Understanding quantum separation and togetherness in 1D ultracold atoms

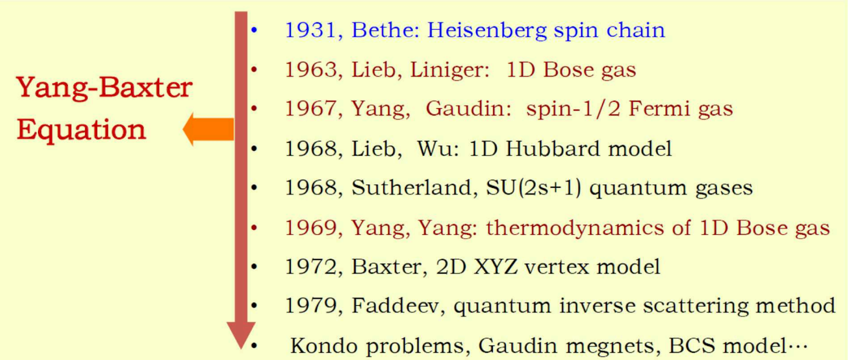
Xi-Wen Guan

WIPM, April 2018

Bethe Ansatz 波函数:

$$\psi(x_1, x_2, \cdots, x_N) = \sum_P A[Q, P] \exp\left[i(k_{P_1} x_{Q_1} + \cdots + k_{P_N} x_{Q_N})\right]$$

Yang-Baxter 方程：



$$Y^{ab}_{jk} Y^{bc}_{ik} Y^{ab}_{ij} = Y^{bc}_{ij} Y^{ab}_{ik} Y^{bc}_{jk}$$

Exact integrability in quantum field theory and statistical systems

Review of Modern Physics, 53, 253, 1981

H. B. Thacker

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

The Potts model

Review of Modern Physics, 54, 235, 1982

F. Y. Wu*

Institut für Festkörperforschung der Kernforschungsanlage Jülich, D-5170 Jülich, West Germany

This is a tutorial review on the Potts model aimed at bringing out in an organized fashion the essential and important properties of the standard Potts model. Emphasis is placed on exact and rigorous results, but other aspects of the problem are also described to achieve a unified perspective. Topics reviewed include the mean-field theory, duality relations, series expansions, critical properties, experimental realizations, and the relationship of the Potts model with other lattice-statistical problems.

Solution of the Kondo problem

Review of Modern Physics, 55, 331, 1983

N. Andrei

Department of Physics and Astronomy, Rutgers University, New Brunswick, NJ 08903

K. Furuya and J. H. Lowenstein

Department of Physics, New York University, New York, NY 10003

This review covers in great detail the Bethe-ansatz approach to the solution of various versions of the Kondo problem.

Importance of being integrable

REVIEWS OF MODERN PHYSICS, VOLUME 76, JULY 2004

***Colloquium*: Exactly solvable Richardson-Gaudin models for many-body quantum systems**

J. Dukelsky*

Instituto de Estructura de la Materia, CSIC, Madrid, Spain

REVIEWS OF MODERN PHYSICS, VOLUME 83, OCTOBER–DECEMBER 2011

One dimensional bosons: From condensed matter systems to ultracold gases

M. A. Cazalilla

Centro de Fisica de Materiales, Paseo Manuel de Lardizabal, 5.
E-20018 San Sebastian, Spain
and Donostia International Physics Center, Paseo Manuel de Lardizabal, 4.

REVIEWS OF MODERN PHYSICS, VOLUME 85, OCTOBER–DECEMBER 2013

Fermi gases in one dimension: From Bethe ansatz to experiments

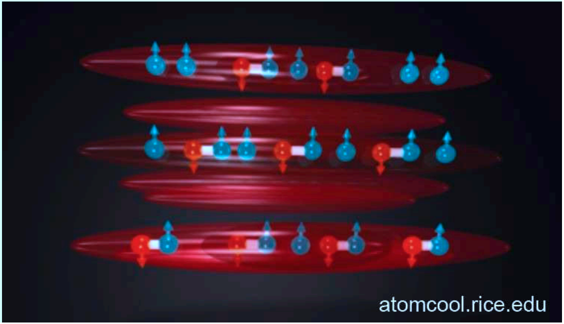
Xi-Wen Guan*

State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics,
Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences,
Wuhan 430071, China
and Department of Theoretical Physics, Research School of Physics and Engineering,

1D Bose gas, coupled tubes

1D Fermi gas, spin imbalance

Boson-Fermi mixtures..

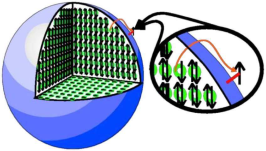


- Fractional excitations (anyons, spinons, magnons ...)
- Quasi-long range order (correlation functions)
- Quantum liquids (Luttinger liquid, spin charge separation)
- Quantum criticality (scaling laws, field theory)
- Quantum dynamics, transport properties
- Conformal Field Theory, gauge field, Yang-Mills theory
- Quantum metrology . . .

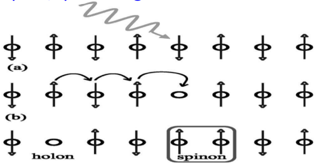
Cazalilla, Citro, Giamarchi, Orignac, & Rigol, *Rev. Mod. Phys.* **83**, 1405 (2011)
Guan, Batchelor, & Lee, Rev. Mod. Phys. **85**, 1633 (2013)

Fermi Liquids vs **Luttinger liquid**

Electronic metal, Kondo impurities, Helium-3, heavy fermions etc.



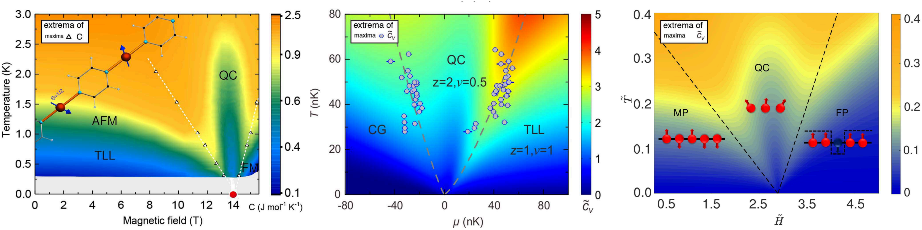
1D correlated electronic systems, spin liquids, quantum gases etc.



Microscopic difference: **quasiparticles** **spin-charge separation**

Macroscopic similarity: **specific heat linearly depends on temperature *T***
susceptibility and compressibility are independent of *T*

Quantum criticality of the energy fluctuation: specific heat



(a) (b) (c)

- (a) quantum criticality of spins
- (b) quantum criticality of spinless bosons
- (c) Spin and charge separation with quantum criticality

Yang, Chen, ... Guan, Yuan, Pan, Phys. Rev. Lett. 119, 165701 (2017)
He, Jiang, Yu, Lin, Guan, Phys. Rev. B 96, 220401(R) (2017)
Breunig, et al., Sci. Adv. 2017; 3:eaao3773

Outline

- Lecture I. Quantum liquid and quantum criticality in one dimension
  - Lieb-Liniger Bose gas
  - Spin-1/2 Heisenberg chain
- Lecture II. Spin charge separation in one dimension
  - Two component Fermi gas
  - Two component Bose gas

I. Quantum liquid and quantum criticality in one dimension

- E H Lieb & W Liniger 1963: δ-function Bose gas
- Continuum field theory problem of bosons with δ-function interaction

$$H = \int_0^L dx \left[\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)\right]$$

$$\left[\Psi(x,t), \Psi^\dagger(y,t)\right] = \delta(x-y), \quad [\Psi(x,t), \Psi(y,t)] = \left[\Psi^\dagger(x,t), \Psi^\dagger(y,t)\right] = 0$$

- N-particle eigenstate

$$|\Phi >= \int_0^L dx^N \chi(x_1, \cdots, x_N) \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N)|0>$$

- wave function

$$\chi = \sum_{\mathcal{P}} A(\mathcal{P}) \mathrm{e}^{\mathrm{i}(k_{\mathcal{P}_1} x_1 + \ldots + k_{\mathcal{P}_N} x_N)}$$

- Lieb-Liniger model

$$H = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j}^N \delta(x_i - x_j)$$

- two-particles

$$\chi(x_1,x_2) = \theta(x_2-x_1)\left[A_{P_1P_2}e^{\mathrm{i}(k_1x_1+k_2x_2)} + A_{P_2P_1}e^{\mathrm{i}(k_2x_1+k_1x_2)}\right] + \theta(x_1-x_2)\left[A_{P_1P_2}e^{\mathrm{i}(k_1x_2+k_2x_1)} + A_{P_2P_1}e^{\mathrm{i}(k_2x_2+k_1x_1)}\right]$$

- discontinuity of the derivative

$$X = \frac{1}{2}(x_1+x_2),\ Y = x_2 - x_1$$

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial X^2} - 2\frac{\partial^2}{\partial Y^2}\right]\chi + 2c\delta(Y)\chi = E\chi$$

$$\left.\frac{\partial\chi}{\partial Y}\right|_{Y=0^+} - \left.\frac{\partial\chi}{\partial Y}\right|_{Y=0^-} = c\chi|_{Y=0}$$

- two-body scattering relation

$$\frac{A_{P_2P_1}}{A_{P_1P_2}} = -\frac{c-\mathrm{i}(k_2-k_1)}{c+\mathrm{i}(k_2-k_1)} = -e^{\mathrm{i}\theta(k_2-k_1)} = Y_{12}(k_2-k_1)$$

$$\theta(x) = -2tan^{-1}\frac{x}{c},\ -\pi < \theta(x) < \pi,\ A_{P_2P_1}(k_2,k_1) = Y_{12}(k_2-k_1)A_{P_1P_2}(k_1,k_2)$$

periodic boundary conditions for three particles

$x_1 < x_2 < x_3$

$$\begin{aligned}
\chi(x_1, x_2, x_3) \quad = \quad & A_{123} e^{\mathrm{i}(k_1x_1+k_2x_2+k_3x_3)} + A_{213} e^{\mathrm{i}(k_2x_1+k_1x_2+k_3x_3)} \\
& + A_{132} e^{\mathrm{i}(k_1x_1+k_3x_2+k_2x_3)} + A_{312} e^{\mathrm{i}(k_3x_1+k_1x_2+k_2x_3)} \\
& + A_{231} e^{\mathrm{i}(k_2x_1+k_3x_2+k_3x_3)} + A_{321} e^{\mathrm{i}(k_3x_1+k_2x_2+k_1x_3)}
\end{aligned}$$

$x_2 < x_3 < x_1$

$$\begin{aligned}
\chi(x_2, x_3, x_1) \quad = \quad & A_{123} e^{\mathrm{i}(k_1x_2+k_2x_3+k_3x_1)} + A_{213} e^{\mathrm{i}(k_2x_2+k_1x_3+k_3x_1)} \\
& A_{132} e^{\mathrm{i}(k_1x_2+k_3x_3+k_2x_1)} + A_{312} e^{\mathrm{i}(k_3x_2+k_1x_3+k_2x_1)} \\
& A_{231} e^{\mathrm{i}(k_2x_2+k_3x_3+k_1x_1)} + A_{321} e^{\mathrm{i}(k_3x_2+k_2x_3+k_1x_1)}
\end{aligned}$$

$$\mathrm{Yang-Baxter\,equation:} \qquad \begin{aligned} & \chi(0, x_2, x_3) = \chi(x_2, x_3, L) \\ & A_{123} = A_{231} e^{ik_1L}, \; A_{213} = A_{132} e^{ik_2L}, \; A_{312} = A_{123} e^{ik_3L} \end{aligned}$$

- Many-particle wave function($x_1 < x_2 < \cdots < x_N$)

$$\chi = \sum_P A_{P_1,\ldots,P_N} \exp \mathrm{i}(k_{P_1} x_1 + \ldots + k_{P_N} x_N)$$

- Periodic conditions

$$\chi(x_1, x_2, \ldots, x_N) = \chi(x_2, \ldots, x_N, x_1)$$

$$\sum_P A_{P_1,P_2,\cdots,P_N} e^{\mathrm{i}(k_{P_1} x_1 + k_{P_2} x_2 + \cdots k_{P_N} x_N)} = \sum_{P'} A_{P_2,\cdots,P_N,P_1} e^{\mathrm{i}(k_{P_2} x_2 + \cdots k_{P_N} x_N + k_{P_1} x_1)}$$

$$P = \begin{pmatrix} k_1 & k_2 & \cdots & k_N \\ P_1 & P_2 & \cdots & P_N \end{pmatrix}, \quad P' = \begin{pmatrix} k_1 & k_2 & \cdots & k_{N-1} & k_N \\ P_2 & P_3 & \cdots & P_N & P_1 \end{pmatrix}$$

$$A_P = A_{P'} e^{\mathrm{i} k_{P1} L}, \;\; A_P = \left[-e^{\mathrm{i}\theta(k_{P_1} - k_{P_2})} \right] A_{P_2 P_1 \ldots P_N} = (-1)^{N-1} e^{\mathrm{i} \sum_j \theta(k_{P1} - k_j)} A_{P'}$$

$$e^{\mathrm{i} k_{P1} L} = (-1)^{N-1} e^{\mathrm{i} \sum_j \theta(k_{P1} - k_j)}$$

- Lieb-Liniger equation

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^{N} k_j^2, \;\; \exp(\mathrm{i} k_j L) = -\prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}\, c}{k_j - k_\ell - \mathrm{i}\, c}$$

- Wave function

$$\psi(x_1, x_2, \ldots, x_N) = \sum_p (-1)^p \Big[\prod_{1 \le i < j \le N} \Big(1 + \frac{\mathrm{i}k_{pj} - \mathrm{i}k_{pi}}{c} \Big) \Big] \exp\Big(\sum_{j=1}^{N} \mathrm{i}k_{pj} x_j \Big)$$

In general, $N!$ plane waves are N-fold products of individual exponential phase factors $e^{\mathrm{i}k_i x_j}$, where the N distinct wave numbers, k_i, are permuted among the N distinct coordinates, x_j. Each of the $N!$ plane waves have an amplitude coefficient in each of regions.

- repulsive interaction

$$K_i L = 2\pi I_i + \sum_{\ell=1}^{N} \theta(k_i - k_\ell), \quad I = -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots \frac{N-1}{2}$$

- thermodynamic limit

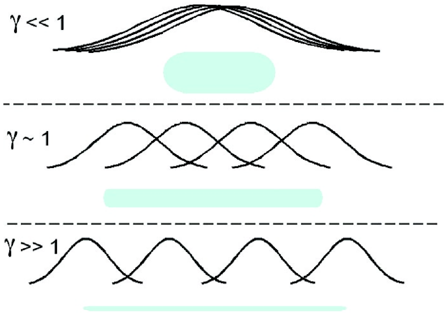
$$\begin{aligned} \frac{2\pi dI(k)}{Ldk} &= 1 + \frac{1}{L}\sum_{\ell=1}^{N} \frac{2c}{c^2 + (k-k_\ell)^2} \\ \rho(k) &= \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-Q}^{Q} dq \, \frac{2c\rho(q)}{c^2 + (k-q)^2} \end{aligned}$$

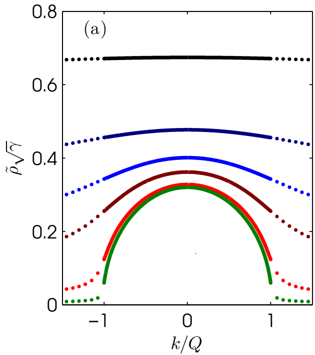
- attractive interaction

$$k_{\pm j} \approx \pm \mathrm{i}\frac{c}{2}[(N - 2j + 1) + \delta_j], \; j = 1, 2, \ldots, N/2$$

$$\Psi(x_1, \ldots, x_N) \approx \left(\sqrt{(N-1)!}/\sqrt{2\pi}\right) |c|^{(N-1)/2} \exp\left\{ \frac{c}{2} \sum_{1 \le i < j \le N} |x_i - x_j| \right\}$$

$$E = -\frac{1}{12} c^2 N(N^2 - 1)$$





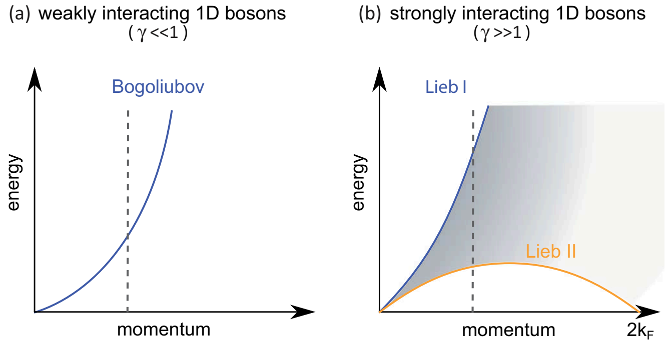
Hermite polynomials

$$H''(q) - 2qH' + 2NH(q) = 0$$

semicircle-law

$$\rho(\mathbf{k}) = \frac{\mathbf{1}}{\pi\sqrt{\gamma}}\sqrt{\mathbf{1} - \frac{\mathbf{k}^2}{\mathbf{4}|\gamma|\mathbf{n}^2}}$$

Gaudin 1971; Batchelor, Guan & McGuire 2004
Guan, Int. J. Mod. Phys. B 2014
Jiang, Chen, Guan, Chinese Physics B, 24, 5 (2015)



particle-hole exitations

$$\Delta E(k_{\rm e}) = \varepsilon(k_{\rm e}) = k_{\rm e}^2 - \mu + \int_{-Q}^{Q} a(k_{\rm e} - k')\varepsilon(k'), \quad \Delta E(k_{\rm h}) = -\varepsilon(k_{\rm h})$$

$\gamma \ll 1$, Bogoliubov theory
$\gamma \gg 1$ create a Fermi surface with $k_F = \pi\rho$

thermodynamic limit

$$\rho(k) = \frac{1}{2\pi} + \int_{-Q}^{Q} a(k-q)\rho(q)\mathrm{d}q, \ \ \frac{E}{L} = \int_{-Q}^{Q} \rho(k)k^2\mathrm{d}k$$

strong coupling

$$E_\infty \equiv \lim_{N,L\to\infty} \frac{E}{L} \approx \frac{1}{3}n^3\pi^2 \left(1 - \frac{4}{\gamma} + \frac{12}{\gamma^2} + \frac{\left(\frac{32}{15}\pi^2 - 32\right)}{\gamma^3}\right)$$

quasiparticle dispersion

$$\epsilon_P = v_s P + \frac{P^2}{2m^*} + O(P^3); \ \ \frac{m}{m^*} = (1 - \gamma\partial_\gamma)\frac{1}{\sqrt{K}}$$

Bogoliubov dispersion

$$\epsilon_P = v_s P\sqrt{1 + \frac{P^2}{4m^2v^2}}, \ \ \epsilon_P = v_s P, \ \ P \to 0$$

high momenta

$$\epsilon_P = \frac{P^2}{2m} + 2\gamma\frac{\hbar^2 n^2}{m} - \frac{\pi^2\hbar^2 n^2}{2m} + O(1/P^2)$$

finite-size correction

$$E(N,L) = LE_\infty - \frac{\pi C v_s}{6L} + O\left(\frac{1}{L^2}\right)$$

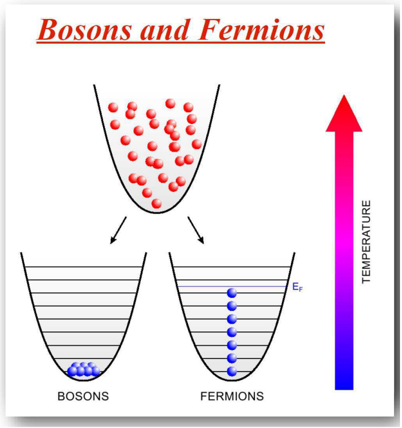
Ristivojevic, PRL 113, 015301 (2014)
Gangardt, Shlyapnikov, NJP. 5, 79 (2003)

quasiparticle dispersion

$$\begin{aligned}
\epsilon_{\mathbf{P}}^{\pm} &= v_s|P| \pm \frac{P^2}{2m^*} + \frac{\lambda}{6}|P|^3 +; \pm \frac{\nu}{24}P^4 \ldots \\
\frac{m}{m^*} &= (1-\gamma\partial_\gamma)\frac{1}{\sqrt{K}} \\
\lambda &= ?? \\
\nu &= ??
\end{aligned}$$

"The spectrum at arbitrary momentum is fully determined by the properties of the ground state".

Ristivojevic, et al.



- Maxwell-Boltzmann distribution:
  $< n_j > \propto 1/\left(\exp\left(\left(\epsilon_j - \mu\right)/T\right)\right)$
- Bose-Einstein distribution:
  $< n_j >= \left(\exp\left[(\epsilon_j - \mu)/T\right] - 1\right)^{-1}$
- Fermi-Dirac distribution:
  $< n_j >= \left(\exp\left[(\epsilon_j - \mu)/T\right] + 1\right)^{-1}$

Quantum statistics:

1. quantum man-body systems
2. microscopic state energy E_i
3. partition function $Z = \sum_{i=1}^{\infty} W_i e^{-E_i/(k_B T)}$
4. free energy $F = -k_B T \ln Z$

- Bethe ansatz equations at finite temperatures

$$\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \frac{c}{\pi}\int \frac{\rho(k')dk'}{c^2 + (k-k')^2}$$

- entropy

$$\begin{aligned} W &= \frac{(L(\rho+\rho^h)dk)!}{(L\rho dk)!(L\rho^h dk)!}, \ S = \int dS = \int \ln dW \\ s &= \frac{S}{L} = \int \left[(\rho+\rho^h)\ln(\rho+\rho^h) - \rho\ln\rho - \rho^h\ln\rho^h\right] dk \\ &= \int \left[(\rho+\rho^h)\ln\left(1+\frac{\rho}{\rho^h}\right) - \rho\ln\left(\frac{\rho}{\rho^h}\right)\right] dk \end{aligned}$$

- partition function

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr}(e^{-H/T}) = \sum_{\rho,\rho^h} W(\rho,\rho^h) e^{-E(\rho,\rho^h)/T} \\ \mathcal{Z} &= \sum_{\rho,\rho^h} e^{-(E(\rho,\rho^h) - S(\rho,\rho^h)T)/T} \end{aligned}$$

- Gibbs ensemble

$$0 = \frac{\delta G}{L} = \frac{\delta E}{L} - \mu\delta n - T\delta s, \ \frac{E}{L} = \int k^2\rho(k)dk, \ n = \int \rho(k)dk$$

$$\delta s = \int (\delta\rho + \delta\rho^h)\ln\left(1+\frac{\rho}{\rho^h}\right) - \delta\rho\ln\left(\frac{\rho}{\rho^h}\right) dk$$

$$\delta\rho(k) + \delta\rho^h(k) = \frac{c}{\pi}\int \frac{\delta\rho(k')dk'}{c^2+(k-k')^2}$$

- Yang-Yang method: dressed energy $\epsilon(k) = T\ln\frac{\rho_h(k)}{\rho(k)}$

$$\begin{aligned} 0 &= \frac{\delta G}{L} \\ &= \int\left[k^2\delta\rho - \mu\delta\rho + T\delta\rho\ln\left(\frac{\rho}{\rho^h}\right) - \frac{c}{\pi}\int\frac{\delta\rho dq}{c^2+(k-q)^2}\ln\left(1+\frac{\rho}{\rho^h}\right)\right]dk \\ &= \int\left[k^2 - \mu + T\ln\left(\frac{\rho}{\rho^h}\right) - \frac{Tc}{\pi}\int\frac{dq}{c^2+(k-q)^2}\ln\left(1+\frac{\rho}{\rho^h}\right)\right]\delta\rho dk \end{aligned}$$

- Yang-Yang equation

$$\varepsilon(k) = k^2 - \mu - \frac{Tc}{\pi}\int\frac{dq}{c^2+(k-q)^2}\ln\left(1+e^{-\varepsilon(q)/T}\right)$$

- pressure

$$\begin{aligned}
p &= \int_{-\infty}^{\infty} \left\{ \left(\mu - k^2\right) \rho(k) + T \left[\left(\rho(k) + \rho^h(k)\right) \right.\right. \\
&\quad \left.\left. \times \ln\left(\rho(k) + \rho^h(k)\right) - \rho(k) \ln \rho(k) - \rho^h(k) \ln \rho^h(k) \right] \right\} dk \\
&= \int_{-\infty}^{\infty} \left\{ \left[\left(\mu - k^2\right) - \ln \frac{\rho(k)}{\rho^h(k)} + T \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right] \rho(k) \right. \\
&\quad \left. + T\rho^h(k) \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right\} dk \\
&= \int_{-\infty}^{\infty} \left\{ T\rho(k) \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right. \\
&\quad \left. - \frac{Tc}{\pi} \int \frac{dq}{c^2 + (k-q)^2} \ln\left(1 + \frac{\rho(q)}{\rho^h(q)}\right) + T\rho^h(k) \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right\} dk \\
&= T \int_{-\infty}^{\infty} \left\{ \left[\rho(k) + \rho^h(k) - \frac{Tc}{\pi} \int \frac{\rho(q)dq}{c^2 + (k-q)^2} \right] \right. \\
&\quad \left. \times \ln\left(1 + \frac{\rho(k)}{\rho^h(k)}\right) \right\} dk \\
&= \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln\left(1 + e^{-\varepsilon(k)/T}\right) dk
\end{aligned}$$

E H Lieb & W Liniger δ-function Bose gas: simpler is better

- Wave function

$$\psi(x_1, x_2, \ldots, x_N) = \sum_P (-1)^P \Big[\prod_{1 \le i < j \le N} \Big(1 + \frac{\mathrm{i}k_{Pj} - \mathrm{i}k_{Pi}}{c} \Big) \Big] \exp \Big(\sum_{j=1}^{N} \mathrm{i}k_{Pj} x_j \Big)$$

- Lieb-Liniger equations

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^{N} k_j^2, \quad \exp(\mathrm{i}k_j L) = - \prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}\, c}{k_j - k_\ell - \mathrm{i}\, c}$$

- Yang-Yang thermodynamics

$$\varepsilon(k) = k^2 - \mu + \int_{-\infty}^{\infty} a(k - k')\varepsilon_-(k')\mathrm{d}k', \qquad p = -\frac{1}{2\pi} \int \varepsilon_-(k)\mathrm{d}k$$

$$\varepsilon_-(k) = -T \ln[1 + \mathrm{e}^{-\varepsilon(k)/T}], \qquad a(x) = \frac{1}{2\pi} \frac{2c}{c^2 + x^2}$$

- fundamental concepts:
  cooperative and collective behaviour, continuum excitation, quasi-long range correlations, quantum liquid, quantum criticality, quantum dynamics, thermodynamics in and out of equilibrium . . .

- Wave function

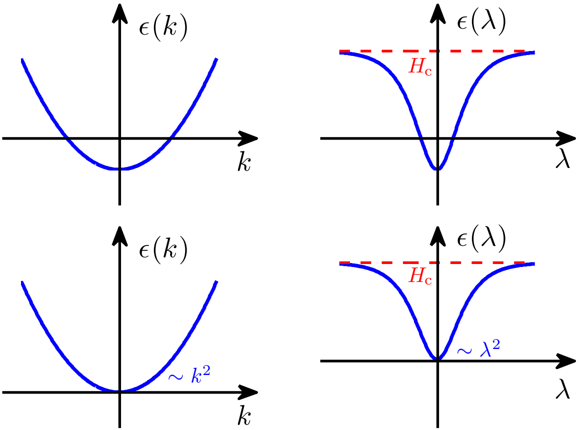
$$\psi_{\{k_i\}}(x_1, x_2, \ldots, x_N) = \sum_p A(k_{p_1}, \ldots, k_{p_N}) e^{i\sum_j k_{p_j} x_j}$$

- M-body local correlation function

$$\begin{aligned} g_M &= \frac{\langle\Omega|(\Psi^\dagger(0))^M(\Psi(0))^M|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \langle(\Psi^\dagger(0))^M(\Psi(0))^M\rangle \\ &= \frac{N!}{(N-M)!}\frac{\int|\psi(0,\cdots,0,x_{M+1},\cdots,x_N)|^2 dx_{M+1}\cdots dx_N}{\int|\psi(x_1,\cdots,x_N)|^2 dx_1\cdots dx_N} \end{aligned}$$

- Non-local 2M-point correlation function and density-density correlation

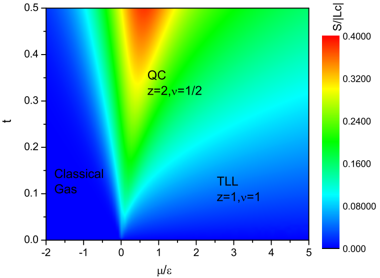
$$\begin{aligned} g_M &= \frac{\langle\Omega|\Psi^\dagger(x_1')\cdots\Psi^\dagger(x_M')\Psi(x_1)\cdots\Psi(x_M)|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \cdots \\ \langle\Psi^\dagger(x)\Psi(0)\rangle &= \rho_0\left[\frac{1}{\rho_0 d(x|L)}\right]^{\frac{1}{2K}}\{b_0+\ldots\} \\ \langle\rho(x)\rho(0)\rangle &= \rho_0^2\left\{1-\frac{K}{2\pi^2}\left[\frac{1}{\rho_0 d(x|L)}\right]^2+\sum_{m=1}^{\infty}a_m\left[\frac{1}{\rho_0 d(x|L)}\right]^{2m^2K}\cos(2\pi m\rho_0 x)\right\} \end{aligned}$$



- Quantum Liquid

- Quantum criticality

Guan, Batchelor and Lee, Rev. Mod. Phys. 85, 1633 (2013)



Boltzmann Statistics	$p_c = \sqrt{\frac{m}{2\pi\hbar^2}} T^{-\frac{3}{2}} \mathrm{e}^{\frac{\mu}{T}},\ p = p_0 + \frac{T^{\frac{3}{2}}}{\sqrt{2\pi}} \mathcal{Z}^2 p_2$	for $T \to \infty$
Fermi Statistics	$p = -\sqrt{\frac{m}{2\pi\hbar^2}} T^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(-\mathrm{e}^{\frac{\mu}{T}}),$	for $c \to \infty$
Bose Statistics	$p = \sqrt{\frac{m}{2\pi\hbar^2}} T^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(\mathrm{e}^{\frac{\mu}{T}}),$	for $c \to 0,$
Fractional Statistics	$(1+\omega_i) \prod_j \left(\frac{\omega_j}{1+\omega_j}\right)^{\alpha_{ji}} = e^{(\epsilon_i - \mu_i)/T}$	for $c,\ T \neq 0$

Haldane, Phys. Rev. Lett. (1991)
Wu, Phys. Rev. Lett. (1994)
Batchelor, Guan, Olkers, Rev. Rev. Lett. (2006)
Jiang, Chen, Guan, CPB (2015)

- Luttinger liquid: $T/(\frac{\hbar^2 n^2}{2m}) \ll 1$

$$F(T) \approx E_0 - \frac{\pi C(k_B T)^2}{6\hbar v_s}$$

- Sommerfeld Expansion

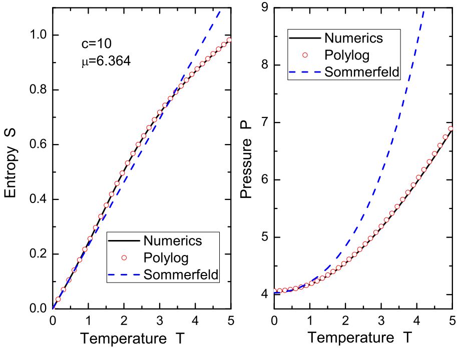
$$\begin{aligned}
p &= \frac{T}{2\pi}\left\{k \ln(1+e^{-\epsilon(k)/T})|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{(\frac{\hbar^2 k}{m} - A'(k,T))ke^{-\epsilon(k)/T}}{1+e^{\epsilon(k)/T}} \frac{1}{T}\right\} \\
&= \frac{1}{\sqrt{\frac{\hbar^2 k^2}{2m}}} \int_0^{\infty} \frac{\sqrt{\epsilon_O} d\epsilon_O}{1+e^{\frac{\epsilon_O - A}{k_B T}}} \\
A(k,T) &= \mu + \frac{2pc}{c^2+k^2} - \frac{4\mu^{5/2}}{15\pi c^3} \\
p &= \frac{2}{3\pi} A^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{T}{A}\right)^2 + \cdots\right]
\end{aligned}$$

- Luttinger liquid: $T/(\frac{\hbar^2 n^2}{2m}) \ll 1$

$$F(T) \approx E_0 - \frac{\pi C (k_B T)^2}{6\hbar v_c}$$

- Quantum criticality

$$\begin{aligned}
n &\approx -\frac{c\tilde{T}^{\frac{1}{2}}}{2\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(-e^{\tilde{A}_0/\tilde{T}})\left[1 - \frac{\tilde{T}^{\frac{1}{2}}}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(-e^{\tilde{A}_0/\tilde{T}}) + \frac{\tilde{T}}{\pi}\mathrm{Li}_{\frac{1}{2}}(-e^{\tilde{A}_0/\tilde{T}})^2 \right.\\
&\quad \left. - \frac{\tilde{T}^{\frac{3}{2}}}{\pi^{\frac{3}{2}}}\mathrm{Li}_{\frac{3}{2}}(-e^{\tilde{A}_0/\tilde{T}})^3 + \frac{3\tilde{T}^{\frac{3}{2}}}{2\sqrt{\pi}}\mathrm{Li}_{\frac{3}{2}}(-e^{\tilde{A}_0/\tilde{T}})\right]\\
\kappa &\approx -\frac{c}{2\varepsilon_0\sqrt{\pi}}\frac{1}{\sqrt{\tilde{T}}}\mathrm{Li}_{-\frac{1}{2}}(-e^{\tilde{A}_0/\tilde{T}})\left[1 - \frac{3\tilde{T}^{\frac{1}{2}}}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(-e^{\tilde{A}_0/\tilde{T}}) + \frac{3\tilde{T}}{\pi}\mathrm{Li}_{\frac{1}{2}}(-e^{\tilde{A}_0/\tilde{T}})\right]\\
\frac{c_{\mathrm{v}}}{T} &= -\frac{3}{8\sqrt{\pi}}T^{-\frac{1}{2}}f_{\frac{3}{2}} + \frac{1}{2\sqrt{\pi}}T^{-\frac{1}{2}}\frac{A}{T}f_{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}}T^{-\frac{1}{2}}\left(\frac{A}{T}\right)^2 f_{-\frac{1}{2}} + O\left(\frac{1}{c}\right)
\end{aligned}$$

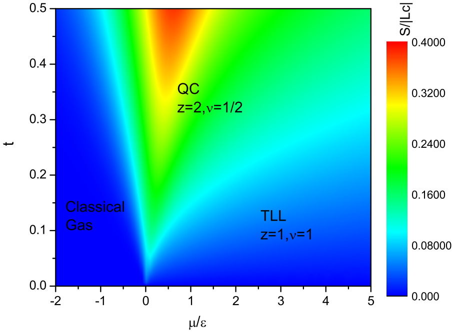


$$F(T) = F(0) - \frac{\pi C (K_B T)^2}{6\hbar v_s} + O(T^4)$$

$$E(L,N) = Le_\infty - \frac{\hbar \pi C v_s}{6L} + O(1/L^2)$$

$$v_s \approx \frac{\hbar \pi n}{m}\left(1 - \frac{8}{\gamma} + \frac{40}{\gamma^2} + \frac{160}{\gamma^3}\left(\frac{\pi^2}{15} - 1\right)\right)$$

Guan, Batchelor JPA 2011

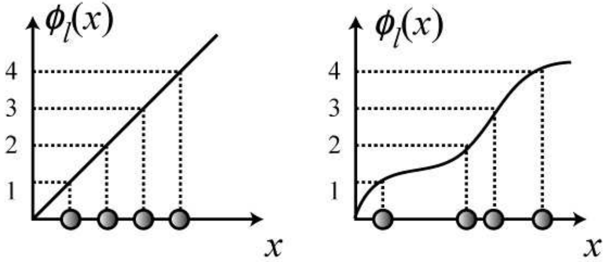


Luttinger liquid : $p(T,\mu) = p_0 + \frac{\pi T^2}{6 v_s}, \quad c_v = \frac{\pi T}{3 v_s}$

Quantumcriticality : $n(T,\mu) \approx n_0 + T^{d/z+1-1/\nu z} \mathcal{F}\left(\frac{\mu - \mu_c}{T^{1/\nu z}}\right), \quad \xi \sim |\mu - \mu_c|^{-\nu}, \quad \Delta \sim \xi^{-z}$

$$c_v/T = T^{d/z+1-2/\nu z} \mathcal{K}\left(\frac{\mu - \mu_c}{T^{1/\nu z}}\right), \quad z = 2,\ \nu = 1/2$$

M. P. A. Fisher, *et al.* PRB 40, 546 (1989)
Guan & Batchelor, J. Phys. A 2011
Jiang, Chen, Guan, Chin. Phys. B (2015)



- Idea of the Bosonization(replace the operators in terms of auxiliary function $\tilde{\phi}(x_i) = 2\pi n$)

$$\rho(x) = \sum_i \delta(x - x_i) \rightarrow \rho(x) = \sum_n \delta(\tilde{\phi}(x) - 2\pi n)$$

$$\rho(x) = \partial_x \tilde{\phi}_\ell(x) \sum_n \delta(\tilde{\phi}_\ell(x) - 2\pi n)$$

- slow variation of the field ϕ: $\tilde{\phi}_\ell(x) = 2\pi\rho_0 x - 2\phi(x)$

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right] \sum_p e^{ip(2\pi\rho_0 x - 2\phi(x))}$$

- Bose field: $\Psi(x) = [\rho(x)]^{1/2} e^{-\mathrm{i}\theta(x)}$, $\left[\Psi(x), \Psi^\dagger(x')\right] = \mathrm{i}\delta(x - x')$

$$\Pi = \frac{1}{\pi}\nabla\theta(x), \quad \left[\phi(x), \nabla\theta(x')\right] = \mathrm{i}\delta(x - x')$$

- Bosonization approach

$$\Psi^\dagger(x) = \sqrt{\rho(x)}e^{i\theta(x)}$$

- Effective Hamiltonian

$$H = \int dx \left(\frac{\pi v_s K}{2} \Pi^2 + \frac{v_s}{2\pi K} (\partial_x \phi)^2 \right) = \frac{\hbar v_s}{2} \int dx \left(\pi K \Pi^2 + \frac{1}{\pi K} (\partial_x \phi)^2 \right)$$

- Luttinger parameters for weak and strong coupling regimes

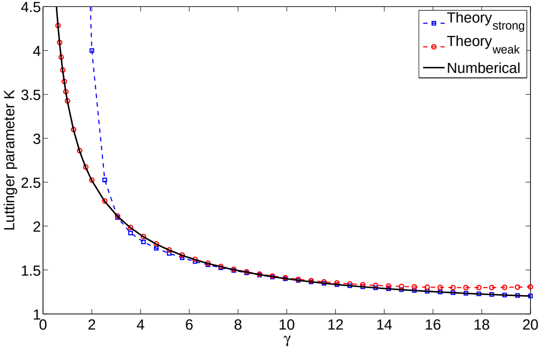
$$K = \frac{v_s}{v_N}, \quad v_N = \frac{L}{\pi\hbar} \left(\frac{\partial^2 E}{\partial N^2} \right)_{N=N_0} = \frac{1}{\pi\hbar} \left(\frac{\partial^2 \mu}{\partial n^2} \right)_{n=n_0}, \quad V_J = v_s K = \pi L \frac{\partial^2}{\partial \alpha} E$$

- sound velocity

$$v_s \approx \frac{\hbar n}{m}\sqrt{\gamma}\left(1 - \frac{\sqrt{\gamma}}{2\pi}\right)$$

$$v_s \approx \frac{\hbar \pi n}{m}\left(1 - \frac{8}{\gamma} + \frac{40}{\gamma^2} + \frac{160}{\gamma^3}\left(\frac{\pi^2}{15} - 1\right)\right)$$

Giamarchi, T. *Quantum Physics in one dimension* (Oxford University Press, Oxford, 2004)



Luttinger parameter

$$K = \frac{\pi}{\sqrt{3e - 2\gamma\frac{de}{d\gamma} + \frac{1}{2}\gamma^2\frac{d^2e}{d\gamma^2}}}$$

strong coupling

$$K = 1 + \frac{4}{\gamma} + \frac{4}{\gamma^2} - \frac{16\pi^2}{3\gamma^3} + \frac{32\pi^2}{3\gamma^4} + O\left(\gamma^{-5}\right)$$

weak coupling

$$K = \pi\left(\gamma - \frac{1}{2\pi}\gamma^{3/2}\right)^{-1/2}$$

$$\langle\Psi^\dagger(x)\Psi(0)\rangle = \rho_0\left[\frac{1}{\rho_0 d(x|L)}\right]^{\frac{1}{2K}}\left\{b_0+\sum_{m=1}^{\infty} b_m\left[\frac{1}{\rho_0 d(x|L)}\right]^{2m^2K}\cos\left(2\pi m\rho_0 x\right)\right\}$$

$$\langle\rho(x)\rho(0)\rangle = \rho_0^2\left\{1-\frac{K}{2\pi^2}\left[\frac{1}{\rho_0 d(x|L)}\right]^2+\sum_{m=1}^{\infty} a_m\left[\frac{1}{\rho_0 d(x|L)}\right]^{2m^2K}\cos\left(2\pi m\rho_0 x\right)\right\}$$

$$\langle\rho(x)\rho(0)\rangle_T = \rho_0^2-\frac{(TZ/v_F)^2}{2\sinh^2(\pi Tx/v_F)}+\sum_l A_l e^{2\mathrm{i}Zlk_F}\left(\frac{\pi T/v_F}{\sinh(\pi Tx/v_F)}\right)^{2l^2z^2}$$

Caux *et al.* PRA 2006, Panfil, *et al.* PRA (2014)
Cazalilla, J. Phys. B 37, S1 (2004)

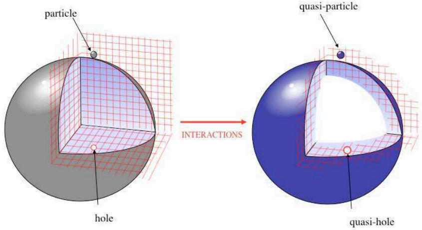
Wilson ratio

The Wilson ratios, defined as the ratios of the magnetic susceptibility/compressibility to specific heat divided by temperature are dimensionless constants at the renormalization fixed point of these systems. The values of the ratio indicate interaction effects and quantifies spin/particle number fluctuations.

$$G = E - N\mu - MH - TS$$

$$\langle \delta M^2 \rangle = \Delta^D k_B T \chi, \quad \langle \delta N^2 \rangle = \Delta^D k_B T \kappa$$

$$R_W^s = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi}{c_v/T}, \quad R_W^c = \frac{\pi^2 k_B^2}{3}\frac{\kappa}{c_v/T}$$



(a) Fermi Liquid (b) Landau Fermi Liquid

- effective Hamiltonian

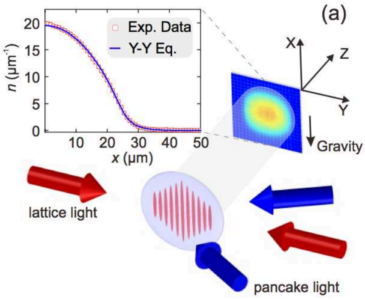
$$H = \int dx \left(\frac{\pi v_s K}{2} \Pi^2 + \frac{v_s}{2\pi K} (\partial_x \phi)^2 \right)$$

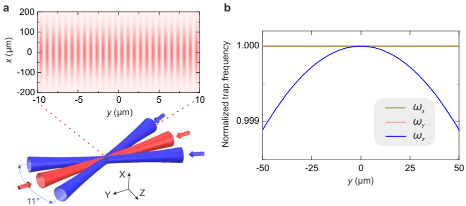
- Wilson ratio

$$R_W^s = \frac{4}{3} \left(\frac{\pi k_B}{\mu_B g} \right)^2 \frac{\kappa}{c_v / T}$$

- Luttinger liquid vs Fermi liquid

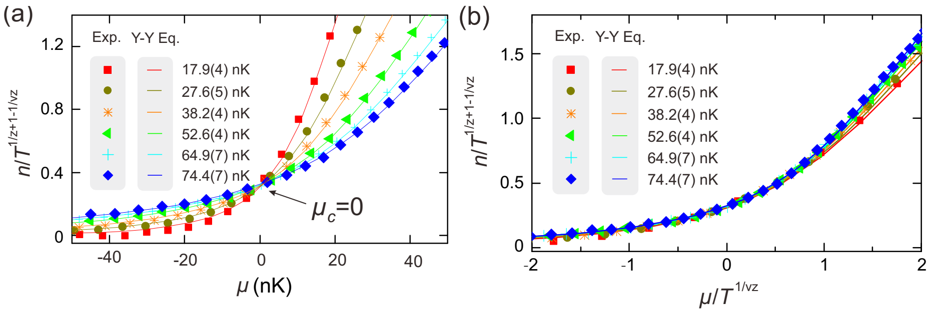
$$\kappa = \frac{1}{\hbar \pi v_N}, \qquad c_v = \frac{\pi k_B^2 T}{3} \frac{1}{\hbar v_s} \qquad \mathbf{R_w = K}$$





An array of 1D tubes is created by a blue-detuned pancake and a red-detuned lattices($\sim 4 \times 10^4$ 87Rb atoms): $\omega_x = 2\pi \times 22.2(1)$Hz; $\omega_\perp = \sqrt{\omega_y \omega_z} = 2\pi \times 7.99(1)$k Hz, $T = 18 - 74$nK

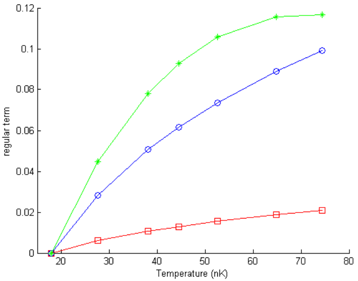
in collaboration with Zhen-Sheng Yuan's group at USTC, Phys. Rev. Lett. 119, 165701 (2017)



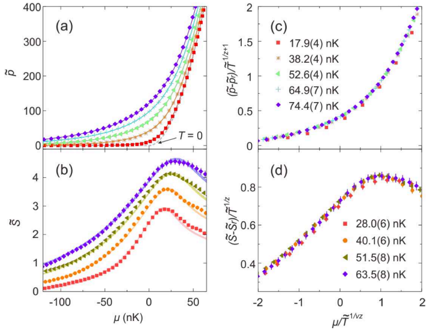
$$\tilde{n}(t,\tilde{\mu}) \approx \tilde{n}_0(\tilde{\mu},t) + t^{\frac{d}{z}+1-\frac{1}{\nu z}}\mathcal{F}\left(\frac{\tilde{\mu}-\tilde{\mu_c}}{t^{\frac{1}{\nu z}}}\right)$$

$$\xi \sim |\mu-\mu_c|^{-\nu}, \quad \Delta \sim \xi^{-z} \sim |\mu-\mu_c|^{z\nu}$$

- scaling functions read off $z = 2.3^{+0.6}_{-0.3}$ and $\nu = 0.56^{+0.07}_{-0.08}$
- equation of state, compressibility, specific heat, speed of sound
- Wilson ratio determines the Luttinger parameter



regular parts for $n_r(T)$, $p_r(T)$, $S_r(T)$

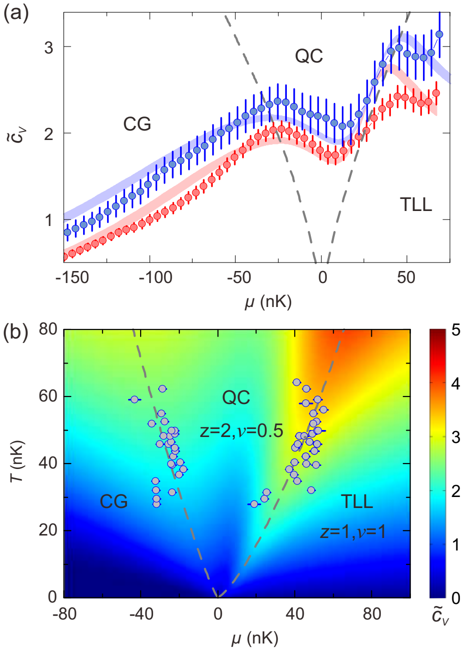


scaling: $\tilde{p} = p/[\hbar^2 c^3/(2m)]$, $\tilde{S} = S/(k_B c)$

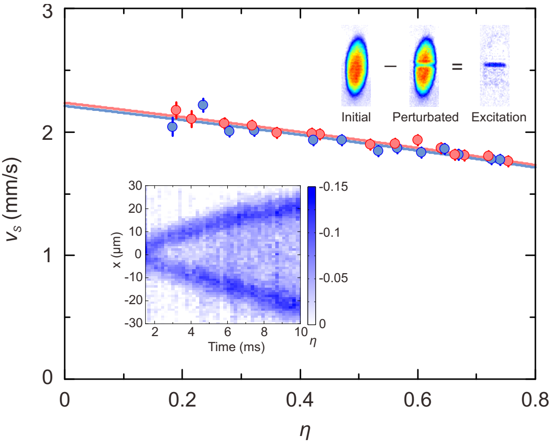
$$p(\mu, T) = p_r(\mu, T) + T^{\frac{1}{z}+1}\mathcal{G}\left(\frac{\mu-\mu_c}{T^{1/\nu z}}\right)$$

$$S(\mu, T) = S_r(\mu, T) + T^{\frac{1}{z}}\mathcal{H}\left(\frac{\mu-\mu_c}{T^{1/\nu z}}\right)$$

- scaling functions read off $z = 2.3^{+0.6}_{-0.3}$ and $\nu = 0.56^{+0.07}_{-0.08}$
- equation of state, compressibility, specific heat, speed of sound
- Wilson ratio determines the Luttinger parameter

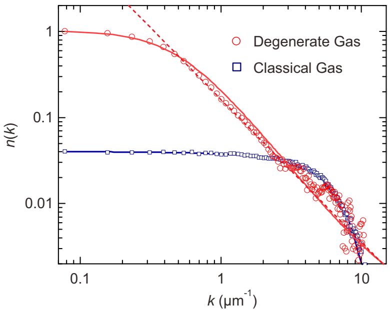


(a)
QC
CG
TLL
$\tilde{c}_V$
μ (nK)
(b)
QC
z=2,ν=0.5
CG
TLL
z=1,ν=1
T (nK)
μ (nK)
$\tilde{c}_V$



The central atoms can be transfer from $|F=1, m_F=-1\rangle$ to $|F=2, m_F=-2,-1,0\rangle$, then removed the atoms in $|F=2\rangle$ to create a density dip. We finally extrapolated $v_s(\eta\to 0)$ from the finite perturbation ratio η base on the relation $v_s(\eta)=v_s(0)\sqrt{1-\eta/2}$.

$$R_{\mathrm{W}}^{\kappa}=\frac{\pi^2 k_B^2}{3}\frac{\kappa}{c_V/T},\quad R_{\mathrm{W}}^{\kappa}=K=v_s/v_N=\hbar\pi n/(mv_s),\ \text{for Luttinger liquid}$$



After switching off the optical confinements; the cloud expands in a weak magnetic potential along y, the trapping frequency $\omega_y = 2\pi \times 10.0(2)$Hz. After a quarter period of oscillation, the momentum distribution is mapped to the spatial density profile $k = m\omega_y y/2\pi$ by focusing technique. $n(k)$ exhibits a power-law decay $n(k) \sim 1/k^{(1-1/2K)}$ at intermediate momenta ($1/l_\phi \le k \le 20/l_\phi$), where phase correlation length $l_\phi = \hbar v_s K/(\pi k_B T)$. Classical gas: for $T = 209(1)$nk:

Luttinger parameter: $K = 15.9$ for $T = 40(1)$nK;

$$n(k) \simeq A(K)\mathrm{Re}[\Gamma(1/4K + \mathrm{i}kl_\phi/2K)/\Gamma(1 - 1/4K + \mathrm{i}kl_\phi/2K)]$$

$$R_{\mathrm{W}}^{\kappa} = \frac{\pi^2 k_B^2}{3}\frac{\kappa}{c_V/T}, \quad R_{\mathrm{W}}^{\kappa} = K = v_s/v_N = \hbar\pi n/(mv_s)$$

M. A. Cazalilla, J. Phys. B 37, S1 (2004)
Yang, Chen, ... Guan, Yuan, Pan, Phys. Rev. Lett. 119, 165701 (2017)

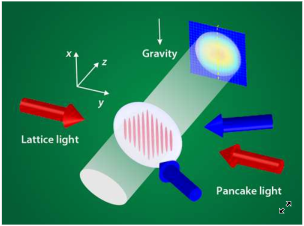
APS physics | Physics

Viewpoint: Theory for 1D Quantum Materials Tested with Cold Atoms and Superconductors

Thierry Giamarchi, Department of Quantum Matter Physics, University of Geneva, 24 Quai Ernest-Ansermet, CH-1211 Geneva 4, Switzerland

October 18, 2017 • *Physics* 10, 115

The Tomonaga-Luttinger theory describing one-dimensional materials has been tested with cold atoms and arrays of Josephson junctions.



Philip Krantz, Krantz NanoArt, adapted by APS/Alan Stonebraker

Figure 1: Sketch of the experimental setup used by Yang *et al.* Arrays of rubidium-87 atoms, cooled and trapped by laser beams, exhibit Tomonaga-Luttinger liquid (TLL) behavior.

"Without a doubt, this research will open new chapters in the TLL field by inspiring studies that examine how other perturbations (coupling between different 1D chains, spin-orbit coupling, and the like) can lead to novel and potentially exotic states in 1D materials".
–Viewpoint by Thierry Giamarchi, Physics, 10, 115 (2007)

Yang, Chen, ... Guan, Yuan, Pan Phys. Rev. Lett. 119, 165701 (2017)
Cedergren, Ackroyd, et al. Phys. Rev. Lett. 119, 167701 (2017)