{3}{2}}}{4\sqrt{J}} \mathrm{Li}_{\frac{3}{2}}\left(-e^{\frac{A}{T}}\right), \quad b_2 = -\frac{1}{2}\frac{\sqrt{\pi}T^{\frac{5}{2}}}{(16J)^{\frac{3}{2}}} \mathrm{Li}_{\frac{5}{2}}\left(-e^{\frac{A}{T}}\right)$$

- b_1: free fermion behaviour due to the dilute magnons, indicating free fermion QC.
- b_2: interaction effect between magnons.

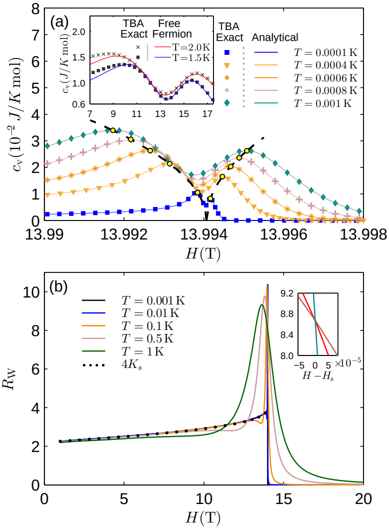
He, Jiang, Yu, Lin, Guan, PRB B 96, 220401(R) (2017)



beyond Free Fermion $\qquad \epsilon(k) \approx \frac{\hbar^2 k^2}{2m^*} + \Delta - g\mu_B M^z H$

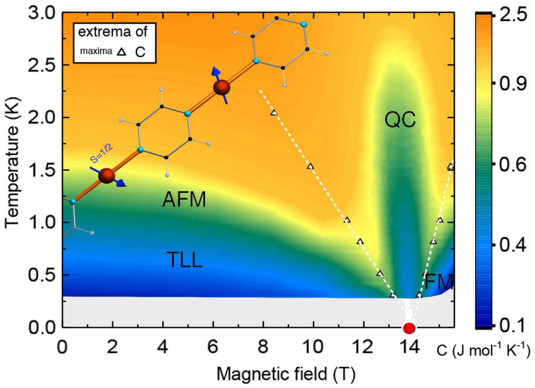
density of magnons $\qquad n_{\mathrm{magnon}} = M_s/N - M^z = \frac{\sqrt{2m^* T}}{\pi} \int_0^\infty \frac{\mathrm{d}x}{e^{x^2 - \frac{\mu}{T}} + 1}$

effective of mass $\qquad m^* \approx \frac{1}{2J}\left(1 - \frac{T^{1/2}}{\sqrt{\pi J}} \int_0^\infty \frac{\mathrm{d}x}{e^{x^2 - \frac{\mu}{T}} + 1}\right)$



$$f = E_0 - \pi T^2/(6v_s) + O(T^3), \quad v_s = \frac{1}{2\pi}\frac{d\varepsilon_1(\lambda)/d\lambda}{\rho_0(\lambda)}\Big|_{\lambda=Q}$$

$$R_W = 4K_s, \quad K_s = \pi v_s \chi/(g\mu_B)^2$$



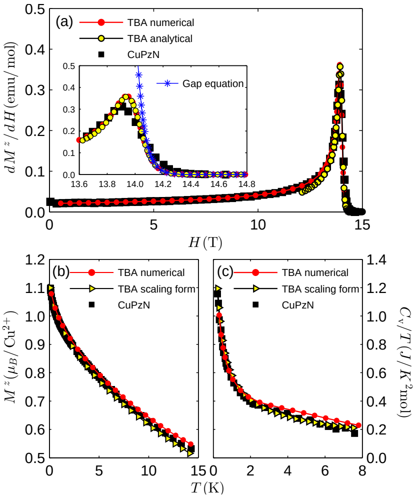
Quantum criticality of $Cu(C_4H_4N_2)(NO_3)_2$

Criticality : $n(T,\mu) \approx n_0 + T^{d/z+1-1/\nu z}\mathcal{F}\left(\frac{\mu-\mu_c}{T^{1/\nu z}}\right), \;\; \xi \sim |\mu-\mu_c|^{-\nu}, \;\; \Delta \sim \xi^{-z}$

$$c_v/T = T^{d/z+1-2/\nu z}\mathcal{K}\left(\frac{\mu-\mu_c}{T^{1/\nu z}}\right), \quad z=2, \; \nu=1/2$$

He, Jiang, Yu, Lin, Guan, PRB B 96, 220401(R) (2017)

Breunig, et al. Sci. Adv. 2017; 3:eaao3773

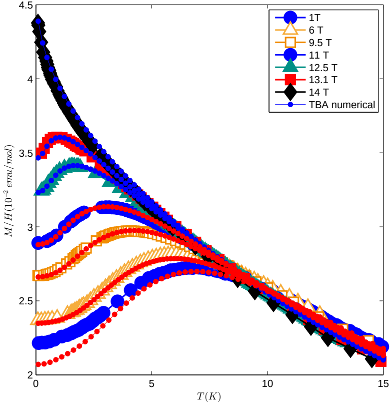


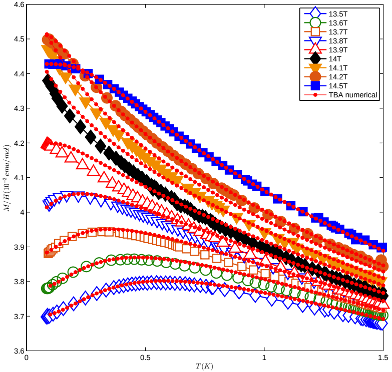
$1 - M^z/M_s = D(1 - H/H_s)^{1/\delta}, \quad \delta = 2, \quad D = 4/\pi$

$(M_s/L - M^z)/H \propto T^{\beta}, \quad c_v/T \propto T^{-\alpha}, \quad \alpha = \beta = 1/2$

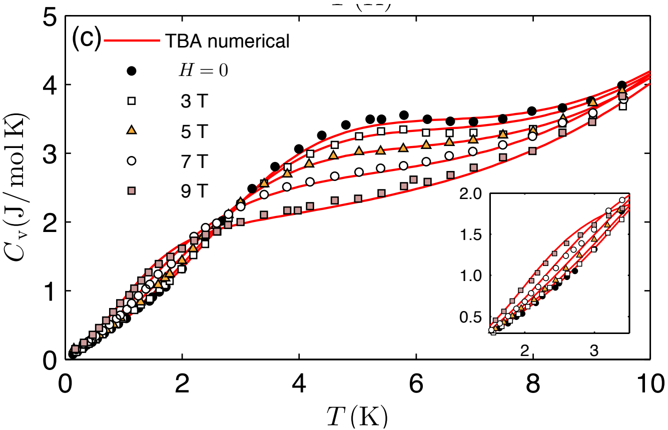
$\alpha + \beta\,(1 + \delta) = 2$

Bethe Ansatz for ideal spin chain material





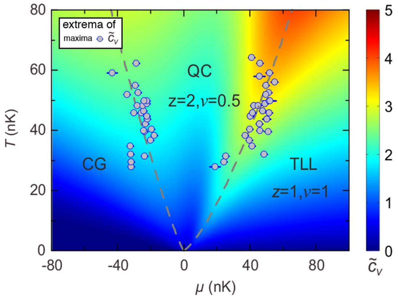
- $Cu(C_4H_4N_2)(NO_3)_2$ spin-1/2 chain at $T = 0.008$K
- theory: $J = 10.81$K, $g = 2.3$, $H_s = 13.9941$(T)
- experiment: $J = 10.8(1)$K, $g = 2.3(1)$, $H_s = 13.97$(T)
- interchain effect is observed for low temperatures and low field ($H = 1$ T)

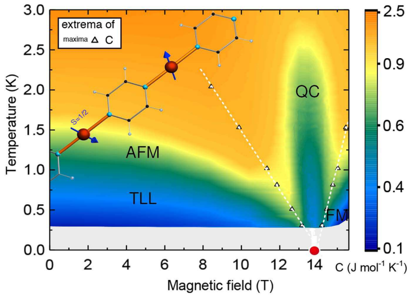


H → H$_s$ $$f \approx \frac{T^{\frac{3}{2}}}{2\sqrt{J\pi}} \mathrm{Li}_{\frac{3}{2}}\left(-e^{\frac{A}{T}}\right) - \frac{T^{\frac{5}{2}}}{16\sqrt{\pi}J^{\frac{3}{2}}} \mathrm{Li}_{\frac{5}{2}}\left(-e^{\frac{A}{T}}\right)$$

$$A = 4J - H - \frac{b_1}{\pi} + \frac{b_2}{\pi}$$

Luttinger liquid $$c_v = T/(3v_s)$$





Quantum criticality of bosons

Quantum criticality of spins

QC of spins $\quad n_m = M_s - M^z \approx -\frac{T^{1/2}}{2\sqrt{\pi}} f_{\frac{1}{2}}(\mu_s), \qquad \mu_s = 4J - g\mu_B M^z H$

$T = 0 \quad n_m = \frac{1}{\pi\sqrt{J}} \mu_s^{\frac{1}{2}}, \qquad J = \frac{h^2}{2m}$

Equivalence $\quad n_m = n_{\rm boson}, \qquad \mu_s = \mu_{\rm boson} = \mu$

$$T_{\rm magnon} = T_{boson} = \frac{1}{y_1}\mu, \qquad T_{\rm spinon} = T_d = \frac{1}{y_2}\mu$$

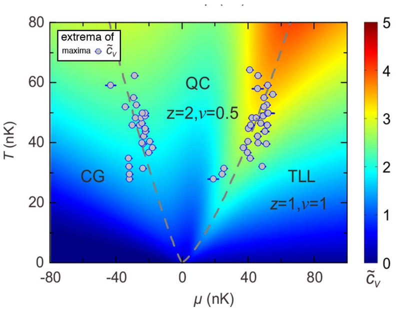
Yang, Chen, ... Guan, Yuan, Pan, Phys. Rev. Lett. 119, 165701 (2017)
Breunig, et al., Sci. Adv. 2017; 3:eaao3773

Outline

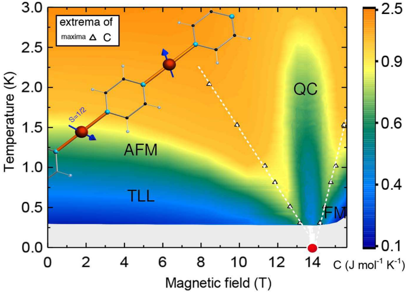
- Lecture I. Quantum liquid and quantum criticality in one dimension
  - Lieb-Liniger Bose gas
  - Spin-1/2 Heisenberg chain
- Lecture II. Spin charge separation in one dimension
  - Two component Fermi gas
  - Two component Bose gas

”Models are to be used, not believed”

Giamarchi, Quantum physics in one dimension, Oxford Scientific Publication



Quantum criticality of bosons



Quantum criticality of spins

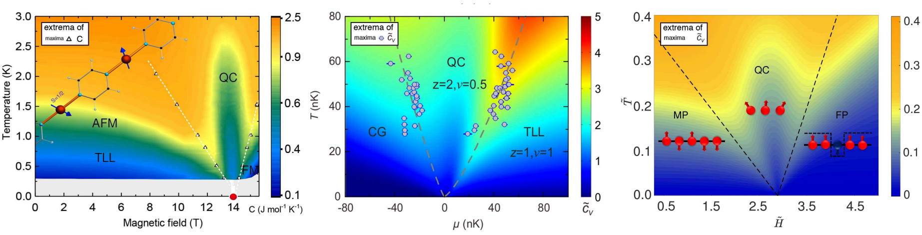
QC of spins $\quad n_m = M_s - M^z \approx -\frac{T^{1/2}}{2\sqrt{\pi}} f_{\frac{1}{2}}(\mu_s), \qquad \mu_s = 4J - g\mu_B M^z H$

$T = 0 \quad n_m = \frac{1}{\pi\sqrt{J}}\mu_s^{\frac{1}{2}}, \qquad J = \frac{h^2}{2m}$

Equivalence $\quad n_m = n_{\mathrm{boson}}, \qquad \mu_s = \mu_{\mathrm{boson}} = \mu$

$$T_{\mathrm{magnon}} = T_{boson} = \frac{1}{y_1}\mu, \qquad T_{\mathrm{spinon}} = T_d = \frac{1}{y_2}\mu$$

Yang, Chen, ... Guan, Yuan, Pan, Phys. Rev. Lett. 119, 165701 (2017)
Breunig, et al., Sci. Adv. 2017; 3:eaao3773

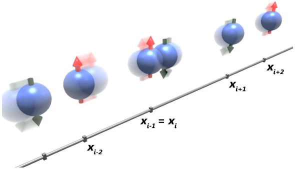


(a) (b) (c)

Spin and Charge meet at quantum criticality

on going research

- Universal description of Luttinger liquid and quantum criticality
- Quantum scalings beyond the free fermion theory
- Ferromagnetism in ultracold atoms



Yang-Gaudin model

$$
\begin{aligned}
\mathcal{H} &= \sum_{j=\downarrow,\uparrow} \int_0^L \phi_j^\dagger(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\phi_j(x)dx \\
&+ g_{1D}\int_0^L \phi_\downarrow^\dagger(x)\phi_\uparrow^\dagger(x)\phi_\uparrow(x)\phi_\downarrow(x)dx \\
&- \frac{H}{2}\int_0^L \left(\phi_\uparrow^\dagger(x)\phi_\uparrow(x) - \phi_\downarrow^\dagger(x)\phi_\downarrow(x)\right) dx
\end{aligned}
$$

- H: effective magnetic field
- $g_{\mathrm{1D}} = -\frac{\hbar^2 c}{m}$, $c = -2/a_{\mathrm{1D}}$, $a_{\mathrm{1D}} = -\frac{a_\perp^2}{a_{\mathrm{3D}}} + Aa_\perp$

Yang Phys. Rev. Lett. **19**, 1312 (1967)
Gaudin, Phys. Lett. **24**, 55 (1967)
Guan, Batchelor and Lee, Rev. Mod. Phys. 85, 1633 (2013)

- Bethe wave function ($0 < x_{Q1} < \ldots < x_{Qi} < x_{Qj} \ldots < x_{QN} < L$)

$$\psi = \sum_P A_{\sigma_1 \ldots \sigma_N}(P_1, \ldots, P_N | Q_1, \ldots, Q_N) \exp \mathrm{i}(k_{P1} x_{Q1} + \ldots + k_{PN} x_{QN})$$

- Bethe wave function ($0 < x_{Q1} < \ldots < x_{Qj} < x_{Qi} \ldots < x_{QN} < L$)

$$\psi' = \sum_P A_{\sigma_1 \ldots \sigma_N}(P_1, \ldots, P_N | Q_1, \ldots, Q_N) \exp \mathrm{i}(\ldots + k_{Pi} x_{Qj} + k_{Pj} x_{Qi} + \ldots)$$

- BA: I. continuity $\psi_{x_{Qi} = x_{Qj}^-} = \psi_{x_{Qi} = x_{Qj}^+}$

$$\sum_P A_{\sigma_1 \ldots \sigma_N}(P_i, P_j | Q_i, Q_j) \exp \mathrm{i}(\ldots + k_{Pi} x_{Qi} + k_{Pj} x_{Qj} + \ldots)$$
$$= \sum_P A_{\sigma_1 \ldots \sigma_N}(P_i, P_j | Q_j, Q_i) \exp \mathrm{i}(\ldots + k_{Pi} x_{Qj} + k_{Pj} x_{Qi} + \ldots)$$

$$A_{\sigma_1 \ldots \sigma_N}(P_i, P_j | Q_i, Q_j) + A_{\sigma_1 \ldots \sigma_N}(P_j, P_i | Q_i, Q_j)$$
$$= A_{\sigma_1 \ldots \sigma_N}(P_i, P_j | Q_j, Q_i) + A_{\sigma_1 \ldots \sigma_N}(P_j, P_i | Q_j, Q_i)$$

- discontinuity

$$X = \frac{1}{2}(x_1 + x_2), \ Y = x_2 - x_1$$

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial X^2} - 2\frac{\partial^2}{\partial Y^2}\right]\Psi + 2c\delta(Y)\Psi = E\Psi$$

$$\frac{\partial \psi}{\partial Y}\bigg|_{Y=0^+} - \frac{\partial \psi}{\partial Y}\bigg|_{Y=0^-} = c\psi|_{Y=0}$$

$$\begin{aligned}\frac{\mathrm{i}}{2}(k_{Pj} - k_{Pi})\left[A_{\sigma_1\ldots\sigma_N}(P_iP_j|Q_iQ_j) - A_{\sigma_1\ldots\sigma_N}(P_jP_i|Q_iQ_j)\right] \\ -\frac{\mathrm{i}}{2}(k_{Pj} - k_{Pi})\left[A_{\sigma_1\ldots\sigma_N}(P_iP_j|Q_jQ_i) - A_{\sigma_1\ldots\sigma_N}(P_jP_i|Q_jQ_i)\right] \\ = c[A_{\sigma_1\ldots\sigma_N}(P_iP_j|Q_iQ_j) + A_{\sigma_1\ldots\sigma_N}(P_jP_i|Q_iQ_j)]\end{aligned}$$

- two-body scattering relation $(A_{\sigma_1\ldots\sigma_N}(P_iP_j|Q_iQ_j) = [T_{ij}]^{\sigma'_1\ldots\sigma'_N}_{\sigma_1\ldots\sigma_N} A_{\sigma'_1\ldots\sigma'_N}(P_iP_j|Q_jQ_i))$:

$$\begin{aligned}\mathrm{i}(k_{Pj} - k_{Pi})\left[A_{\sigma_1\ldots\sigma_N}(P_iP_j|Q_iQ_j) - [T_{ij}]^{\sigma'_1\ldots\sigma'_N}_{\sigma_1\ldots\sigma_N} A_{\sigma'_1\ldots\sigma'_N}(P_jP_i|Q_iQ_j)\right] \\ = c\left[A_{\sigma_1\ldots\sigma_N}(P_iP_j|Q_iQ_j) + I^{\sigma'_1\ldots\sigma'_N}_{\sigma_1\ldots\sigma_N} A_{\sigma'_1\ldots\sigma'_N}(P_jP_i|Q_iQ_j)\right]\end{aligned}$$

- Two-body scattering relation

$$A_{\sigma_1\dots\sigma_N}(P_iP_j|Q_iQ_j) = \left[\frac{\mathrm{i}(k_{Pj}-k_{Pi})T_{ij}+cI}{\mathrm{i}(k_{Pj}-k_{Pi})-c}\right]_{\sigma_1\dots\sigma_N}^{\sigma'_1\dots\sigma'_N} A_{\sigma'_1\dots\sigma'_N}(P_jP_i|Q_iQ_j)$$

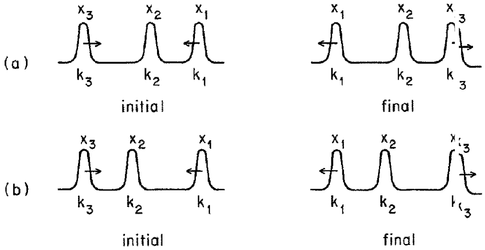
$$Y_{ij}(u) = \frac{\mathrm{i}uT_{ij}+cI}{\mathrm{i}u-c}$$

$$Y_{ij}(k_{Pj}-k_{Pi}) = \frac{\mathrm{i}(k_{Pj}-k_{Pi})T_{ij}+cI}{\mathrm{i}(k_{Pj}-k_{Pi})-c}$$

$$T_{\sigma_1\sigma_2} = -P_{\sigma_1\sigma_2} = \frac{1}{2}\left(1+\vec{\sigma}_1\vec{\sigma}_2\right)$$

Integrability: Nondiffraction

- Two-body scattering: the states with initial momenta k_1 and k_2 scattered into states with final momenta k_1' and k_2', if it is elastic, we request $k_1' + k_2' = k_1 + k_2$ and $k_1'^2 + k_2'^2 = k_1^2 + k_2^2$. These two solutions are *reflections* of each other in the (k_1, k_2) plane.
- However for particles ($N > 2$), it is not sufficient to have both energy and momentum conserved for the model to be solvable since the number of variables exceed the number of equations. Consider the case where ($N = 3$). H. Bethe proposed an hypothesis: Outgoing waves only consist of reflected waves, namely, no diffracted waves (Gu & Yang, Commun. Math. Phys. **122**,105(89)):



$(a) : A_{123} \to A_{213} \to A_{231} \to A_{321}$ $\quad (b) : A_{123} \to A_{132} \to A_{312} \to A_{321}$

Group of Quantum Integralable System
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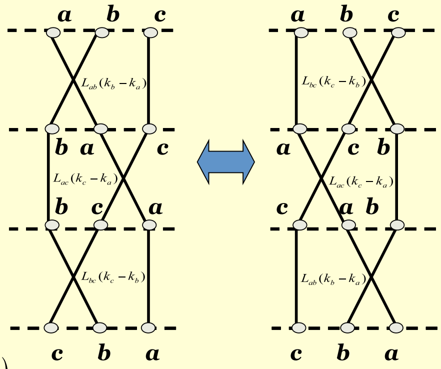
CN Yang's 1967

- **The many-body scattering matrix reduces to a product of many two-body scattering matrices.** Outgoing waves only consist of reflected waves, namely, no diffracted waves.

$$A_{21} = S_{12}(k_1 - k_2)A_{12}$$

$$S_{bc}(k_c - k_b)S_{ac}(k_c - k_a)S_{ab}(k_b - k_a)$$
$$= S_{ab}(k_b - k_a)S_{ac}(k_c - k_a)S_{bc}(k_c - k_b)$$



Schrodinger Eigenvalueproblems → **Bethe ansatz eqs**

$$A_{123}(k_1,k_2,k_3|Q) = [Y_{12}(k_2-k_1)]^{213} A_{213}(k_2,k_1,k_3|Q)$$
$$= [Y_{12}(k_2-k_1)]^{213}[Y_{23}(k_3-k_1)]^{231} A_{231}(k_2,k_3,k_1|Q)$$
$$= [Y_{12}(k_2-k_1)]^{213}[Y_{23}(k_3-k_1)]^{231}[Y_{12}(k_3-k_2)]^{321} A_{321}(k_3,k_2,k_1|Q)$$

$$A_{123}(k_1,k_2,k_3|Q) = [Y_{23}(k_3-k_2)]^{132} A_{132}(k_1,k_3,k_2|Q)$$
$$= [Y_{23}(k_3-k_2)]^{132}[Y_{12}(k_3-k_1)]^{312} A_{312}(k_3,k_1,k_2|Q)$$
$$= [Y_{23}(k_3-k_2)]^{132}[Y_{12}(k_3-k_1)]^{312}[Y_{23}(k_2-k_1)]^{321} A_{321}(k_3,k_2,k_1|Q)$$

Yang-Baxter equation

$$Y_{12}(k_2-k_1)Y_{23}(k_3-k_1)Y_{12}(k_3-k_2) = Y_{23}(k_3-k_2)Y_{12}(k_3-k_1)Y_{23}(k_2-k_1)$$

Properties of $Y-$matrix

$$Y_{ab}(u)Y_{cd}(v) = Y_{cd}(v)Y_{ab}(u)$$
$$Y_{ab}(u)Y_{ba}(-u) = 1$$

- periodic boundary conditions: $\psi(x_1,\ldots,x_i,\ldots,x_N)=\psi(x_1,\ldots,x_i+L,\ldots,x_N)$

$$
\begin{aligned}
&A_\sigma(P_i,P_1,\ldots,P_N|Q_i,Q_1,\ldots,Q_N)\\
&\quad=\exp(\mathrm{i}k_iL)A_{\sigma'}(P_1,\ldots,P_N,P_i|Q_1,\ldots,Q_N,Q_i)
\end{aligned}
$$

- gives

$$
\begin{aligned}
&Y_{12}(k_1-k_i)Y_{23}(k_2-k_i)\ldots Y_{i-1,i}(k_{i-1}-k_i)A_\sigma(P_1,\ldots,P_i,\ldots,P_N|Q_i,Q_1,\ldots,Q_N)\\
&\quad=T_{12}T_{23}\ldots T_{N-1,N}Y_{N-1,N}(k_i-k_N)Y_{N-2,N-1}(k_i-k_{N-1})\ldots Y_{i,i+1}(k_i-k_{i+1})\\
&\quad\times\exp(\mathrm{i}k_iL)A_\sigma(P_1,\ldots,P_i,\ldots,P_N|Q_i,Q_1,\ldots,Q_N)
\end{aligned}
$$

- further

$$
\begin{aligned}
&T_{i-1,i}\ldots T_{23}T_{12}Y_{12}(k_1-k_i)Y_{23}(k_2-k_i)\ldots Y_{i-1,i}(k_{i-1}-k_i)A_E(P|Q)\\
&\quad=T_{i,i+1}T_{i+1,i+2}\ldots T_{N-1,N}Y_{N-1,N}(k_i-k_N)\ldots Y_{i,i+1}(k_i-k_{i+1})\exp(\mathrm{i}k_iL)A_E(P|Q)
\end{aligned}
$$

- LHS:

$$\begin{aligned}\mathrm{LHS} &= T_{i-1,i}\dots T_{23}R_{12}(k_1-k_i)T_{23}T_{23}Y_{23}(k_2-k_i)\dots Y_{i-1,i}(k_{i-1}-k_i)A_E(P|Q)\\ &= T_{i-1,i}\dots T_{34}R_{13}(k_1-k_i)T_{34}T_{34}R_{23}(k_2-k_i)Y_{34}(k_3-k_i)\dots\\ &\quad\times Y_{i-1,i}(k_{i-1}-k_i)A_E(P|Q)\\ &\vdots\\ &= R_{1,i}(k_1-k_i)R_{2,i}(k_2-k_i)\dots R_{i-1,i}(k_{i-1}-k_i)A_E(P|Q)\end{aligned}$$

- RHS:

$$\mathrm{RHS} = R_{i,N}(k_i-k_N)R_{i,N-1}(k_i-k_{N-1})\dots R_{i,i+1}(k_i-k_{i+1})\exp(\mathrm{i}k_iL)A_E(P|Q)$$

- Using the R-operator property:

$$\begin{aligned}Y_{ab}(u)Y_{cd}(v) &= Y_{cd}(v)Y_{ab}(u)\\ Y_{ab}(u)Y_{bc}(u+v)Y_{ab}(v) &= Y_{bc}(v)Y_{ab}(u+v)Y_{bc}(u)\\ Y_{ab}(u)Y_{ba}(-u) &= 1\end{aligned}$$

- The eigenvalue problem: algebraic Bethe ansatz

$$\mathfrak{R}_i(k_i)A_E(P|Q) = \exp(\mathrm{i}k_iL)A_E(P|Q)$$
$$\mathfrak{R}_i(k_i) = R_{i+1,i}(k_{i+1} - k_i)\dots R_{N,i}(k_N - k_i)R_{1,i}(k_1 - k_i)\dots R_{i-1,i}(k_{i-1} - k_i)$$

- quantum transfer matrix : $\tau(u) = \mathrm{Tr}_a(\mathcal{T}_N(u))$ and $L_i(k_i - u) \equiv R_{i,a}(k_i - u)$

$$\mathcal{T}_N(u) = L_N(k_N - u)\dots L_2(k_2 - u)L_1(k_1 - u)$$

$$
\begin{aligned}
&\tau(u)|_{u=k_i} \\
&= \mathrm{Tr}_a(\mathcal{T}_N(u))|_{u=k_i} \\
&= \mathrm{Tr}_a(R_{N,a}(k_N - k_i) \ldots R_{i+1,a}(k_{i+1} - k_i)P_{i,a} \\
&\quad \times R_{i-1,a}(k_{i-1} - k_i) \ldots R_{2,a}(k_2 - k_i)R_{1,a}(k_1 - k_i)) \\
&= \mathrm{Tr}_a(P_{i,a}R_{N,i}(k_N - k_i) \ldots R_{i+1,i}(k_{i+1} - k_i) \\
&\quad \times R_{i-1,a}(k_{i-1}k - i) \ldots R_{2,a}(k_2 - k_i)R_{1,a}(k_1 - k_i)) \\
&= \mathrm{Tr}_a(R_{i-1,a}(k_{i-1}k - i) \ldots R_{2,a}(k_2 - k_i)R_{1,a}(k_1 - k_i) \\
&\quad \times P_{i,a}R_{N,i}(k_N - k_i) \ldots R_{i+1,i}(k_{i+1} - k_i)) \\
&= \mathrm{Tr}_a(P_{i,a}R_{i-1,i}(k_{i-1}k - i) \ldots R_{2,i}(k_2 - k_i)R_{1,i}(k_1 - k_i) \\
&\quad \times R_{N,i}(k_N - k_i) \ldots R_{i+1,i}(k_{i+1} - k_i)) \\
&= \mathrm{Tr}_a(P_{i,a})R_{i+1,i}(k_{i+1} - k_i) \ldots R_{N,i}(k_N - k_i) \\
&\quad \times R_{1,i}(k_1 - k_i) \ldots R_{i-1,i}(k_{i-1} - k_i) \\
&= \mathfrak{R}_i(k_i)
\end{aligned}
$$

- Quantum Inverse Scattering Method: $R_{12}(u) = P_{12} Y_{12}(u)$

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$

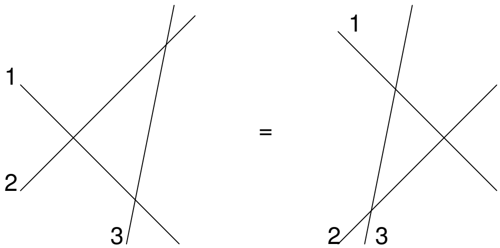


Figure: Graphical representation of the Yang-Baxter relation for three particle scattering.

- **Lax operator:** $R_{a1,n}(u) = L_n(u)$, $\check{R}(u-v) = PR(u-v)$

$$\check{R}(u-v)L_n(u) \otimes L_n(v) = L_n(v) \otimes L_n(u)\check{R}(u-v)$$

- **Monodromy matrix:** $\mathcal{T}_N(u) = \overset{\curvearrowleft}{\prod}_{i=1}^{N} L_i(u) = L_N(u)L_{N-1}(u)\dots L_1(u)$

$$\check{R}(u-v)\mathcal{T}_N(u) \otimes \mathcal{T}_N(v) = \mathcal{T}_N(v) \otimes \mathcal{T}_N(u)\check{R}(u-v)$$

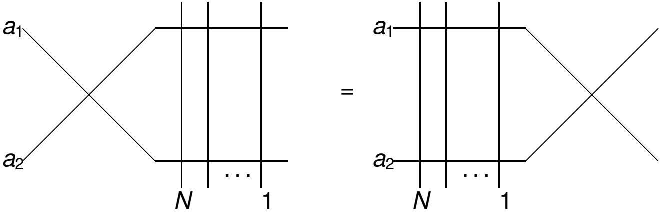


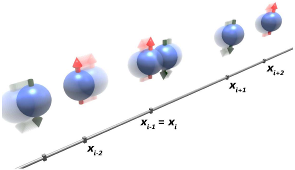
Figure: The “train” argument.

- Transfer matrix $\tau(u) = \mathrm{Tr}_a(\mathcal{T}_N(u))$

$$[\tau(u), \tau(v)] = 0$$

- Conserved quantities $\tau(u) = \tau_0 + H_1 u + H_2 u^2 + \ldots$

$$\begin{aligned}
\left.\frac{d}{du}\tau(u)\right|_{u=0} &= \sum_{i=1}^{N} \mathrm{Tr}_a \left[L_{a,N}(u) \ldots \frac{dL_{a,i}(u)}{du} \ldots L_{a,1}(u) \right]_{u=0} \\
&= \sum_{i=1}^{N} \mathrm{Tr}_a \left[P_{a,N} \ldots P_{a,i+1} \frac{dL_{a,1}(0)}{du} P_{a,i-1} \ldots P_{a,1} \right] \\
&= \sum_{i=1}^{N} \mathrm{Tr}_a \left[P_{a,N} \ldots P_{a,i+2} \frac{dL_{i+1,i}(0)}{du} P_{i+1,i} P_{a,i+1} \ldots P_{a,1} \right] \\
&= \mathrm{Tr}_a(P_{a,1} \ldots P_{a,N}) \sum_{i=1}^{N} P_{i+1,i} \frac{dL_{i+1,i}(0)}{du}
\end{aligned}$$



- Bethe Ansatz equations

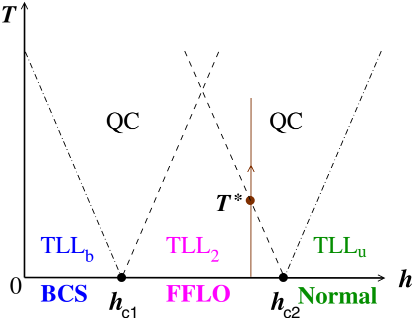
$$\exp(\mathrm{i}k_j L) = \prod_{\ell=1}^{M} \frac{k_j - \Lambda_\ell + \mathrm{i}\,c/2}{k_j - \Lambda_\ell - \mathrm{i}\,c/2},$$

$$\prod_{\ell=1}^{N} \frac{\Lambda_\alpha - k_\ell + \mathrm{i}\,c/2}{\Lambda_\alpha - k_\ell - \mathrm{i}\,c/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}\,c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}\,c}$$

- Energy

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^{N} k_j^2$$

- Repulsive interaction $c > 0$
- Attractive interaction $c < 0$



Phase diagram of attractive Fermi gas in 1D

Guan, Batchelor, Lee & Bortz, Phys. Rev. B 76, 085120 (2007)

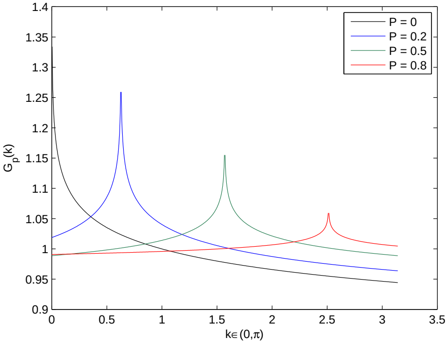
Zhao, Guan, Liu, Batchelor & Oshikawa, Phys. Rev. Lett. 103, 140404 (2009)

Yu, Chen, Lin, Roemer, Guan, Phys. Rev. B 94, 195129 (2016)

Pair correlation function

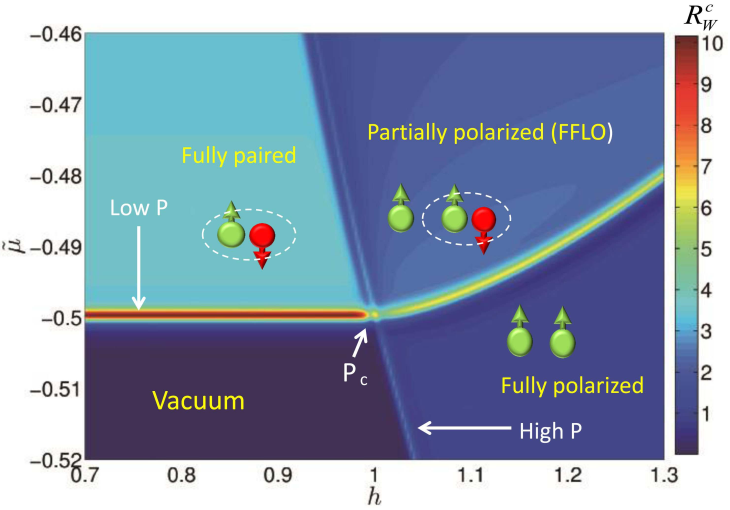
$$\langle \psi_\uparrow^\dagger(x,t)\psi_\downarrow^\dagger(x,t)\psi_\uparrow(0,0)\psi_\downarrow(0,0)\rangle \approx \frac{A_{p,1}\cos\left(\pi(n_\uparrow - n_\downarrow)x\right)}{|x+\mathrm{i}v_u t|^{\theta_1}|x+\mathrm{i}v_b t|^{\theta_2}}$$

$$\theta_1 \approx \frac{1}{2}, \qquad \theta_2 \approx \frac{1}{2} - \frac{(1-P)}{2|\gamma|}$$



The spatial modulations are characteristic of a Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state. The backscattering among the Fermi points of bound pairs and unpaired fermions results in a 1D analog of the FFLO state and displays a microscopic origin of the FFLO nature.

Lee & Guan, Nucl. Phys. B **853**, 125 (2011)



The second Wilson ratio at $t \sim 0.001\epsilon_b$

$$R_W^c = \frac{\pi^2 k_B^2}{3}\frac{\kappa}{c_v/T}, \quad TTL_P : R_W^c = 4K, \quad TTL_F : R_W^c = 1$$

Bethe ansatz equations

$$e^{\mathrm{i}k_i L} = \prod_{j=1}^{M} \frac{k_i - \Lambda_j + \frac{1}{2}\mathrm{i}c}{k_i - \Lambda_j - \frac{1}{2}\mathrm{i}c}, \quad i = 1, \ldots, N$$

$$\prod_{i=1}^{N} \frac{\Lambda_j - k_i + \frac{1}{2}\mathrm{i}c}{\Lambda_j - k_i - \frac{1}{2}\mathrm{i}c} = \prod_{\ell=1}^{N} \frac{\Lambda_j - \Lambda_\ell + \mathrm{i}c}{\Lambda_j - \Lambda_\ell - \mathrm{i}c}, \quad j = 1, \ldots, M$$

- an effective antiferromagnetic Heisenberg spin chain for $c > 0$

$$H = \frac{J}{2} \sum_{i=1}^{N} \hat{S}_i \bullet \hat{S}_{i+1} - h \sum_{i}^{N} s_i^z, \quad J \approx \frac{4E_F}{c}$$

$$E = \frac{\pi^2}{3L^2} N \left(N^2 - 1\right) - \frac{nJ}{2} \sum_{i=1}^{M} \frac{1}{r_i^2 + 1/4} + O(c^{-2})$$

$$\left(\frac{r_i + \frac{i}{2}}{r_i - \frac{i}{2}}\right)^N = -\prod_{j=1}^{M} \frac{r_i - r_j + \mathrm{i}}{r_i - r_j - \mathrm{i}}$$

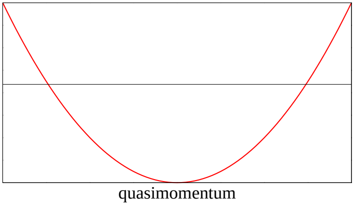
Oelkers, Bathchelor, Bortz and Guan, J. Phys. A 39, 1073 (2006)
Lee, Guan, Sakai and Batchelor, PRB 85, 085414 (2012)
Guan, Batchelor and Lee, Rev. Mod. Phys. 85, 1633 (2013)

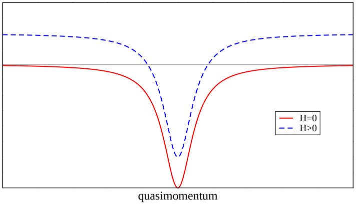
Spin-charge separation in repulsive Fermi gas

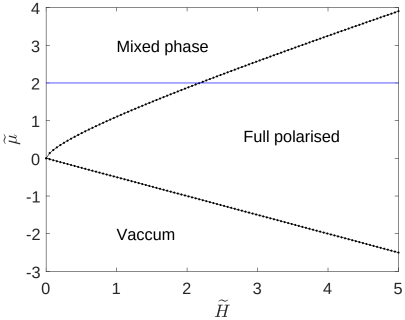
$$\varepsilon(k) = k^2 - \mu - \frac{H}{2} - T\sum_{n=1}^{\infty} a_n * \ln\left(1 + e^{-\phi_n(k)/T}\right)$$

$$\phi_n(\lambda) = nH - Ta_n * \ln\left(1 + e^{-\varepsilon(\lambda)/T}\right) + T\sum_{m=1}^{\infty} T_{nm} * \ln\left(1 + e^{-\phi_m(\lambda)/T}\right)$$

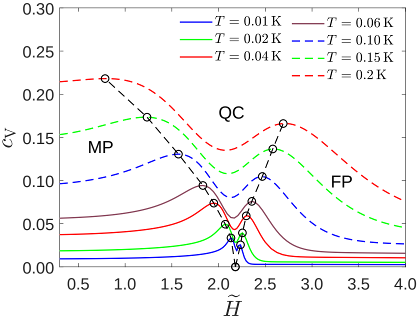
$$p = \frac{T}{2\pi}\int_{-\infty}^{\infty} \ln[1 + \mathrm{e}^{-\varepsilon(k)/T}]\mathrm{d}k$$







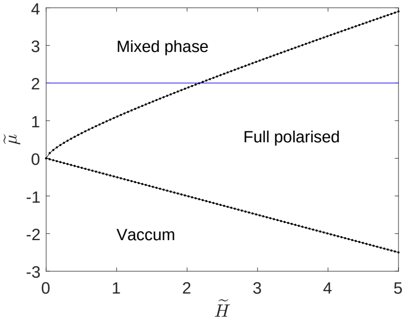
Phase diagram



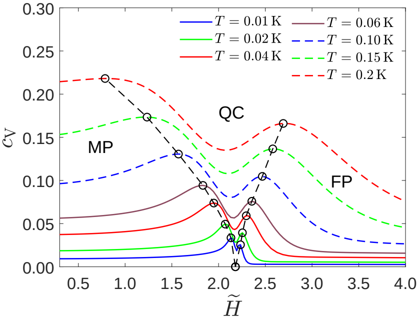
Specific heat in the vicinity of $\tilde{H} = 2.18$

- Spin charge interaction effect at quantum criticality

$$\begin{aligned}
\varepsilon(k) &= k^2 - \mu - f_{xxx}(H, T) \\
f_{xxx}(H, T) &= -\frac{H}{2} - T \sum_{n=1}^{\infty} a_n * \ln\left(1 + e^{-\phi_n(k)/T}\right) \\
\phi_n(\lambda) &= nH - Ta_n * \ln[1 + \mathrm{e}^{-\varepsilon(k)/T}] + T \sum_{m=1}^{\infty} T_{mn} * \ln[1 + \mathrm{e}^{-\phi_m(\lambda)/T}]
\end{aligned}$$



Phase diagram



Specific heat in the vicinity of $\tilde{H} = 2.18$

- Spin charge interaction

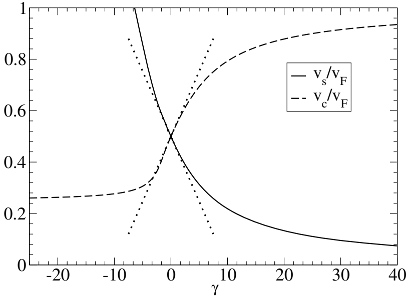
$$p - p_0 = \frac{\pi T^2}{6}\left(\frac{1}{v_c} + \frac{1}{v_s}\right), \qquad v_c = \frac{t_c}{2\pi\rho_c(k_0)}, \quad v_s = \frac{t_s}{2\pi\rho_s(\lambda_0)}$$

$$t_c = \left.\frac{\mathrm{d}\varepsilon(k)}{\mathrm{d}k}\right|_{k=k_0}, \qquad t_s = \left.\frac{\mathrm{d}\phi_1(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_0}$$

- Spin charge interaction effect at QC

$$
\begin{aligned}
p &= \frac{T}{2\pi}\int_{-\infty}^{\infty}\ln(1+\mathrm{e}^{-\varepsilon(k)/T})\mathrm{d}k \\
\varepsilon(k) &= A_1 k^2 - \mu - f_{xxx}, \qquad \phi_1(\lambda) = B_1\lambda^2 - \mu_s
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= 1 + \frac{8T}{\pi c^3}\int_{-\infty}^{\infty}\ln(1+\mathrm{e}^{-\frac{\phi_1(\lambda)}{T}})\mathrm{d}\lambda \\
f_{xxx} &= \frac{H}{2} + \frac{2T}{\pi c}\int_{-\infty}^{\infty}\ln(1+\mathrm{e}^{-\frac{\phi_1(\lambda)}{T}})\mathrm{d}\lambda - \frac{8T}{\pi c^3}\int_{-\infty}^{\infty}\lambda^2\ln(1+\mathrm{e}^{-\frac{\phi_1(\lambda)}{T}})\mathrm{d}\lambda \\
B_1 &= \frac{16p}{c^3} - \frac{T}{\pi c^3}\int_{-\infty}^{\infty}\ln(1+\mathrm{e}^{-\frac{\phi_1(\lambda')}{T}})\mathrm{d}\lambda' \\
\mu_s &= -H + \frac{4p}{c} - \frac{8T}{\pi c^3}\int_{-\infty}^{\infty}k^2\ln(1+\mathrm{e}^{-\frac{\varepsilon(k)}{T}})\mathrm{d}k - \frac{T}{\pi c}\int_{-\infty}^{\infty}\ln(1+\mathrm{e}^{-\frac{\phi_1(\lambda')}{T}})\mathrm{d}\lambda' \\
&\quad + \frac{T}{\pi c^3}\int_{-\infty}^{\infty}\lambda'^2\ln(1+\mathrm{e}^{-\frac{\phi_1(\lambda')}{T}})\mathrm{d}\lambda'
\end{aligned}
$$



- Spins and Charges velocities for $H \to 0$: $\gamma \ll 1$ and $\gamma \gg 1$

$$v_{c,s} = \frac{1}{2} v_F \left(1 + \pm \frac{\gamma}{\pi^2}\right), \qquad \begin{cases} v_c \approx 2\pi n_c \left(1 - \frac{4 \ln 2}{\gamma}\right) \\ v_s \approx \frac{2\pi^3 n_c}{3\gamma} \left(1 - \frac{6 \ln 2}{\gamma}\right) \end{cases}$$

- Spins and Charges velocities for $H \to H_c$ and $\gamma \gg 1$

$$\begin{cases} v_c \approx 2\pi n_c \left(1 - \frac{12}{\pi\gamma} \sqrt{1 - \frac{H}{H_c}}\right) \\ v_s \approx \frac{H_c}{n_c} \sqrt{1 - \frac{H}{H_c}} \end{cases}$$

Oelkers, Bathchelor, Bortz and Guan, J. Phys. A 39, 1073 (2006)
Lee, Guan, Sakai and Batchelor, PRB 85, 085414 (2012)

- Luttinger liquid with backward interaction

$$
\begin{aligned}
H &= H_c + H_\sigma + \frac{2g_1}{(2\pi\alpha)^2}\int dx \cos(\sqrt{8\phi_\sigma}) \\
H_\nu &= \int dx \left(\frac{\pi v_\nu K_\nu}{2}\Pi_\nu^2 + \frac{v_\nu}{2\pi K_\nu}\left(\partial_x \phi_\nu\right)^2 \right)
\end{aligned}
$$

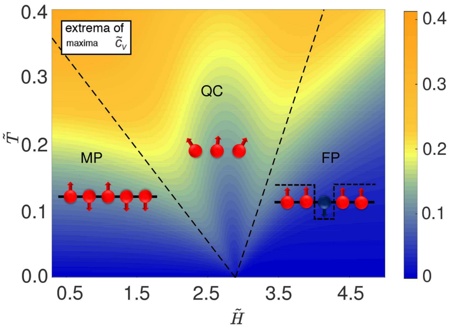
- charge and spin Bose fields $[\phi_\nu(x), \Pi_\mu(y) = \mathrm{i}\delta_{\nu\mu}\delta(x-y)]$

$$
\begin{aligned}
\phi_{c,\sigma} &= \left(\phi_\uparrow \pm \phi_\downarrow\right)/\sqrt{2} \\
\Pi_{c,\sigma} &= \left(\Pi_\uparrow \pm \Pi_\downarrow\right)/\sqrt{2}
\end{aligned}
$$

- Fermi liquid signature

$$
\begin{aligned}
F &= E_0 - \frac{\pi T^2}{6}\left(\frac{1}{v_s} + \frac{1}{v_c}\right) \\
R_W &= \frac{2v_c}{v_c + v_\sigma}, \qquad \kappa = 2K_c/\pi v_c
\end{aligned}
$$

Giamarchi, Quantum Physics in One dimension (Oxford University Press, Oxford 2004)



- On-going research: Beyond free fermion theory, i.e. spin and change interacting effect

$$f = E_0 - \frac{\pi T^2}{6}\left(\frac{1}{v_s} + \frac{1}{v_c}\right)$$

$$h_s = 2nJ; \quad \chi \approx \frac{1}{J\pi^2} = \frac{1}{2\pi v_s}; \quad v_s = \pi J/2$$

$$1 - \frac{m_z}{m_s} = D\left(1 - \frac{h}{h_s}\right)^{\delta}; \quad \frac{(m_s - m_z)}{h} \sim T^{\beta}; \quad \frac{c_v}{T} \sim T^{-\alpha}$$

$$\alpha + \beta(1+\delta) = 2, \qquad \alpha = \beta = 1/2, \qquad \delta = 2$$

$$T_{magnon} = T_{boson} = \frac{1}{y_1}\mu, \qquad T_{spinon} = T_d = \frac{1}{y_2}\mu$$

III. Spin charge separation in interacting bosons

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Polarization of Interacting Bosons with Spin

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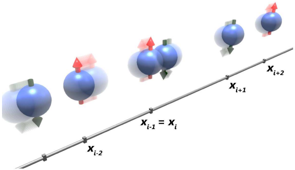
[2]*NEC Research Institute, 4 Independence Way, Princeton, New Jersey 08540*
(Received 1 July 2002; published 7 November 2002)

We prove that in the absence of explicit spin-dependent forces one of the ground states of interacting bosons with spin is always fully polarized. Generally, this state is degenerate with other states, but one can specify the exact degeneracy. For $T > 0$, the magnetization and zero-field susceptibility exceed that of a pure paramagnet. The results are relevant to experimental work on triplet superconductivity and condensation of atoms with spin. They eliminate the possibility, raised in some theoretical speculations, that the ground state or positive temperature state might be antiferromagnetic.

- Ferromagnetic spin-spin interaction enhances magnetization

Eisenberg and Lieb, PRL 89, 220403 (2002)
Erhard et al. PRL 92, 160406 (2004); Fuchs et al, PRL, 95, 150402 (2005)
Li, Gu, Ying, and U. Eckern, Europhys. Lett. 61, 368 (2003)
Guan, Batchelor, Takahashi, Phys. Rev. A 76, 043617 (2007)

Ferromagnetism in two-component spinor Bose gas



two-component Bose gas

$$\begin{aligned}\mathcal{H} &= \sum_{j=\downarrow,\uparrow}\int_0^L \phi_j^\dagger(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\phi_j(x)dx \\ &+g_{1D}\sum_{j,j'}\int_0^L \phi_j^\dagger(x)\phi_{j'}^\dagger(x)\phi_{j'}(x)\phi_j(x)dx \\ &-\frac{H}{2}\int_0^L\left(\phi_\uparrow^\dagger(x)\phi_\uparrow(x)-\phi_\downarrow^\dagger(x)\phi_\downarrow(x)\right)dx\end{aligned}$$

- H: effective magnetic field; $g_{\rm 1D} = -\frac{\hbar^2 c}{m}$, $c = -2/a_{\rm 1D}$, $a_{\rm 1D} = -\frac{a_\perp^2}{a_{\rm 3D}} + Aa_\perp$

Sutherland, Phys. Rev. Lett. 20, 98 (1968)

Li, Gu, Ying, and U. Eckern, Europhys. Lett. 61, 368 (2003)

Guan, Batchelor, Takahashi, Rev. A 76, 043617 (2007) scattering matrix

$$S_{ab} = \begin{pmatrix} \frac{k-\mathrm{i}/\lambda}{k+\mathrm{i}/\lambda} & 0 & 0 & 0 \\ 0 & \frac{k}{k+\mathrm{i}/\lambda} & \frac{-\mathrm{i}/\lambda}{k+\mathrm{i}/\lambda} & 0 \\ 0 & \frac{-\mathrm{i}/\lambda}{k+\mathrm{i}/\lambda} & \frac{k}{k+\mathrm{i}/\lambda} & 0 \\ 0 & 0 & 0 & \frac{k-\mathrm{i}/\lambda}{k+\mathrm{i}/\lambda} \end{pmatrix}$$

- strong coupling regime $Lc \gg 1$

$$E = \frac{\pi^2}{3L^2} N \left(N^2 - 1 \right) \left[1 - \frac{4N}{Lc} + \frac{2}{Lc} \Omega_M^N \right] + O(c^{-2})$$

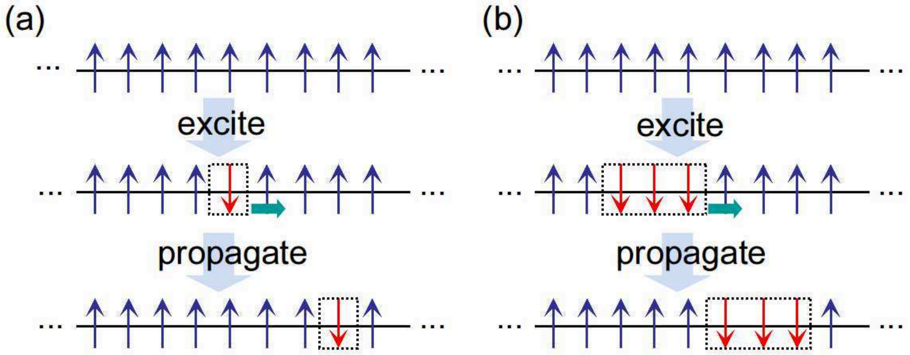
$$k_i = \frac{\pi}{L} n_i \left(1 - \frac{1}{Lc} \Omega_M^N \right), \quad n_i = \pm 1, \pm 3, \ldots, \pm (N-1), \text{ for odd } N$$

- effective Heisenberg chain with a ferromagnetic exchange coupling J

$$H = \frac{J}{2} \sum_{i=1}^{N} \hat{S}_i \bullet \hat{S}_{i+1} - h \sum_i s_i^z$$

$$E = \frac{\pi^2}{3L^2} N \left(N^2 - 1 \right) \left(1 - \frac{4N}{Lc} \right) - \frac{nJ}{2} \sum_{i=1}^{M} \frac{1}{r_i + 1/4} + O(c^{-2})$$

$$J \approx -\frac{4E_F}{c}, \qquad \left(\frac{r_i - \frac{i}{2}}{r_i + \frac{i}{2}} \right)^N = -\prod_{j=1}^{M} \frac{r_i - r_j - \mathrm{i}}{r_i - r_j + \mathrm{i}}$$


(a)
...
...
excite
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propagate
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(b)
...
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excite
...
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propagate
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...

Collective modes for spin degree

$$\varepsilon(Q) = J\,(1 - \cos Q)\,, \qquad \epsilon(q) \approx \frac{q^2}{2m^*}, \text{ for } p \to 0$$

Where Q is the spin wave vector $-\pi < Q < \pi$. The momentum of magnon $q = \hbar n Q$.

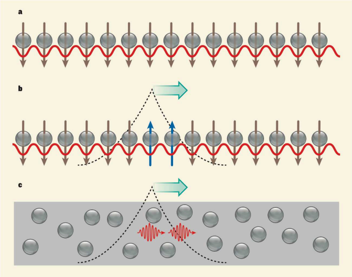
weak coupling: quasi condensate

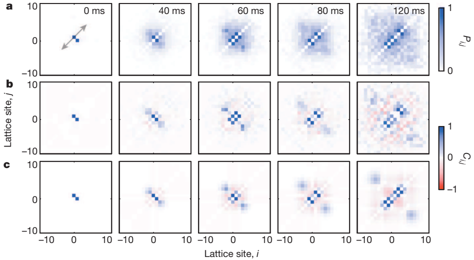
$$\begin{aligned} k_j &\approx \frac{2c}{L}\sum_{\ell=1}^{N}\frac{1}{k_j - k_\ell} + \frac{c}{L\lambda_1} + \frac{c}{L\lambda_1^2}k_j \\ E(q) &= E_0 + q^2, \qquad \frac{m}{m^*} \approx 1 - \frac{4\sqrt{\gamma}}{3\pi} \end{aligned}$$

strong coupling: all bosons are moving as flipping one spin

$$\begin{aligned} k_j &\approx \left(\frac{2\pi n_j}{L} + \frac{2q}{Lc} + \frac{c}{L\lambda_1}\right)\left(1 - \frac{2N}{Lc} + \frac{c}{L\lambda_1^2}\right) \\ n_j &= \pm 1, \pm 3, \ldots, \pm (N-1)/2, \;\; \lambda_1 \approx Nc/Lq \\ \frac{m}{m^*} &\approx \frac{1}{N} + \frac{2\pi^2}{3\gamma}\left(1 - \frac{2}{\gamma}\right) \end{aligned}$$

Fuchs, Gangardt, Keilmann and Shlyapnikov, Phys. Rev. Lett. 95, 150402 (2005)
Batchelor, Bortz, Guan and Oelkers, J. Stat. Mech. JSTAT/2006/P03016





- novel concepts: itinerant ferromagnetism, fractional excitations, spin liquids

$$H = -Jex \sum_i \left[\frac{1}{2} \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + \Delta S_i^z S_{i+1}^z \right]$$

- Correlation function $C_{i,j} = P_{i,j} - P_i P_j$, where $P_i = \sum_j P_{i,j}$ is the probability of finding one atom on site i and $P_{i,j}$ is a joint probability of simutaneously detecting atoms on lattice sites i and j.

Fukuhara, et al Nature, 502, 76 (2013)

Fukuhara, et al Nature, Phys. 9, 235 (2013)

Effective field theory

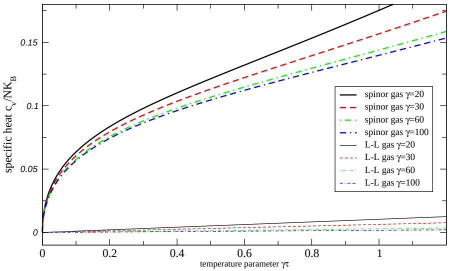
$$
\begin{aligned}
H &= H_{\mathrm{ph}} + H_{XXX} \\
H_{\mathrm{ph}} &= \int dx \left(\frac{\pi v_s K}{2} \Pi^2 + \frac{v_s}{2\pi K} (\partial_x \phi)^2 \right) \\
H &= \frac{J}{2} \sum_{i=1}^{N} \hat{S}_i \bullet \hat{S}_{i+1} - h \sum_{i}^{N} s_i^z
\end{aligned}
$$

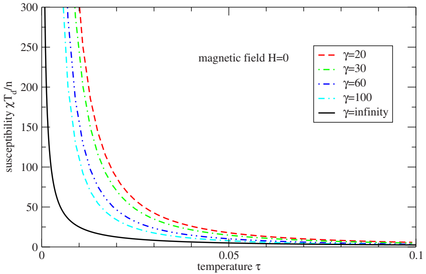
Guan, Batchelor, Takahashi, Phys. Rev. A 76, 043617 (2007)
Matveev and Furusaki, Phys. Rev. Lett. 101, 170403 (2008)

spin charge separation

$$
\begin{aligned}
\epsilon(k) &\approx k^2 - \mu - \frac{2cP(T,H)}{c^2+k^2} + f_{XXX}(T,H) \\
f_{XXX}(T,H) &\approx J\ln 2 - T\int_{-\infty}^{\infty} d\lambda s(\lambda)\ln(1+\eta_1(\lambda)) \\
\ln\eta_1(\lambda) &= \frac{2\pi J}{T}s(\lambda) + s * \ln(1+\eta_2(\lambda)) \\
\ln\eta_n(\lambda) &= s * \ln(1+\eta_{n-1}(\lambda))\ln(1+\eta_{n+1}(\lambda))
\end{aligned}
$$

Guan, Batchelor, Takahashi, Phys. Rev. A 76, 043617 (2007)





$$\frac{c_v}{NK_B} \approx \frac{1.042 \times 3\sqrt{3}(\gamma\tau)^{\frac{1}{2}}}{8(1-\frac{3}{\gamma})\pi} - \frac{3(\gamma\tau)}{2(1-\frac{6}{\gamma})\pi^2} + \frac{0.9 \times 45\sqrt{3}(\gamma\tau)^{\frac{3}{2}}}{32(1-\frac{9}{\gamma})\pi^3} + \frac{\tau}{6(1-\frac{4}{\gamma})}$$

$$\chi \approx \frac{n}{T_d}\left[\frac{\pi^2}{18\gamma\tau^2}\left(1-\frac{6}{\gamma}\right) + \frac{0.5826 \times \sqrt{3}\pi}{6\sqrt{\gamma}\tau^{\frac{3}{2}}}\left(1-\frac{3}{\gamma}\right) + \frac{0.678}{4\tau}\right]$$

- Critical exponents: $c_v \sim T^{-a}$, $\chi \sim T^{-b}$; ferromagnetic: $a = -0.5$, $b = 2$
- paramagnetic: $a = -1$, $b = 1$

Guan, Batchelor, Takahashi, Phys. Rev. A 76, 043617 (2007)

Conclusion and on-going research

- We have studied quantum liquid and spin charge separation phenomenon with quantum criticality.
- Generalized hydrodynamics: spin and charge currents
- Quantum metrology in spin systems with long rang interaction: quantum collapse and revival, disorders, entanglement entropy $\cdots$

Thank you for your attentions!