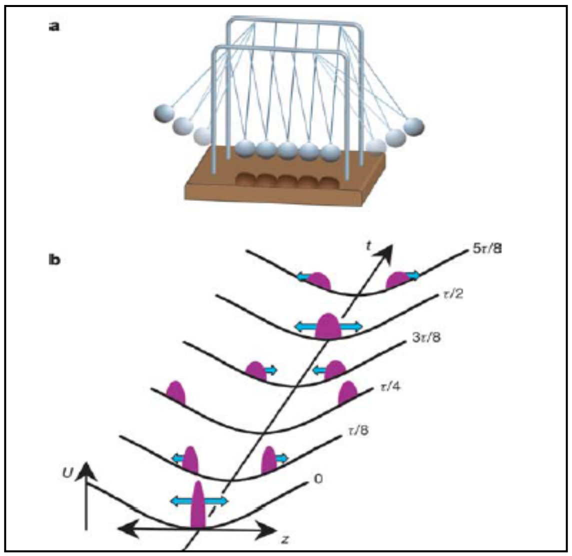
Nonequilibrium Dynamics in Few and Many Body Integrable Quantum Systems



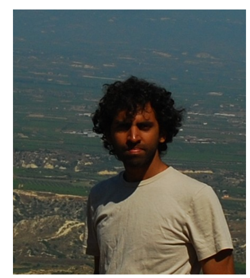
Kinoshita, Wenger, Weiss (Nature '06)

Natan Andrei

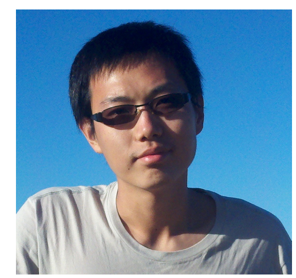
RUTGERS

Huijie Guan

Garry Goldstein



Deepak Iyer



Wenshuo Liu

Quantum Few to Many Body Physics – Wuhan 2018

Time Evolution of systems out of equilibrium

- **Prepare an isolated quantum many-body system in state $|\Phi_0\rangle$,** typically eigenstate of H_0
- **At , $t = 0$ evolve system with $H(t)$:**

If the Hamiltonian is time independent,

$$|\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle$$

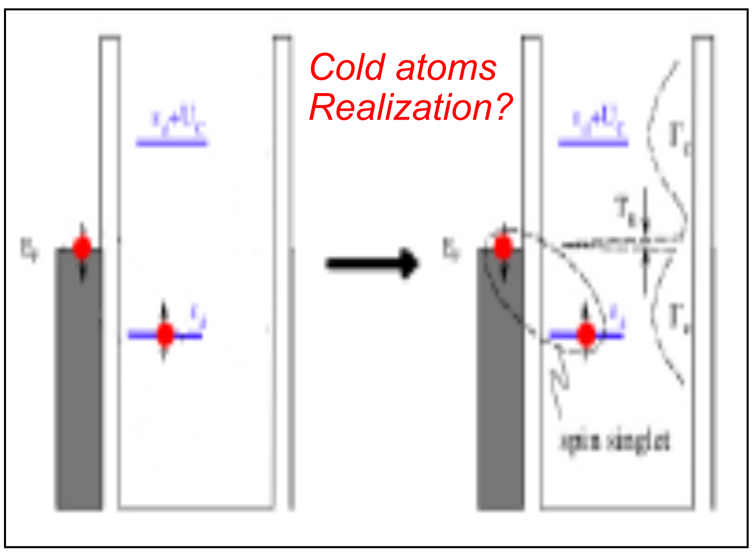
If the Hamiltonian is time dependent,

$$|\Phi_0, t\rangle = Te^{-i\int_0^t H(t')dt'}|\Phi_0\rangle$$

- **Many experiments: cold atom systems, nano-devices, molecular electronics, photonics :** *new systems, old questions*

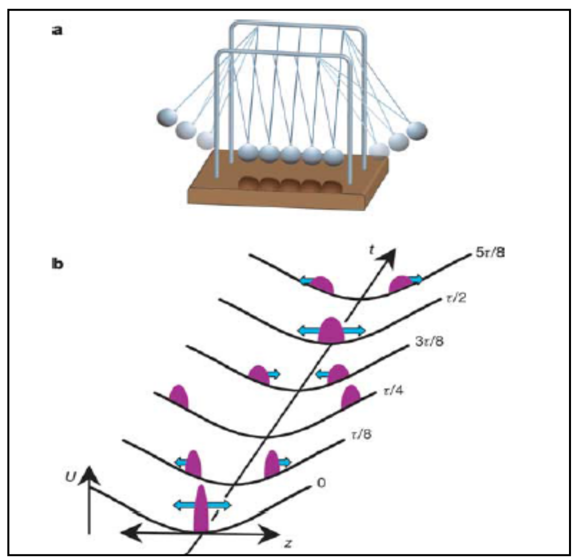
Time evolution of observables: $\langle O(t)\rangle = \langle \Phi_0, t|O|\Phi_0, t\rangle$

- **Manifestation of interactions**



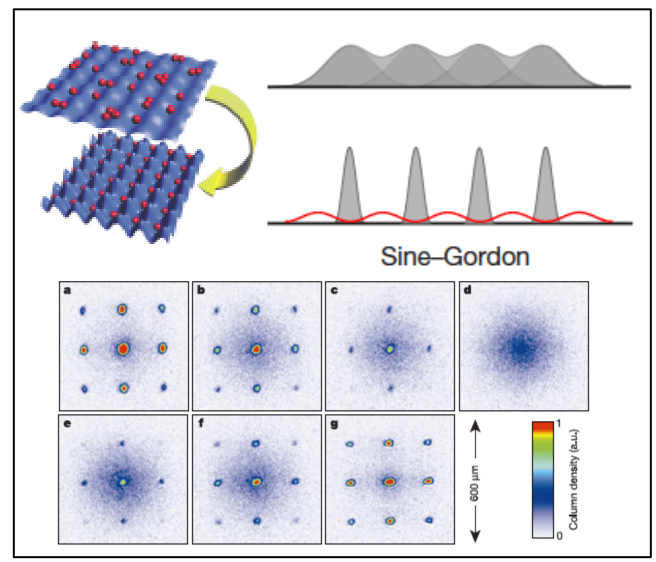
Time evolution of the Kondo peak.
- Time resolved photo emission spectroscopy

Kinoshita et al '06



Newton's Cradle

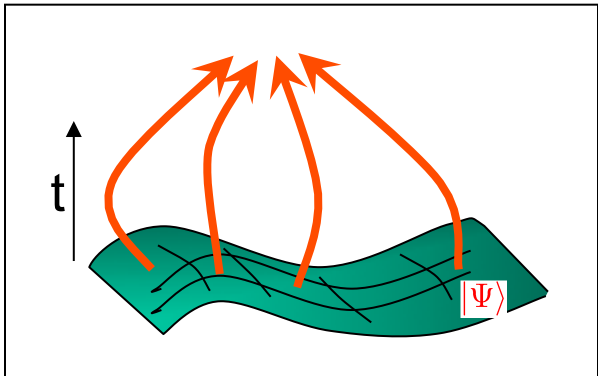
Greiner et al '02 *Bloch et al '08*



Mott insulator ←→ superfluid :2d, 1d

Quenching – long time limit, thermalization

Time evolution and statistical mechanics: equilibrium



$$\langle A(t)\rangle = \langle \Phi_0|e^{iHt} A e^{-iHt}|\Phi_0\rangle \underset{t\to\infty}{\longrightarrow} \bar{A}_{\Phi_0}$$

- **Long time limit and thermalization:**

- is there a density operator ρ such that $\bar{A} = Tr(\rho A)$?
- does it depend only on $E_0 = \langle \Phi_0|H|\Phi_0\rangle$, not on $|\Phi_0\rangle$?

$$\langle A(t)\rangle = \sum_{\alpha,\beta} \langle \Phi_0|\alpha\rangle A_{\alpha\beta} \langle \beta|\Phi_0\rangle e^{i(E_\alpha - E_\beta)t} \underset{t\to\infty}{\rightarrow} \sum_\alpha |\langle \Phi_0|\alpha\rangle|^2 A_{\alpha\alpha}$$

$$\overset{?}{=} \langle A\rangle_{\text{microcan}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{\alpha\in\Delta E} A_{\alpha\alpha}$$

- **Gibbs Ensemble**

- Non integrable models: ETH $A_{\alpha\alpha} = \langle \alpha|A|\alpha\rangle = f_A(E_\alpha)$, with f smooth function of E_α

Deutsch '92, Srednicki '94

→ Gibbs ensemble: $\langle A(t\to\infty)\rangle = \frac{1}{Z} Tr(e^{-\beta H} A)$, with β det. by $\langle H\rangle_{t=0} = \frac{1}{Z} Tr(e^{-\beta H} H)$

- **Generalized Gibbs ensemble (GGE)**

- Integrable models: local conserved charges, I_n, GETH $\langle \alpha|A|\alpha\rangle = f(I_{1,\alpha}, I_{2,\alpha}, \dots)$

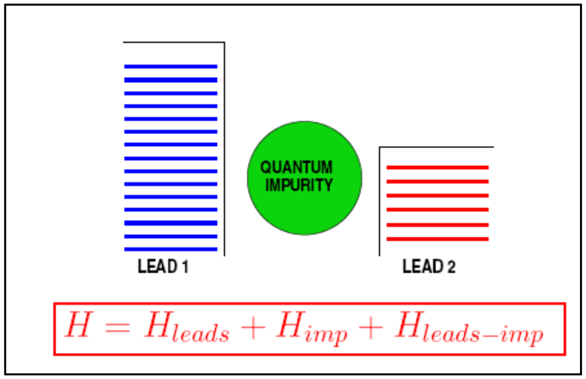
$\langle A(t\to\infty)\rangle = Tr(\hat{\rho} A)$ $\quad \hat{\rho} = Z^{-1}\exp(-\sum_n \beta_n I_n)$ *with* $\langle I_n\rangle_{t=0} = Tr(I_n \hat{\rho})$ *Rigol et al*

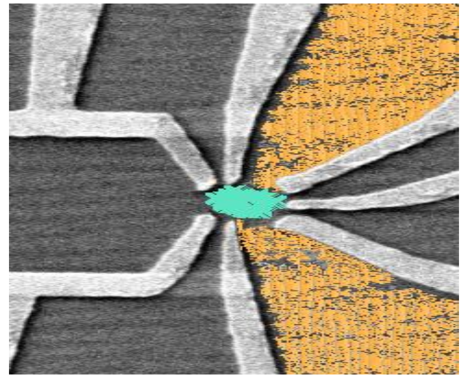
Quenching and non-thermalization (no MBL)

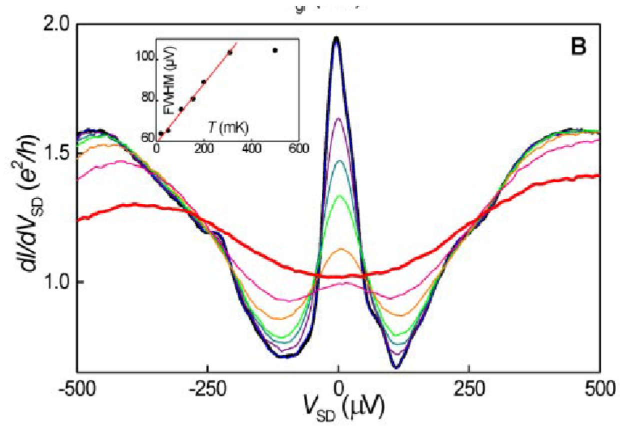
Nonequilibrium currents

Goldhaber-Gordon *et al*, Cronenwett *et al*, Schmid *et al*

- **Two baths or more:**
  - time evolution in a nonequilibrium set up:

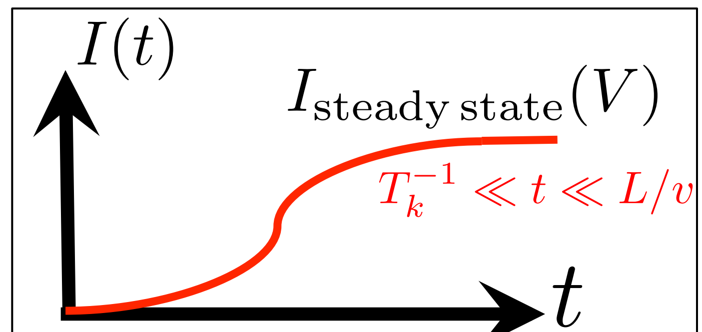






- $t \leq 0$, leads decoupled, system described by: ρ_o
- $t = 0$, couple leads to impurity
- $t \geq 0$, evolve with $H(t) = H_0 + H_1$

Interplay - strong correlations and nonequilibrium

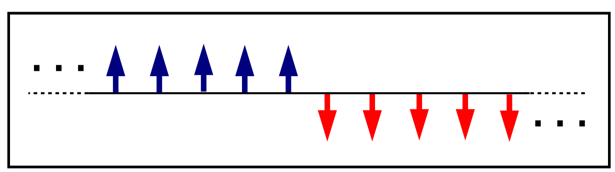


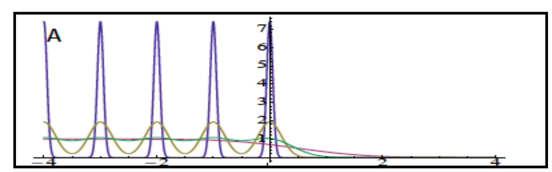
- **What is the time evolution of the current** $\langle I(t) \rangle$ **?**

- **Long time limit: Under what conditions is there a nonequilibrium steady state (NESS)? Dissipation mechanism?**

- **Steady state – is there a non-thermal** ρ_{NESS} **? Voltage dependence?**
- **New effects out of equilibrium? New scales? Phase transitions, universality?**

- **Domain wall:** spin currents, NESS
  $t \to \infty, L \to \infty$





- ***Newton's Cradle*** (no NESS)

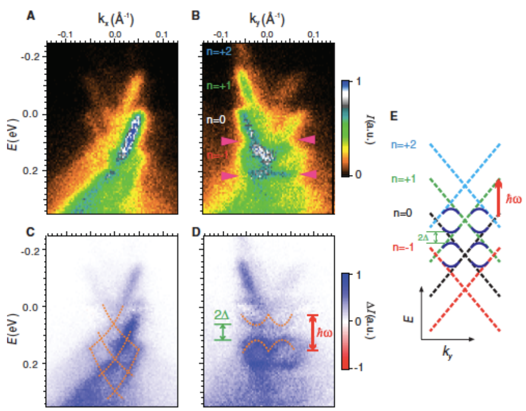
(Periodically) Driven systems

- A *Floquet* Hamiltonian $H(t) = T(t+T)$ has solutions of the form:

$$\psi_\alpha(x,t) = e^{-i\epsilon_\alpha t}\phi_\alpha(x,t)$$

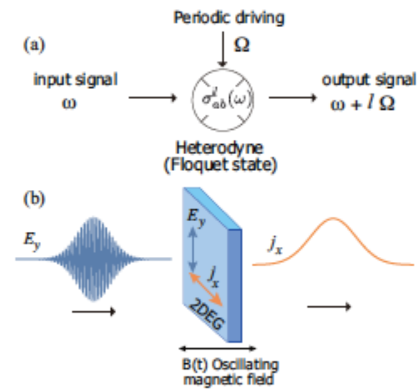
with $\phi_\alpha(x,t)$ periodic and the *quasi energies* ϵ_α determined up to $\omega = 2\pi/T$

- Many experimental realizations: pump-probe, state engineering, photoinduced Floquet- Weyl semimetal phases..



Periodically driven TI:
Floquet – Bloch states

Gedik et al '13



Heterodyne Hall Efect

Oka and Bucciantini '16

- Driven Lieb-Liniger (in a box)

$$H = -\int b^\dagger(x)\partial^2 b(x)\,dx + c\int b^\dagger(x)b(x)b^\dagger(x)b(x)\,dx + f(t)\int x\, b^\dagger(x)b(x)\,dx$$

Floquet spectrum, heating, synchronicity, turbulence..

Outline

Wish to study these questions exactly and confront them with experiment

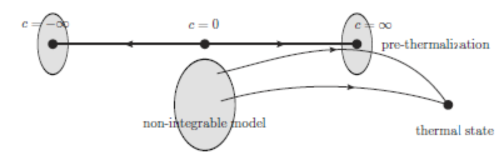
1. Quench evolution in quantum integrable many-body systems

- *Yudson's contour approach (infinite volume system): (Yudson '85)* $L = \infty, N$ fixed

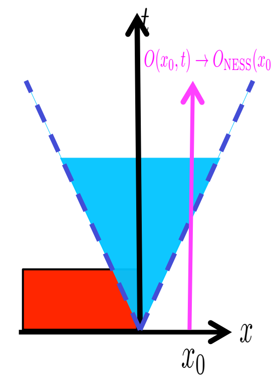
2. **Bosons on the continuous line with short range interactions** *(Lieb-Liniger model)*

A. Finite boson system: $N-\text{finite}, L \to \infty,\ t \ll L/v_{typ}$

Hanbury Brown –Twiss effect and RG flow in time

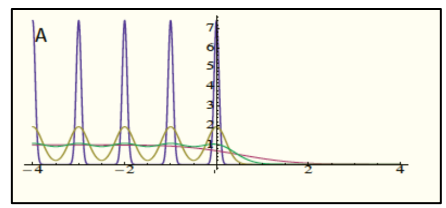


B. Thermodynamic boson system $N, L \to \infty,\ n = N/L$ fixed$,\ t \ll L/v_{typ}$



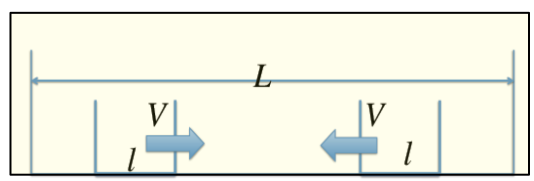
- Generalizing Yudsons' approach to thermodynamic systems

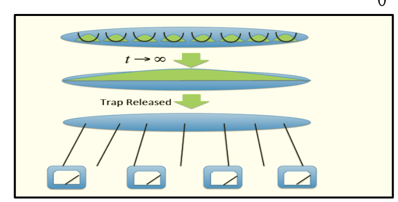
Time evolution of observables – monster formula



- Non equilibration (NESS): the Domain Wall
- Equilibration and GGE for repulsive interaction:

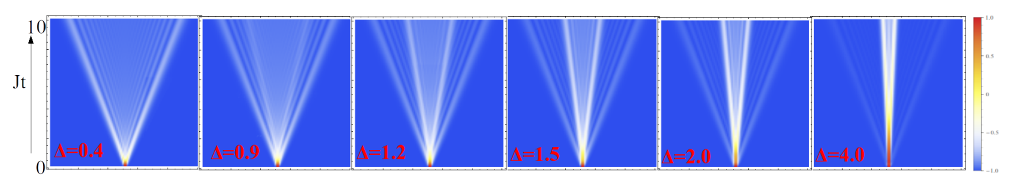
  - Quench from Mott to Lieb-Liniger fluid
  - Newton's Cradle on the average (poor man's version)





3. Quenching the XXZ Heisenberg model

4. Quenching the Gaudin-Yang Model



Quenching in 1-d systems

Physical Motivation:

- **Natural dimensionality of many systems:**
  - **wires, waveguides, optical traps, edges**
- **Impurities: Dynamics dominated by s-waves, reduces to 1D system**
- **Many experimental realizations: ultracold atom traps, nano-systems..**

Special features of 1- d : theoretical

- **Strong quantum fluctuations for any coupling strength**
- **Powerful mathematical methods:**
  - **RG methods, Bosonization, CFT methods, Bethe Ansatz approach**

- Bethe Ansatz approach: allows complete diagonalization of *H*

- Experimentally realizable: *Hubbard model, Kondo model, Anderson model, Lieb-Liniger model, Sine-Gordon model, Heisenberg model, Richardson model..*

- BA $\longrightarrow$ Quench dynamics of many body systems? Exact!

Others approaches: Keldysh, t-DMRG, t-NRG, t-RG, exact diagonalization..

Time Evolution and the Bethe Ansatz

- A given state $|\Phi_0\rangle$ can be formally time evolved in terms of a complete set of energy eigenstates $|F^\lambda\rangle$

$$|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle \quad \longrightarrow \quad |\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle = \sum_\lambda e^{-i\epsilon_\lambda t}|F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$$

If H integrable $\longrightarrow$ eigenstates $|F^\lambda\rangle$ are known via the Bethe-Ansatz

- Use *Bethe Ansatz* to study quench evolution and nonequilibrium

- New technology is necessary:

- **Standard approach: PBC $\longrightarrow$ Bethe Ansatz eqns $\longrightarrow$ spectrum $\longrightarrow$ thermodynamics**

- **Non equilibrium entails *additional* difficulties:**

i. Compute overlaps (form factors) ii. Compute matrix elements iii. Sum over complete basis

Yudson's contour representation (infinite volume)

Instead of $|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$ introduce (directly in infinite volume):

Contour representation of $|\Phi_0\rangle$

$$|\Phi_0\rangle = \int_\gamma d^N\lambda \, |F^\lambda\rangle(F^\lambda|\Phi_0\rangle$$

V. Yudson, sov. phys. *JETP* (1985)

Computed S-matrix of Dicke model

with: $|F^\lambda\rangle$ Bethe eigenstate

$|F^\lambda)$ obtained from Bethe eigenstate by setting $S = I$ - One quadrant suffices

γ contour in momentum space $\{\lambda\}$ determined by **pole structure** of $S(\lambda_i - \lambda_j)$

Note: in the infinite volume limit momenta $\{\lambda\}$ are not quantized
- no Bethe Ansatz equations, $\{\lambda\}$ free parameters

then:

$$|\Phi_0, t\rangle = \int_\gamma d^N\lambda \, e^{-iE(\lambda)t}|F^\lambda\rangle(F^\lambda|\Phi_0\rangle$$

- Describes systems in the zero density limit
- Generalize to thermodynamic systems with finite density

Ultracold Atoms – the Lieb Liniger model

Gas of neutral atoms moving on the line and interacting with short-range interaction

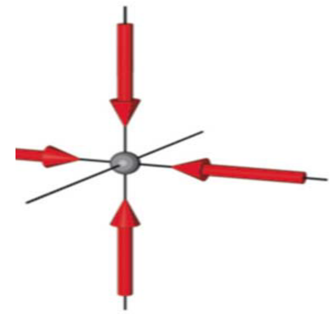
Short range interaction among atoms: $V(x_1 - x_2) = c\delta(x_1 - x_2)$

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l)$$

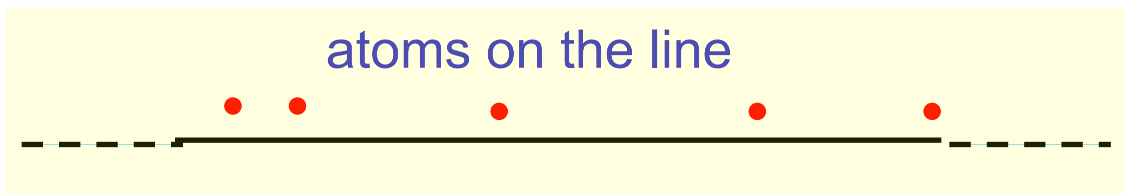
$c > 0$ *repulsive*

$c < 0$ *attractive*

Can be tune by Feshbach resonance



Bloch et al '08



Comment:

- Very short range interaction. Valid for low densities,

$$l = L/N \gg l_{\mathrm{Van\ der\ Vaals}}$$

- The description of physics depends on the scale of observation

Bosonic system – BA solution

The N-boson eigenstatestate (Lieb-Liniger '67)

$$|\lambda_1, \cdots, \lambda_N\rangle = \int_y \prod_{i<j} Z^y_{ij}(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j y_j} \; b^\dagger(y_j)|0\rangle$$

- **Eigenstates labeled by Momenta** $\lambda_1, \cdots, \lambda_N$

- ***Thermodynamics***: *impose PBC* $\longrightarrow$ *BA eqns* $\longrightarrow$ *momenta* $\begin{cases} \textit{real} & c > 0 \\ \textit{n-strings} & c < 0 \end{cases}$

- ***Dynamics*** *(infinite volume): momenta unconstrained*

- **Dynamic factor:** $Z^y_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \quad = \begin{cases} 1 & y_i > y_j \\ S^{ij} & y_i < y_j \end{cases}$

- *The 2-particle S-matrix:* $S_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$ *enters when the particles cross*

- poles of the S-matrix at: $\lambda_i = \lambda_j + ic$

- **The energy eigenvalues**

$$H|\lambda_1, \cdots, \lambda_N\rangle = \sum_j \lambda_j^2 \, |\lambda_1, \cdots, \lambda_N\rangle$$

bosonic system: contour representation

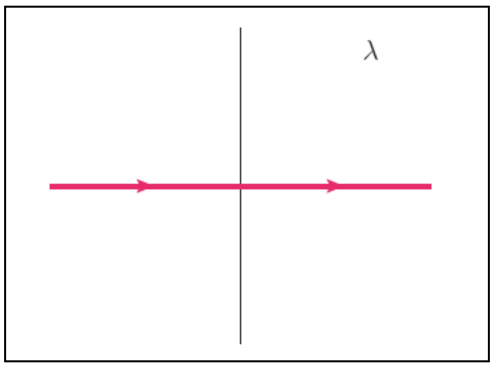
"Central theorem"

denote:

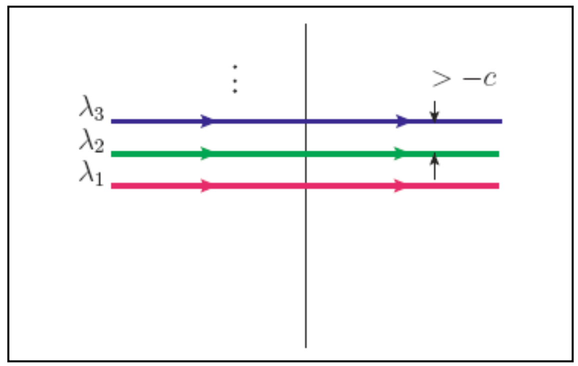
$$\theta(\vec{x}) = \theta(x_1 > x_2 > \cdots . x_N)$$

$$|\Phi_0\rangle = \int_x \Phi_0(\vec{x}) b^\dagger(x_N) \cdots b^\dagger(x_1)|0\rangle =$$

$$= \int_{x,y} \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$



Repulsive c > 0



Attractive c < 0,

contour accounts for strings, bound states

It time evolves to:

$$|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\lambda_j^2 t} e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$

- **Expression contains full information about the dynamics of the system**

What to calculate?

- **We shall study:**

1. Evolution of the density $\rho(x) = b^\dagger(x)b(x)$

$$C_1(x,t) = \langle \Phi_0, t | b^\dagger(x) b(x) | \Phi_0, t \rangle = \int dx_1..dx_N \, |\Phi_0(x_1,..,x_N,t)|^2 \sum_j \delta(x - x_j)$$

- The probability to find the bosons at point x at time t
if at time $t=0$ they started with wave function $\Phi_0(x_1,..,x_N)$

- Can be measured by Time of Flight experiments

- **competition** between quantum broadening and attraction

2. Evolution of noise correlation

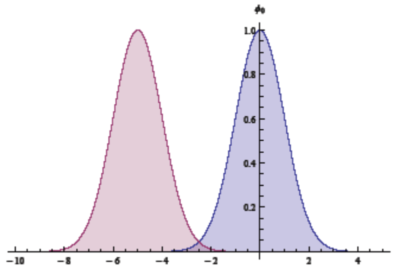
$$C_2(x,y,t) = \frac{\langle \Phi_0, t | \rho(x)\rho(y) | \Phi_0, t \rangle}{C_1(x,t) C_1(y,t)} - 1$$

- Time dependent Hanbury Brown - Twiss effect

Evolution of *few-body* bosonic system: density

- Consider an initial state:

$$\Phi_0(x_1, x_2) = \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} e^{-\frac{(x_1)^2+(x_2+a)^2}{2\sigma^2}}$$

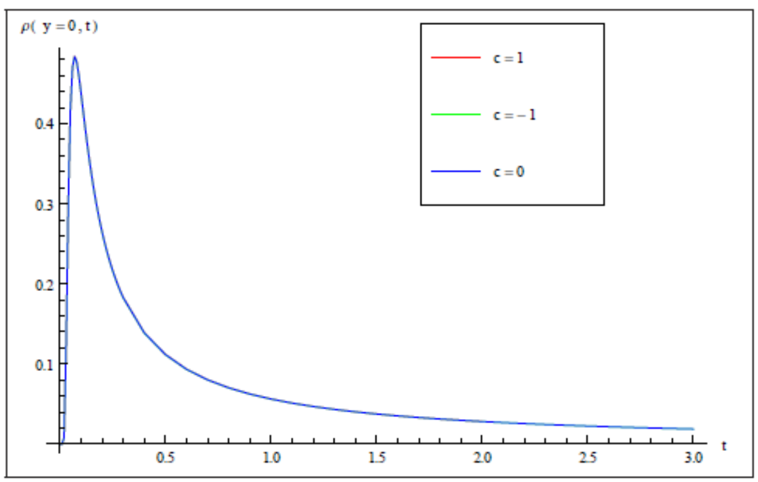


- Its evolution is:

$$\Phi_0(y_1, y_2, t) = \int_x \Phi_0(x_1, x_2)\frac{1}{4\pi i t}\, e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}[1 - c\sqrt{\pi i t}\theta(y_2-y_1)e^{\frac{i}{8t}\alpha^2(x,y,t)}\mathrm{erfc}(\frac{i-1}{4}\frac{i\alpha(x,y,t)}{\sqrt{t}})]$$

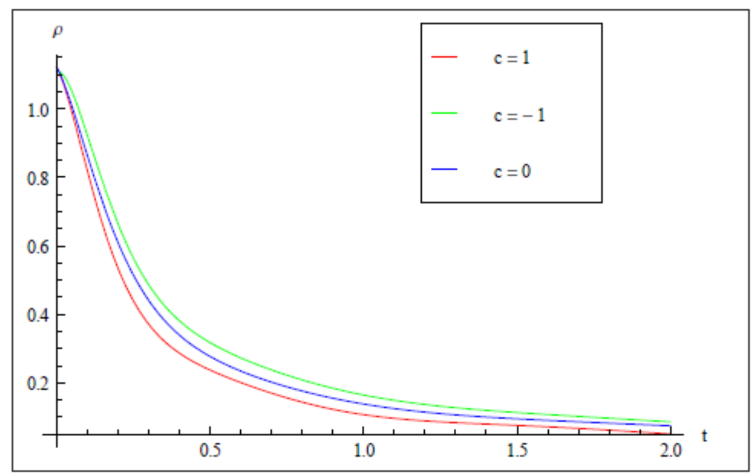
- Compute the evolution of the density $\rho(0,t)$:

***i.* Initial condition:** $a \gg \sigma$



No interaction effects- small initial overlap, then density too low

***ii.* Initial condition:** $a \ll \sigma$

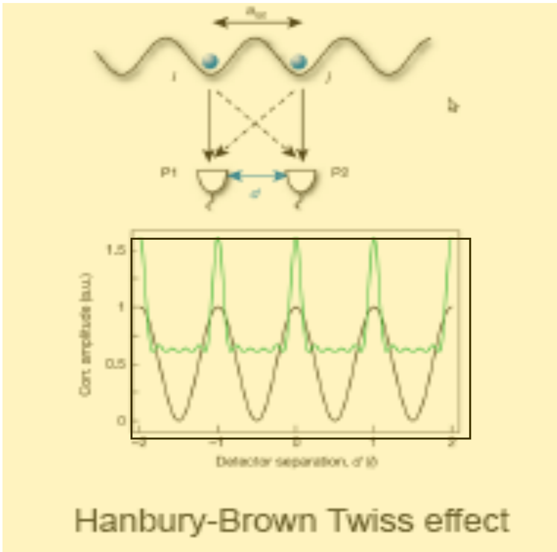


Strong interaction effects: initial overlap

The Hanbury Brown – Twiss effect

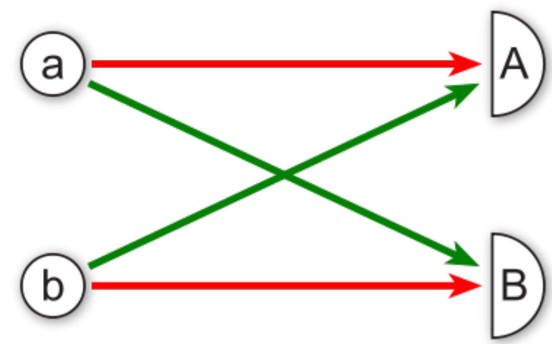
- Hanbury brown-Twiss effect for two bosons

Bloch et al.
RMP '08



Measure: $C_2(x_1, x_2, t)$

- Two stars: a, b



Free bosons $C_2(x, -x) \sim \cos x$

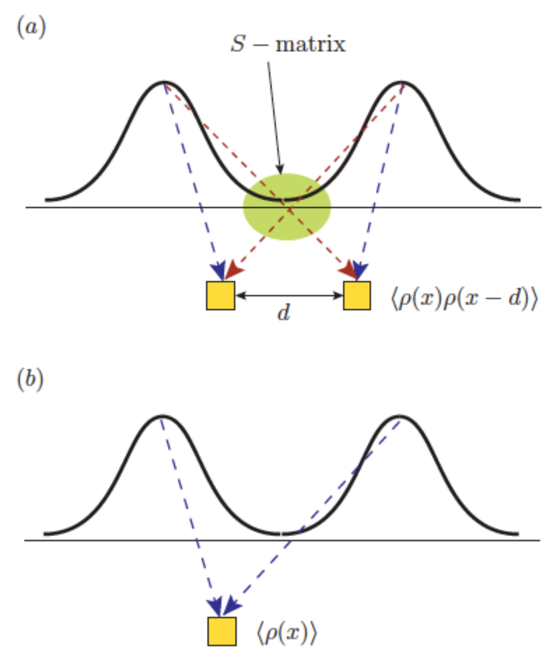
Free Fermions $C_2(x, -x) \sim -\cos x$

Time dependent Hanbury Brown - Twiss effect

- Time dependent
- Many bosons
- More structure: main peaks, sub peaks
- Effects of interactions?

▪ repulsive bosons evolve into fermions

• attractive bosons evolve to a condensate



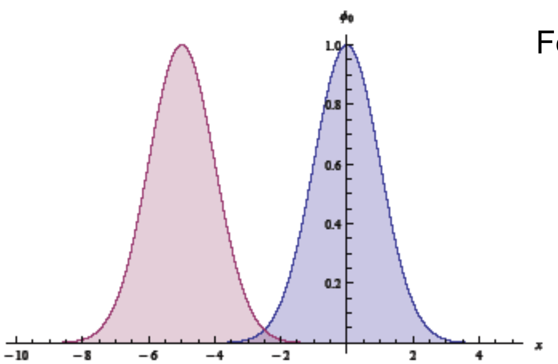
Evolution of repulsive bosons into fermions: HBT

Density - Density correlation: *long time asymptotics - repulsive: Iyer, NA '13*

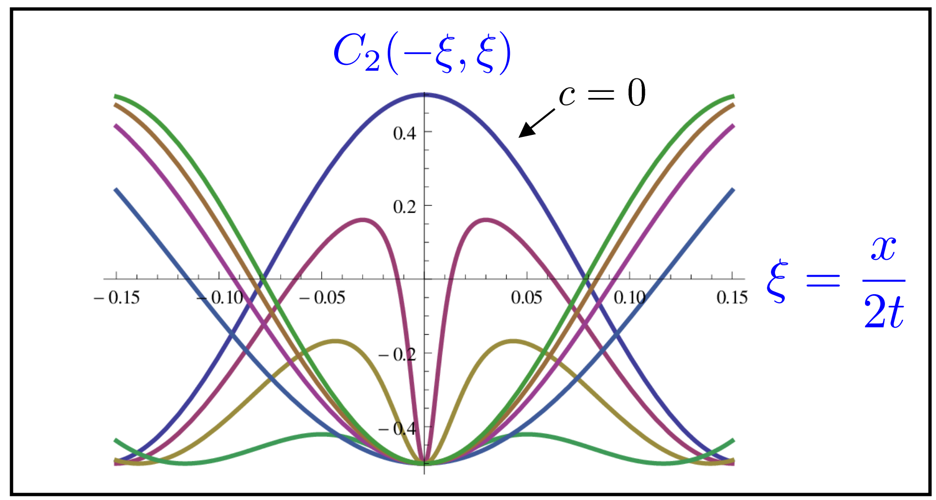
- **Bosons turn into fermions as time evolves (for any $c > 0$)**
- **Can be observed in the noise correlations: (dependence on t only via $\xi = x/2t$)**

$$C_2(x_1, x_2, t) \to C_2(\xi_1, \xi_2) = \frac{\langle \rho(\xi_1)\rho(\xi_2) \rangle}{\langle \rho(\xi_1) \rangle \langle \rho(\xi_2) \rangle} - 1,$$

$c/a = 0, .3, \cdots, 4$



Fermionic correlations evolve



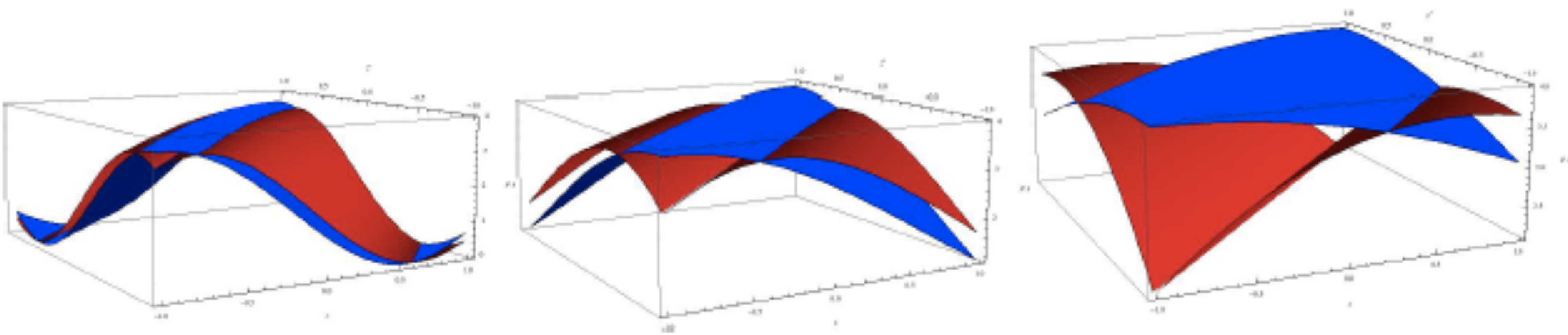
- **Fermionic dip develops for any repulsive interaction on time scale set by $1/c^2$**

Evolution of repulsive bosons into fermions: HBT

Density - Density correlation: *repulsive* *Dwivedi '15*

- **Bosons turn into fermions as time evolves (for any $c > 0$)**
- **Can be observed in the noise correlations:**

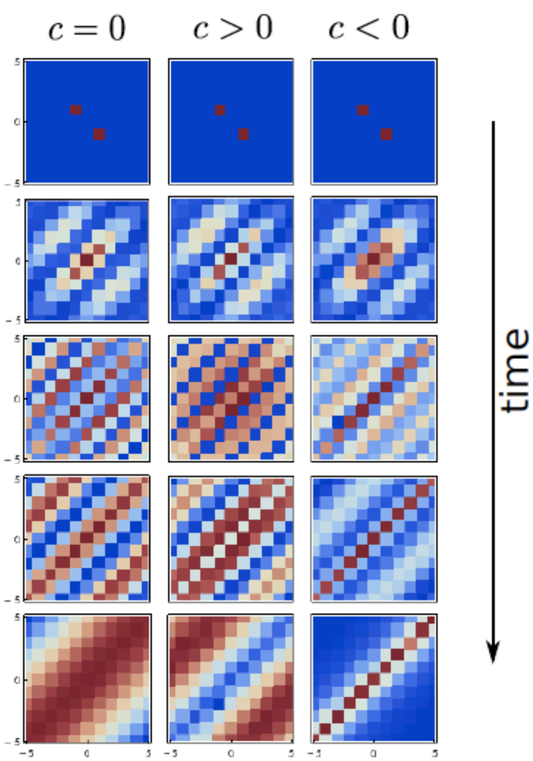
As the density correlation $\langle \rho(x_1)\rho(x_2)\rangle_t$ evolves in time the effective repulsion increases - looks like fermions with the same initial conditions (red- interacting bosons, blue - free bosons)



(a) t=0.25 (b) t=0.5 (c) t=1

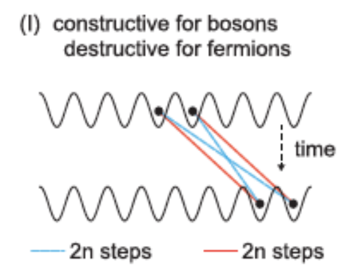
Evolution of Bosons into Fermions

This fermionization also occurs on the lattice *(Iyer, NA '13)*

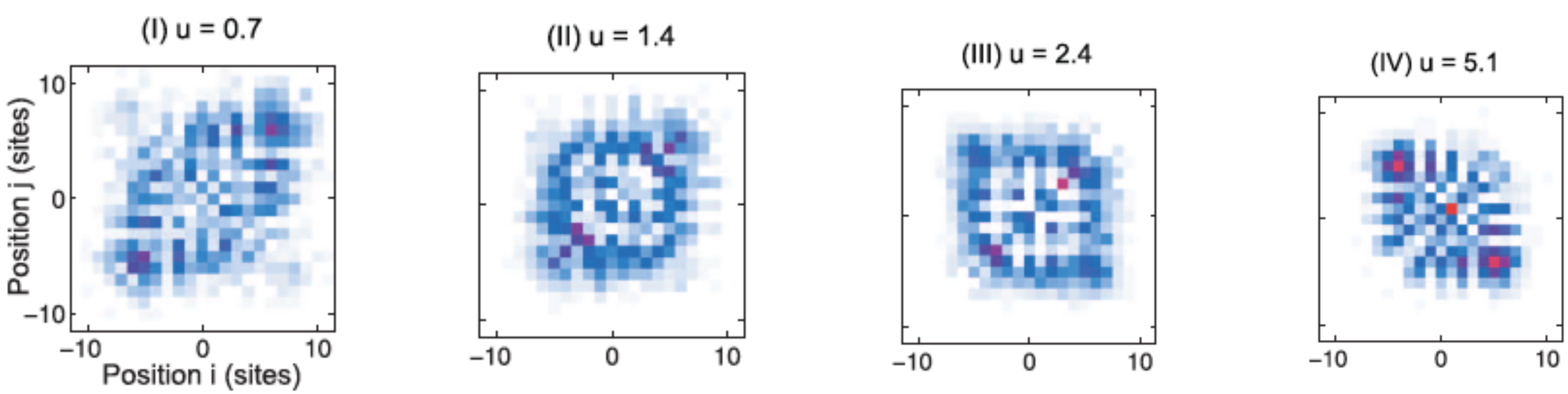


$$H_{BH} = -t\sum_j (b_j^\dagger b_{j+1} + h.c.) + c\sum_j b_j^\dagger b_j (b_j^\dagger b_j - 1)$$

Time evolution of density-density correlation matrix $(\langle \rho(x)\rho(y)\rangle)$ for the $|\Psi_{\rm latt}\rangle$ initial state. Blue is zero and red is positive. The repulsive model shows anti-bunching, i.e., fermionization at long times, while the attractive model shows bunching.

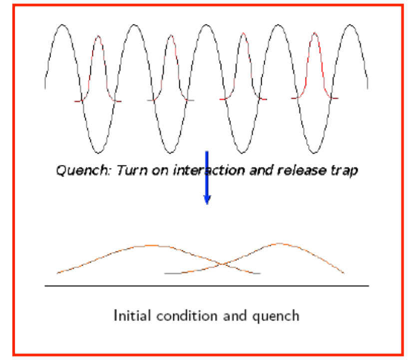


In a recent experiment *(Greiner et al '15)*



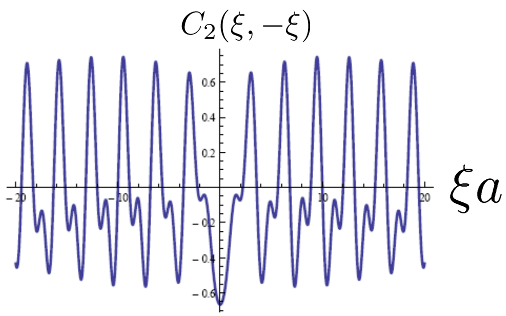
Evolution of *few-body* bosonic system: noise correlations

Noise correlations – starting from a lattice

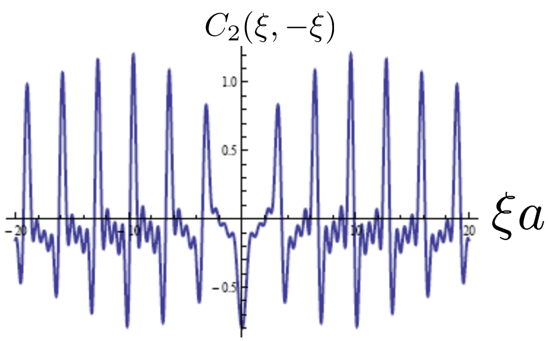


Repulsive bosons

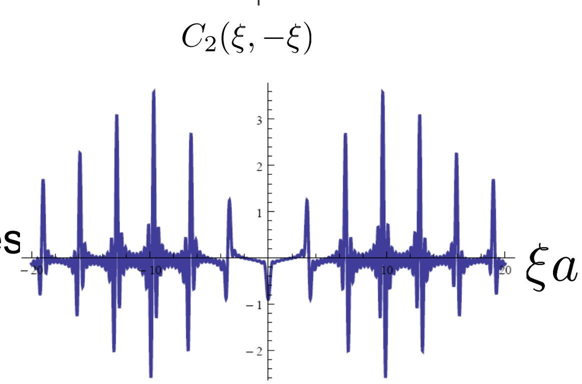
3 particles



5 particles



10 particles

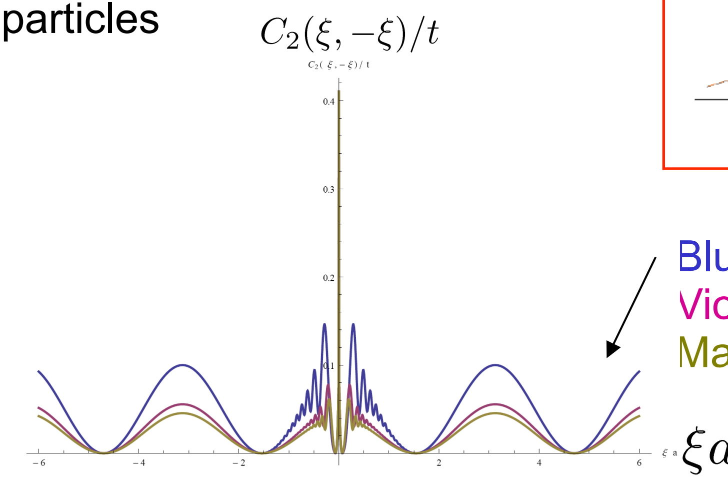


Fermionic dip as $\xi \to 0$

Structure emerges at $\xi a = \sigma$

Attractive bosons

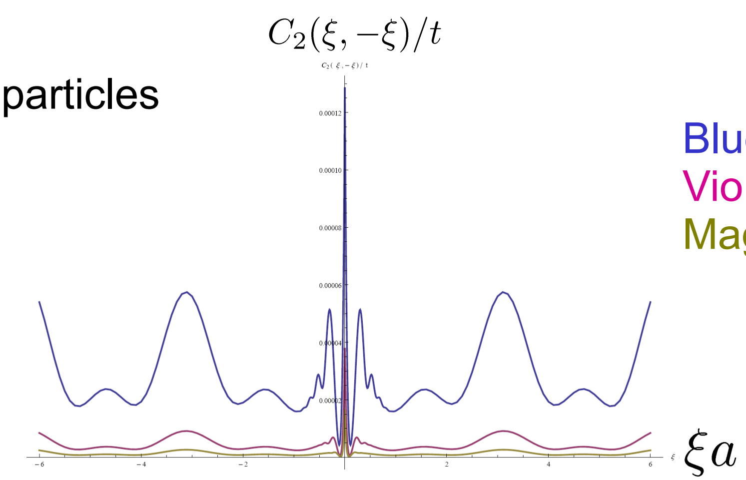
2 particles



Blue - short times
Violet - longer
Magenta - longest

central peaks increase with time - weight in the bound states increases

3 particles



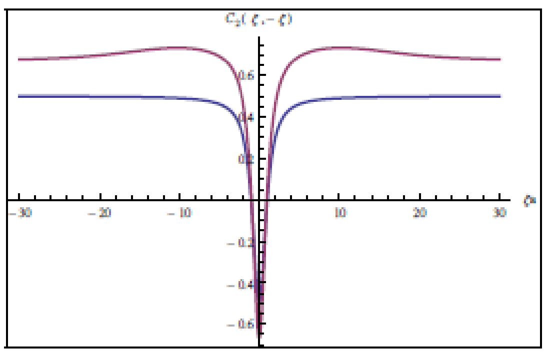
Blue - short times
Violet - longer
Magenta - longest

peaks diffuse – momenta redistribute

Evolution of *few-body* bosonic system: noise correlation

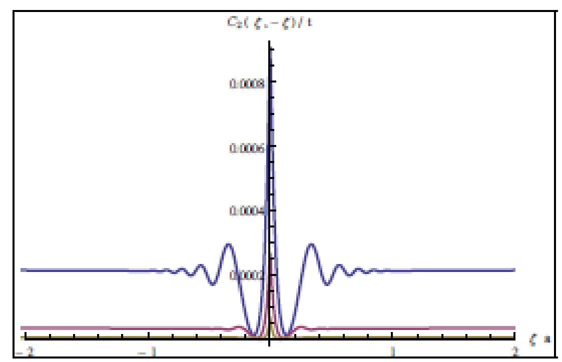
Noise correlations – starting from a condensate:

Repulsive bosons



Two (blue) and three bosons,

Attractive bosons



Three bosons, at times: tc^2 = 20; 40; 60

Emergence of an asymptotic Hamiltonian

Long time asymptotics - repulsive:

- **Bosons turn into fermions as time evolves (for any $c > 0$)**

$$
\begin{aligned}
|\Phi_0, t\rangle &= \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\Sigma_j \lambda_j^2 t - \lambda_j(y_j - x_j)} \prod_j b^\dagger(y_j)|0\rangle \\
&= \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\sqrt{t}\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic\sqrt{t}} e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_j b^\dagger(y_j)|0\rangle \\
&\to \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_{i<j} sgn(y_i - y_j) \prod_j b^\dagger(y_j)|0\rangle \\
&= e^{-iH_0^f t} \int_{x,k} \mathcal{A}_x\, \theta(\vec{x})\Phi_0(\vec{x}) \prod_j c^\dagger(x_j)|0\rangle.
\end{aligned}
$$

$\mathcal{A}_x$ *antisymmetrizer*

where

$H_0^f = -\int_x c^\dagger(x)\partial^2 c(c)$

- In the long time limit repulsive bosons for any $c > 0$ propagate under the influence of* Tonks – Girardeau *Hamiltonian (hard core bosons=free fermions)

- The state equilibrates, does not thermalize

- Argument valid for any initial state Φ_0

- **Scaling argument fails for attractive bosons** (instead, they form bound states)

Evolution of a bosonic system: saddle point app

Long time asymptotics -

Stationary phase approx at large times (carry out λ - integration)

- **Repulsive** – only stationary phase contributions (on real line); (cf. Lamacraft 2011)

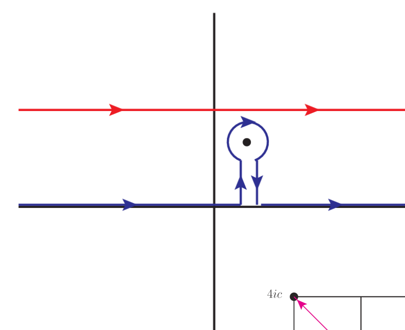
$$\phi\left(\xi \equiv \frac{y}{2t}, x, t\right) = S_\xi \frac{1}{(4\pi i t)^{\frac{N}{2}}} \prod_{i<j} \frac{\xi_i - \xi_j - ic\, sgn(\xi_i - \xi_j)}{\xi_i - \xi_j - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j}$$

- **Attractive** – contributions from stationary phases and poles.

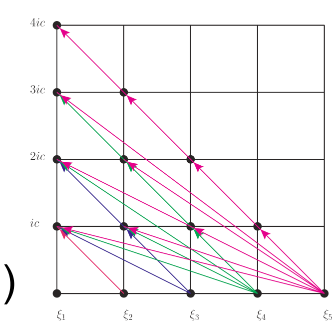
e. g for two particles:

Pole contributions from deformation of contours – formation of bound states

$$\phi(\xi, x, t) = S_\xi\Big[\frac{1}{4\pi i t}\frac{\xi_1 - \xi_2 - ic\, sgn(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j} + \\ + \frac{2c\theta(\xi_2 - \xi_1)}{\sqrt{4\pi i t}} e^{i\xi_1^2 t - i\xi_1 x_1 - i(\xi_1 - ic)^2 t + i(\xi_1 - ic)(2t\xi_2 - x_2)}\Big]$$



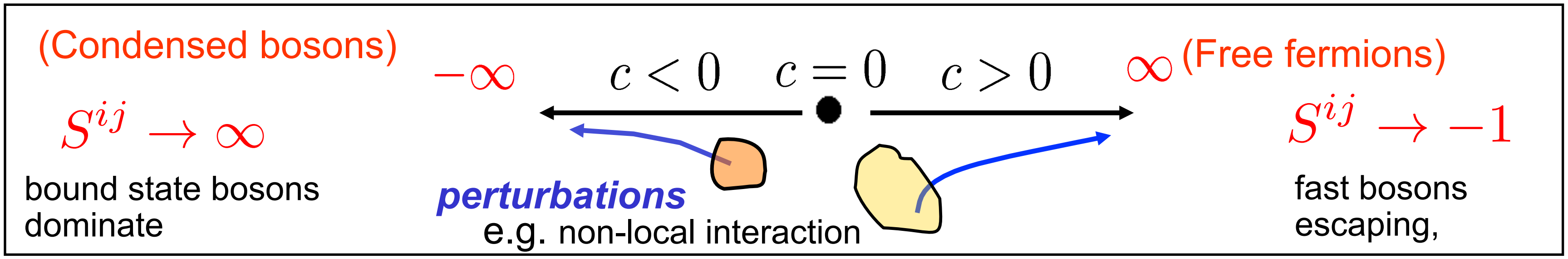
Bound states (string solutions) appear naturally



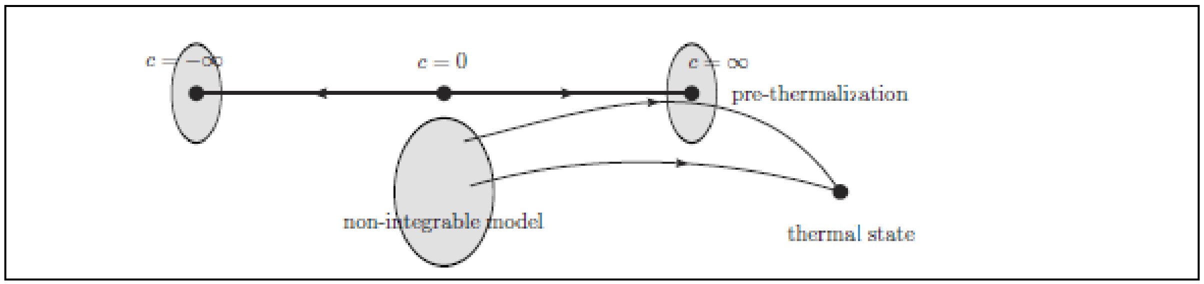
- repulsive correlations depend on $\xi = \frac{y}{2t}$ only (light cone propagation)
- attractive correlations maintain also t dependence (bd. states provide scales)

Evolution of a *few-body* bosonic system

- **Conclusion: coupling constant "*effectively evolves*" with time** $t \sim \ln(D_0/D)$



- **What is beyond** $c = \infty$ **? Thermalization** *(Iyer, NA '13)*



- **Fermionization also occurs on the lattice: *Bose Hubbard model* (not integrable)**

- **The underlying mechanism: the S-matrix** $S^{ij} = \dfrac{k_i - k_j + ic}{k_i - k_j - ic}$

a. **repulsive bosons, fast bosons escaping,** $S^{ij} \to -1$

b. **attractive bosons, bound state bosons,** $S^{ij} \to \infty$

Evolution of *many-body* system: the thermodynamic regime

Thermodynamic regime - $N, L \to \infty, \;\; n = N/L \; fixed, \;\; t \ll L/v_{typ}$

To carry out time evolution need expand initial state $|\Phi^0\rangle$ in eigenstates $|k_1, \cdots, k_N\rangle$

Finite size Yudson representation: *(Goldstein, NA '13)*

Claim: Any initial state -

$$|\Phi_0\rangle = \int_{-L/2}^{L/2} d^N x \;\Phi_0\left(x_1, \cdots, x_N\right) b^\dagger\left(x_N\right), \cdots, b^\dagger\left(x_1\right)|0\rangle$$

Can be expressed in terms of eigenstates (a generalization of Yudson '85) $c > o$

$$|\Phi_0\rangle = \sum_{n_1=-\infty}^{\infty} \ldots \sum_{n_N=-\infty}^{\infty} \mathcal{N}\left(\{k_n\}\right)^{-1} |k_{n_1} \ldots . k_{n_N}\rangle \left(k_{n_1} \ldots k_{n_N}\right| |\Phi_0\rangle \, .$$

Overcomes the difficulty of calculating overlaps

Time evolution of an observable

- Time evolution of generating function of density correlations- $\langle\Phi(t)|\Theta|\Phi(t)\rangle$

$$\Theta = e^{\alpha Q_{xy}} \quad Q_{xy} \equiv \int_x^y b^\dagger(z)\, b(z)\, dz \quad |\Phi\rangle = \int dx_1 \cdots dx_n \Phi(x_1 \cdots x_n) b^\dagger(x_1)\cdots b^\dagger(x_n)|0\rangle$$

Charge between x and y.

- Compute the Green's function

$$G(\Theta, t; x_1 ... x_N; y_1 ... y_N) = \langle 0|b(y_1)..b(y_N)(\, e^{iHt}\,\Theta\, e^{-iHt}\,)b^\dagger(x_1)..b^\dagger(x_N)|0\rangle$$

- Inserted Yudson representation twice - overlaps simple plane waves

$$G\left(\Theta, t; x_1, x_2, ...x_N; y_1, ...y_N\right) =$$
$$= \sum_{n_1,...n_N} \frac{1}{\mathcal{N}\left(k_{n_1}....k_{n_N}\right)} \langle 0|\, b\left(y_1\right)...b\left(y_N\right)|k_{n_1}..k_{n_N}\rangle \times$$
$$\times \langle k_{n_1}...k_{n_N}|\,\Theta\,|q_{n_1},..q_{n_N}\rangle \times \sum_{n_1,..n_N} \frac{1}{\mathcal{N}\left(q_{n_1}...q_{n_N}\right)} \times$$
$$\times \left(q_{n_1}, ...q_{n_N}|\, b^\dagger\left(x_1\right)...b^\dagger\left(x_N\right)|0\right\rangle \prod_i e^{i\left(k_{n_i}^2 - q_{n_i}^2\right)t}$$

- Need matrix elements: $\langle k_{n_1}....k_{n_N}|\,\Theta\,|q_{n_1}...q_{n_N}\rangle$

Korepin, Bogoliubov, Izergin, Slavnov, Kitanine..

Time evolution of an observable

Consider: $Q_{xy} \equiv \int_x^y b^\dagger(z)\, b(z)\, dz$ *Charge between x and y.*

- *Expression valid for all times* $L, N \to \infty, \quad t \ll L/v_{typ}$

$$
\begin{aligned}
&\langle \exp(\alpha Q_{xy}(t)) \rangle = 1 + \int dX dY F_{\alpha,xy}(X,Y,t) \times \\
&\quad \times \left\langle b^\dagger(Y) \exp\left[i \int_X^Y dz \pi b^\dagger(z) b(z) \right] b(X) \right\rangle + \\
&+ \int dX_1 dX_2 dY_1 dY_2 \times F_{\alpha,xy}(X_1,Y_1,t) F_{\alpha,xy}(X_2,Y_2,t) \times \\
&\times \left\langle sgn(Y_2 - Y_1) b^\dagger(Y_1) b^\dagger(Y_2) e^{i \int_{X_1}^{Y_1} dz \pi b^\dagger(z) b(z)} \times \right. \\
&\times \left. sgn(X_2 - X_1) e^{i \int_{X_2}^{Y_2} dz \pi b^\dagger(z) b(z)} b(X_1) b(X_2) \right\rangle - \\
&- \frac{i\alpha}{\pi^2 c} \int dX_1 dY_1 dX_2 dY_2 \{ F_{\alpha,x}(X_1,Y_1,t) G_{\alpha,x}(X_2,Y_2,t) - \\
&- F_{\alpha,y}(X_1,Y_1,t) G_{\alpha,y}(X_2,Y_2,t) \} \langle sgn(y_2 - y_1) \\
&\cdot sgn(x_2 - x_1) \cdot b^\dagger(y_1) b^\dagger(y_2) e^{i \int_{X_1}^{Y_1} dz \pi b^\dagger(z) b(z)} . \\
&\cdot e^{i \int_{X_2}^{Y_2} dz \pi b^\dagger(z) b(z)} b(X_1) b(X_2) \rangle - \\
&- \frac{i\alpha}{2\pi^2 c} \int dX dY G_{\alpha,X}(X,Y,t) \left\langle b^\dagger(Y) e^{i \int_X^Y dz \pi b^\dagger(z) b(z)} . \right. \\
&\cdot \left. \int_{-\infty}^{\infty} dv sgn(x - v) \rho(v) sgn(v - Y) sgn(v - X) \right\rangle + \\
&+ \frac{i\alpha}{2\pi^2 c} \int dX dY G_{\alpha,y}(X,Y,t) \left\langle b^\dagger(Y) e^{i \int_X^Y dz \pi b^\dagger(z) b(z)} . \right. \\
&\cdot \left. \int_{-\infty}^{\infty} dv sgn(y - v) \rho(v) sgn(v - Y) sgn(v - X) \right\rangle + \ldots.
\end{aligned}
$$

Matrix elements: Korepin, Bogoliubov, Izegin, Kitanine…

1. Up to $\alpha^3, 1/c^2$
2. ***All expectation values taken w.r.t initial state***
3. ***Valid for all initial states***

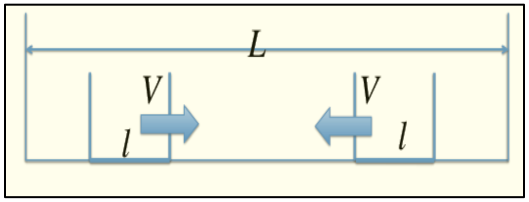
$$F_{\alpha,xy}(X,Y,t) \equiv i \frac{e^\alpha - 1}{2\pi} \frac{\exp\left(-\frac{i}{4t}(Y^2 - X^2)\right)}{Y - X} \times \left[\exp\left(\frac{i(Y-X) \cdot x}{2t} \right) - \exp\left(\frac{i(Y-X) \cdot y}{2t} \right) \right]$$

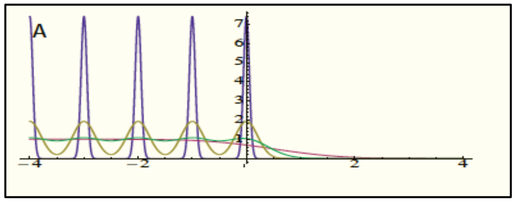
$$G_{\alpha,x}(X,Y,t) \equiv \exp\left(i \frac{(Y-X) x}{2t} \right) \frac{\exp\left(-i(Y^2 - X^2)/2t\right)}{t}$$

Time evolution- flow chart

Initial state translational invariant
example: **Mott insulator**

Initial state not translational invariant
example: **Newton's cradle**, **Domain wall**



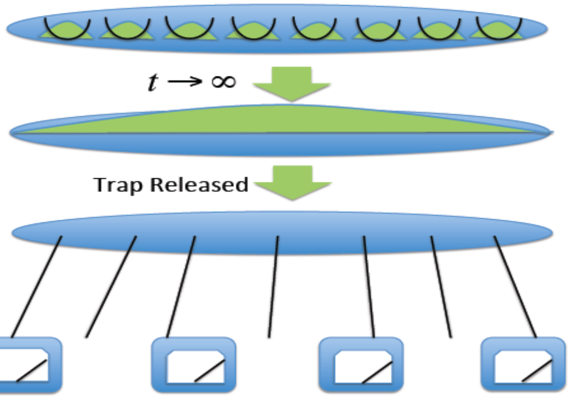


Equilibrates to local GGE (c >0)

Equilibrates but not to local GGE (c < 0)

Goldstein, NA '13

Universal correlations (if low YY entropy)

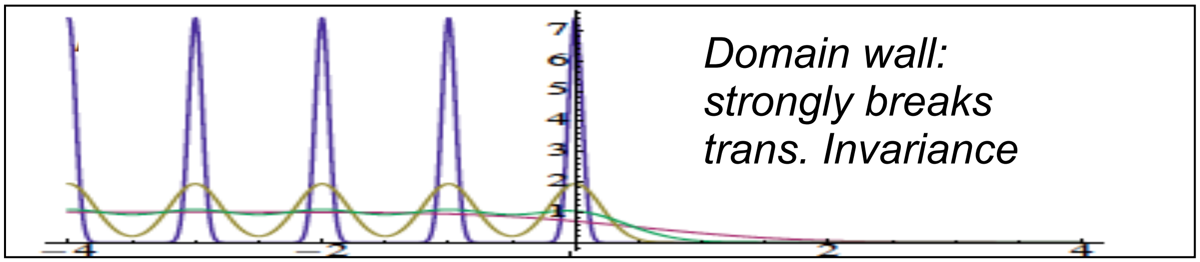


System does not equilibrate:
currents, *local* entropy production

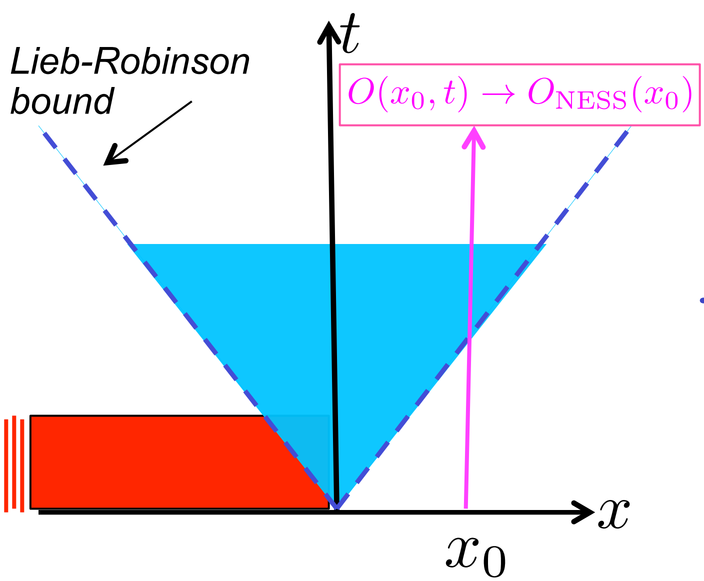
Evolution to NESS: Domain wall

***Example**: time evolution from a non-trans. invariant initial state (no equilibration)*

$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty} \int_{-\infty}^{\infty} \varphi(x+jl)\, b^{\dagger}(x)\, |0\rangle \qquad \text{with: } \varphi(x) = \frac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$$



- System evolves to NESS $\rho(x,t) \to$ *(G Goldstein, NA '13)*



$$\frac{1}{l}\left(1+\sum_{s=1}^{\infty} e^{-\pi^2(8t^2/\sigma+\sigma/2)s^2/l^2}\cos(2\pi s x/l)\right) \qquad x \ll -\sqrt{8t^2/\sigma}$$

Outside the cone

$$\frac{1}{l}\left(\frac{1}{2}-\frac{4\pi}{cl}\right) \qquad \sigma \ll |x| \ll t$$

Nonequilibrium steady state (NESS)
Independent of x, t; interaction effects

$$\frac{1}{l}\left(\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)+\right.$$
$$+\frac{16}{\pi cl}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right)$$
$$\left.+\frac{16}{\pi cl}\frac{\pi}{2}\left(1-\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)\right)Erfc\left(\frac{x}{\sqrt{A}}\right)\right)$$

$x \sim t$

$A \equiv 8t^2/\sigma + \sigma/2$

Crossover regime

$$0 \qquad x \gg \sqrt{8t^2/\sigma}$$

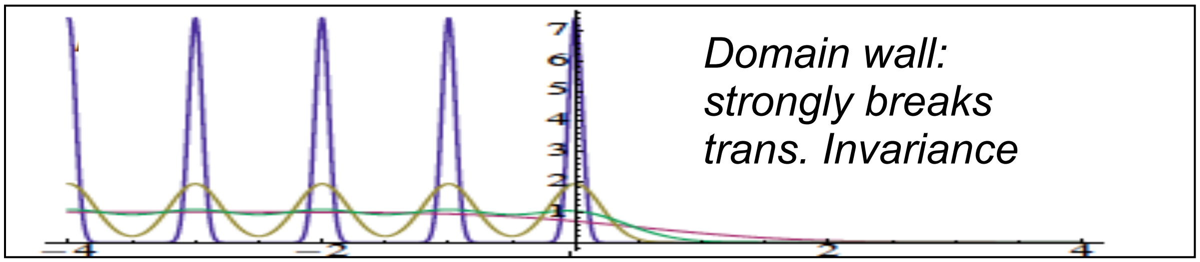
Outside the cone

Evolution to NESS: Domain wall (no thermalization)

***Example**: time evolution from a non-trans. invariant initial state (no equilibration)*

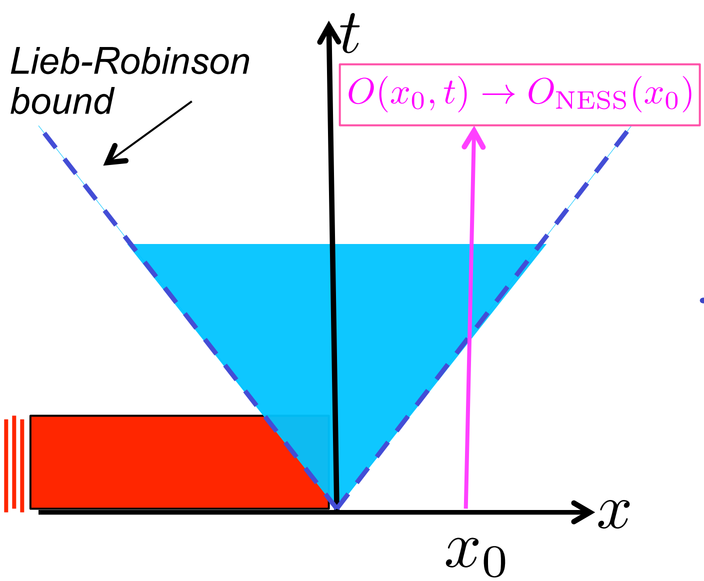
$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty} \int_{-\infty}^{\infty} \varphi(x+jl)\, b^{\dagger}(x)\, |0\rangle$$

with: $\varphi(x) = \dfrac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



Domain wall: strongly breaks trans. Invariance

- System evolves to NESS $\rho(x,t) \rightarrow$ (G Goldstein, NA '13)



$$\frac{1}{l}\left(1+\sum_{s=1}^{\infty} e^{-\pi^2\left(8t^2/\sigma+\sigma/2\right)s^2/l^2}\cos\left(2\pi s x/l\right)\right) \qquad x \ll -\sqrt{8t^2/\sigma}$$

Outside the cone

$$\frac{1}{l}\left(\frac{1}{2}-\frac{4\pi}{cl}\right) \qquad \sigma \ll |x| \ll t$$

Nonequilibrium steady state (NESS)
Independent of x, t; interaction effects

$$\frac{1}{l}\left(\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)+\right.$$
$$+\frac{16}{\pi c l}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right)$$
$$\left.+\frac{16}{\pi c l}\frac{\pi}{2}\left(1-\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)\right)Erfc\left(\frac{x}{\sqrt{A}}\right)\right)$$

$x \sim t$

$A \equiv 8t^2/\sigma + \sigma/2$

Crossover regime

$$0 \qquad x \gg \sqrt{8t^2/\sigma}$$

Outside the cone

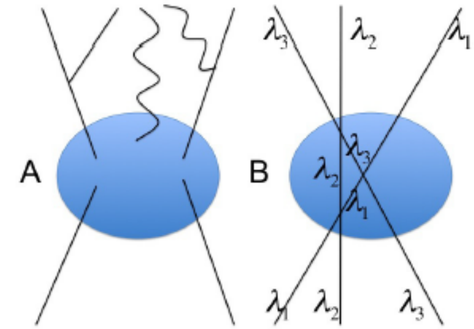
Evolution to GGE: trans. invariance, diagonal ensemble

GGE if: 1. Diagonal ensemble 2. Operator can be expanded

i. For trans. invariant initial states : $\langle \Theta \rangle (t \to \infty) = Tr \rho_D \Theta$

$$\rho_D = \sum p_{\{k\}} \left|\{k_i\}\right\rangle \left\langle \{k_i\}\right| \quad \textit{with} \quad p_k = |\langle \{k\} | \Phi_0 \rangle|^2$$

ii. The diagonal element can be Taylor expanded (~ GETH)



$$\langle \{k_i\} | \Theta | \{k_i\} \rangle = c_0 + c_1 \sum k_i + c_{1,1} \sum k_i k_j + c_2 \sum k_i^2 + ..$$

iii. Thus: $$\langle \Theta \rangle \to c_0 + c_1 \langle I_1 \rangle + c_{1,1} \langle I_1^2 \rangle + c_2 \langle I_2 \rangle + ..$$

with: $\langle I_1 \rangle = \sum p_{\{\lambda\}} \sum k_i, \quad \langle I_1^2 \rangle = \sum p_{\{k\}} \sum k_i k_j, \quad \langle I_2 \rangle = \sum p_{\{k\}} \sum k_i^2$ ...

iv. Equivalently: $$\langle \Theta \rangle = Tr \rho_{GGE} \Theta$$

with $$\rho_{GGE} \sim e^{-\sum_n \beta_n I_n}$$

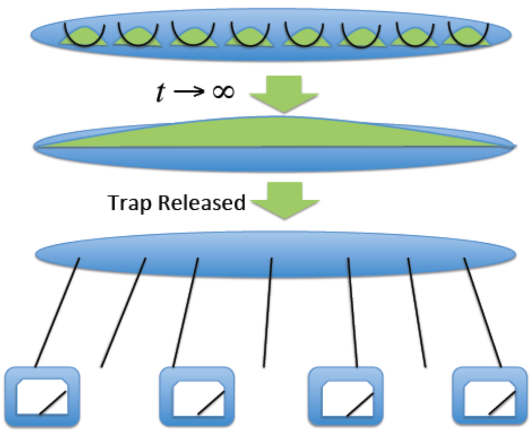
Actually need also short correlations among I_n

Time evolution- interaction quench from a Mott state

Example: **Quenching from a Mott insulator to a Lieb-Liniger Liquid (GGE):**

$$t=0 \quad |\Phi_0\rangle = \prod_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x+jl)\, b^{\dagger}(x)\,|0\rangle$$

with $\varphi(x) = \dfrac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



$t \to \infty \ |\Phi_0\rangle \to \rho_{GGE}$ ***How to describe?***

- **GGE** *corresponds to a pure state:* $tr\left[\Theta\rho_{GGE}\right] = \left\langle \vec{k}_0 \middle| \Theta \middle| \vec{k}_0 \right\rangle$

with the eigenstate $|\vec{k}_0\rangle$ *given by* $\rho_p(k), \rho_t(k)$ *satisfying* *(Caux '13) :*

$$\rho_t(k) = \frac{1}{2\pi} + \frac{1}{2\pi}\int dq K(k,q)\,\rho_p(q) \quad \text{with} \quad K(k,q) = \frac{2c}{c^2+(k-q)^2}$$

- But $\langle I_n\rangle_{t=0} = \langle I_n\rangle_{t\to\infty}$

$$L\int dk\rho_p(k)\,k^n = I_n(t=0) \doteq \frac{L}{l}\left(\frac{2}{\sigma}\right)^{\frac{n}{2}} \frac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}!\right)}$$

$\longrightarrow$ *Final distribution*

$$\begin{cases} \rho_p(k) = \dfrac{\sigma^{\frac{1}{2}}}{\pi^{\frac{1}{2}} l} \exp\left(-\dfrac{k^2\sigma}{2}\right) \\ \rho_t(k) \cong \dfrac{1}{2\pi} \quad \text{for} \quad l \gg \sqrt{\sigma} \end{cases}$$

→ **The occupation probability** $f(k) \equiv \dfrac{\rho_p(k)}{\rho_t(k)} \cong \dfrac{2\sqrt{\pi\sigma}}{l} \exp\left(-\dfrac{k^2\sigma}{2}\right)$

Time evolution – interaction quench from a Mott state

Can compute various correlation functions:

1. $$\left\langle b^{\dagger}(0)\, b^{\dagger}(0)\, b(0)\, b(0)\right\rangle \cong 2\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi} f(k_1)\, f(k_2)\, \frac{(k_2-k_1)^2}{(k_2-k_1)^2+c^2} + \ldots\ldots$$

$$= \frac{2}{l^2} - \frac{2\sqrt{\pi c^2\sigma}}{l^2}\left[\exp\left(\frac{\sigma c^2}{4}\right) Erfc\left(\sqrt{\frac{\sigma c^2}{4}}\right)\right] \rightarrow \begin{cases} \cong \dfrac{2}{l^2} & c^2\sigma \ll 1 \\ \cong \dfrac{1}{l^2 c^4 \sigma^2} & c^2\sigma \gg 1 \end{cases}$$

Suppression of density correlations, measurable by Time of Flight experiments

2. $$\left\langle b^{\dagger}(0)\, b^{\dagger}(0)\, b^{\dagger}(0)\, b(0)\, b(0)\, b(0)\right\rangle \cong 6\int dk_1 dk_2 dk_3 f(k_1)\, f(k_2)\, f(k_3)\, \frac{(k_2-k_1)^2}{(k_2-k_1)^2+c^2}\frac{(k_3-k_1)^2}{(k_3-k_1)^2+c^2}\frac{(k_3-k_2)^2}{(k_3-k_2)^2+c^2}$$

Strong suppression of three body decay rates, measurable through trap loss or third moment of particle number (Bouchoule'10)

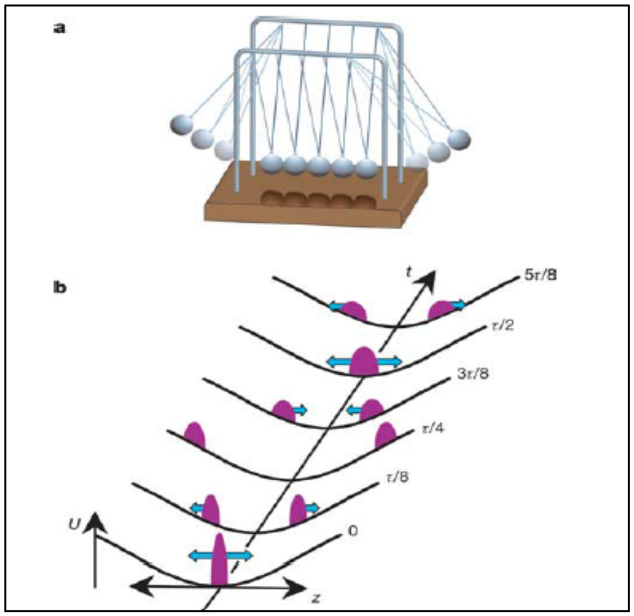
$$\rightarrow \begin{cases} \cong \dfrac{6}{l^3} & c^2\sigma \ll 1 \\ \cong \dfrac{9\times 2^{\frac{9}{2}}}{l^3 c^6 \sigma^3} & c^2\sigma \gg 1 \end{cases}$$

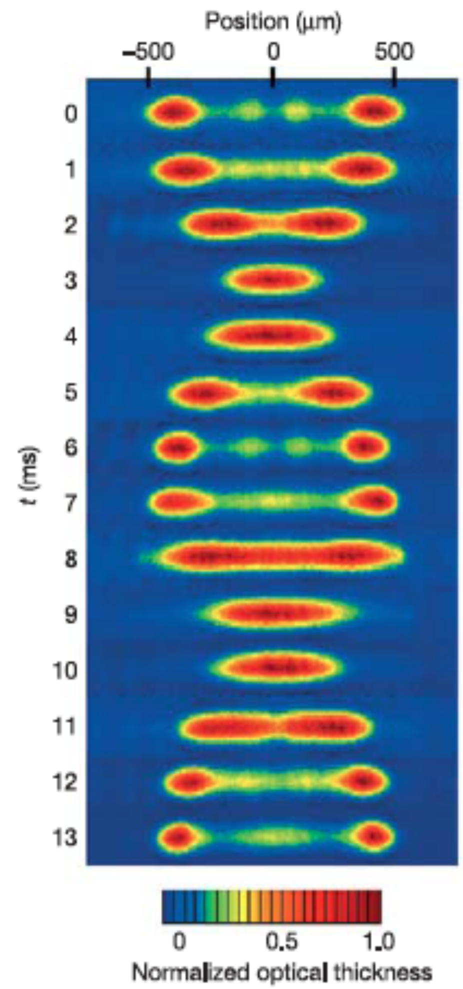
3. $$\left\langle \rho(x)\, \rho(0)\right\rangle \cong \rho^2 + \frac{1}{4\pi^2 e^2 l^2}\exp\left(-\frac{x^2}{\sigma}\right)$$ *for* $l \gg \sqrt{\sigma}$

Gaussian decay of density-density function

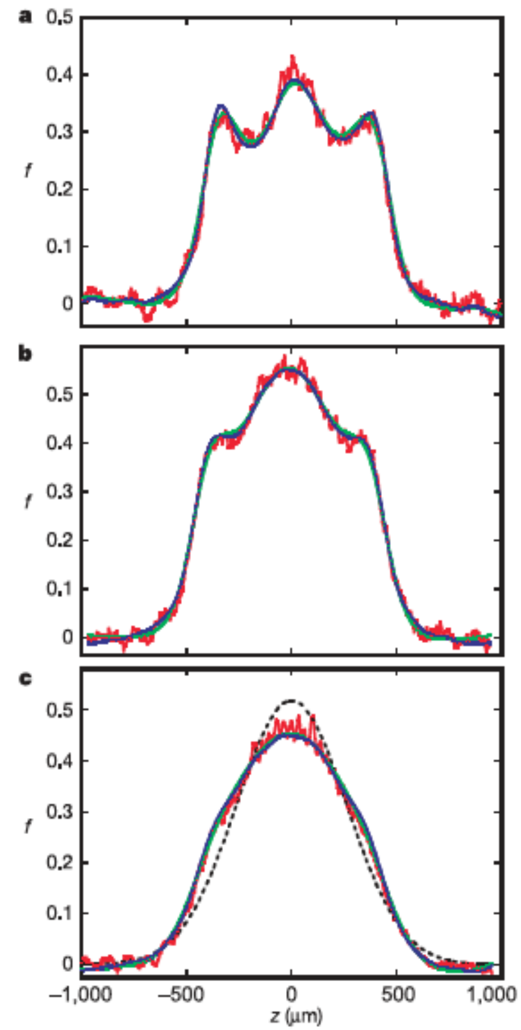
Newton's Cradle - simplified

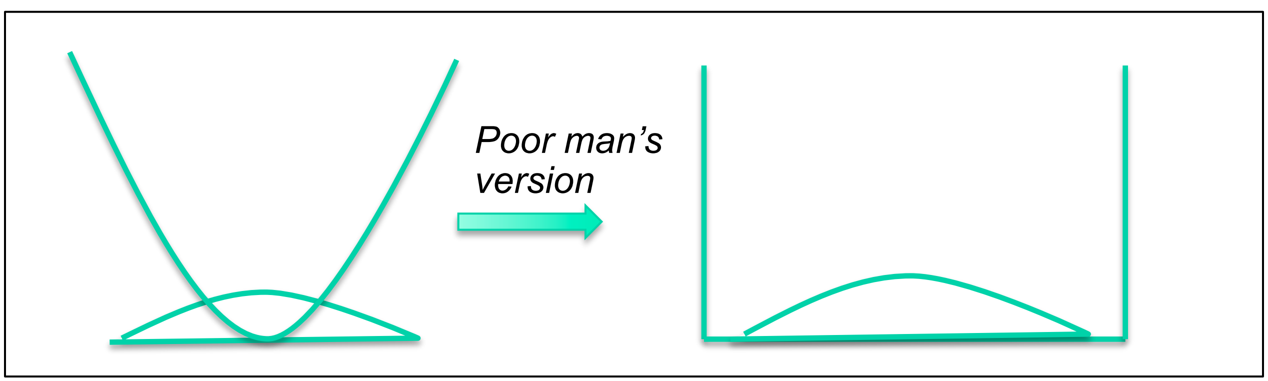
Kinoshita, T. Wenger, D. S. Weiss, Nature (2006)





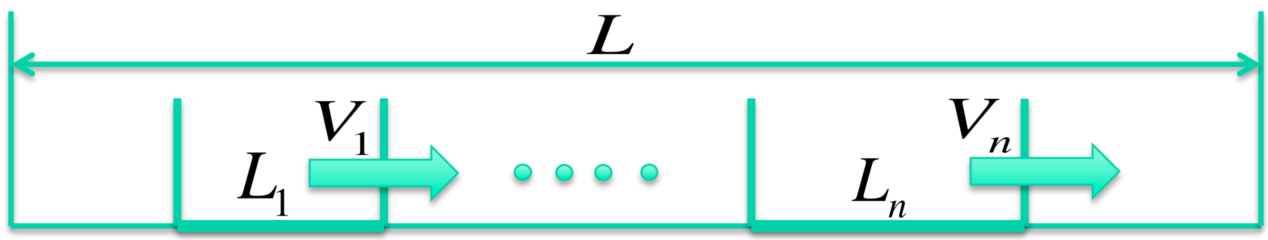
averaged momentum distributions





Newton's Cradle

Initial state: **several moving boxes within a larger container**
– *poor man's version of Weiss' experiment*



System does not equilibrate, but long time average is a diagonal ensemble

$$\langle\Theta\rangle_T \equiv \frac{1}{T}\int_0^T dt\langle\Psi|e^{iHt}\Theta e^{-iHt}|\Psi\rangle = \frac{1}{T}\sum_\lambda\sum_\kappa \frac{e^{i(E_\lambda - E_\kappa)T} - 1}{i\left(E_\lambda - E_\kappa\right)}\langle\Psi \mid \lambda\rangle\langle\lambda|\Theta|\kappa\rangle\langle\kappa \mid \Psi\rangle$$

$$\cong \sum_\lambda \langle\Psi \mid \lambda\rangle\langle\lambda|\Theta|\lambda\rangle\langle\lambda \mid \Psi\rangle$$

GGE for the average - *match* conservation laws:

LL in a box *(Gaudin)*

$$\psi\left(|k_1|,..|k_N|\right) = \sum_{\{\varepsilon\}} C\left\{\varepsilon\right\}\bar{\psi}\left(\varepsilon_1|k_1|,..\varepsilon_N|k_N|\right)$$

$$L\int dk\rho_p^f(k)\,k^{2n} = \sum L_i\int\rho_p^i(k)\left(k+\frac{1}{2}V_i\right)^{2n}$$

All even charges are conserved $\{I_{2n}\}$

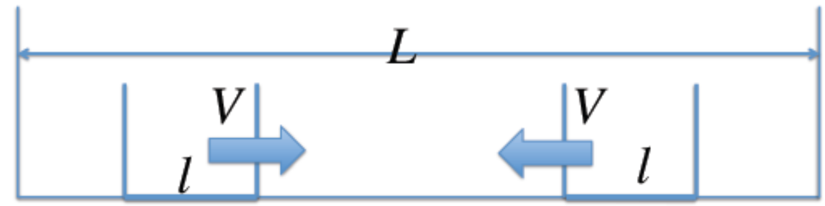
Solution for final quasi-particle density

$$\rho_p^f(k) = \sum\frac{L_i}{2L}\left(\rho_p^i\left(k+\frac{1}{2}V_i\right)+\rho_p^i\left(k-\frac{1}{2}V_i\right)\right)$$

Newton's Cradle in a box

Initial state:

Two boxes of length l each containing N bosons in a given initial state ρ^i moving towards each other at speed V



1. $$\rho^i_{gs}(k) = \theta(-k_F, k_F)\frac{1}{2\pi}\left(1 + \frac{2k_F}{\pi c}\right) + o(\frac{k_F}{c})....$$

2. $$\rho^i_{\mathrm{BEC}}(x) = \frac{\tau \frac{d}{d\tau} a(x,\tau)}{1 + a(x,\tau)}$$ $$x = \frac{k}{c}, \tau = \frac{n}{c}$$

Caux et al '12

$$a(x,\tau) = \frac{2\pi\tau}{x\sinh(2\pi x)} J_{1-2ix}(4\sqrt{\tau}) J_{1+2ix}(4\sqrt{\tau})$$

- The velocity distribution (measurable): $$P(v,t) = \int dx e^{-i\frac{v}{2}x}\langle b^\dagger(x) b(0)\rangle_t$$

- *The field-field correlation given in terms of the occupation probability* $$f(k) = \frac{\rho_p(k)}{\rho_t(k)}$$

$$\langle b^\dagger(x) b(0)\rangle_{t\to\infty} = \int \frac{dk}{2\pi} f(k) e^{-ikx}\omega(k)\exp\left(-x\int du\, f(k)\, p_u(k)\right)$$

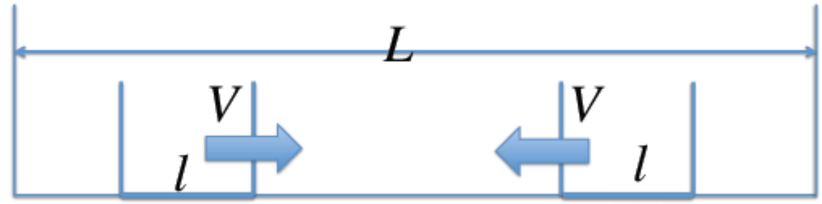
Korepin, Izergin '87

with:

$$2\pi p_u(k) = -\frac{k-u+ic}{u-k+ic}\exp\left(-\int ds\, f(s)\, K(u,s)\, p_s(k)\right) - 1 \quad \omega(k) = \exp\left(-\frac{1}{2\pi}\int dq K(k,q) f(q)\right) \quad K(k,q) = \frac{2c}{(k-q)^2 + c^2}$$

Newton's Cradle in a box

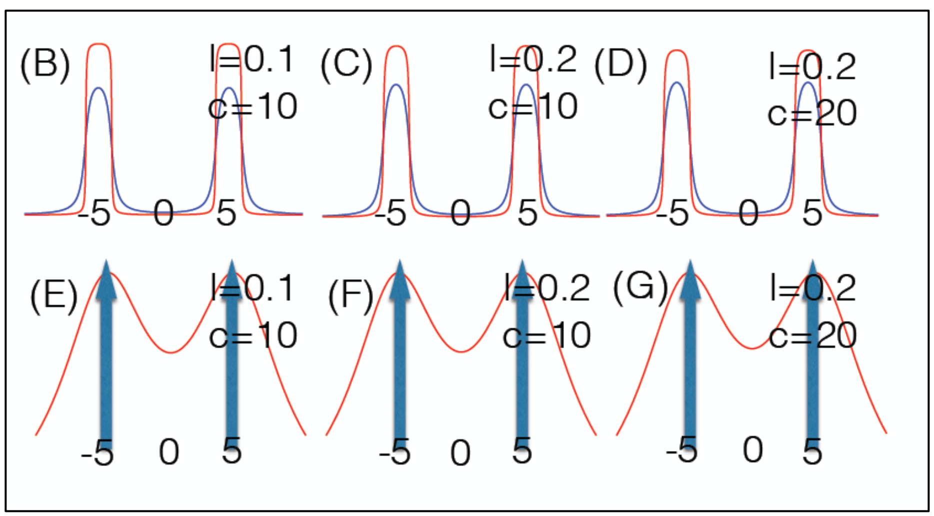
The velocity distribution:



$$P\left(v,t\right)=\int dx e^{-i\frac{v}{2}x}\left\langle b^{\dagger}\left(x\right)b\left(0\right)\right\rangle_{t}$$

ρ^{i} - ground state $\longrightarrow$ $P\left(v\right)\sim A_{L}\frac{\exp\left(-\frac{F_{L}}{\pi c}\right)}{2\pi}\sum_{i,j=\pm}\left(-1\right)^{j}\arctan A_{i,j}\left(v\right)$

ρ^{i} - BEC $\longrightarrow$ $P\left(v\right)\sim nB_{L}\frac{\exp\left(-\frac{G_{L}}{\pi c}\right)}{\pi}\left(\frac{H_{L}}{H_{L}^{2}+\frac{1}{4}\left(v-VK_{L}\right)}+\frac{H_{L}}{H_{L}^{2}+\frac{1}{4}\left(v+VK_{L}\right)}\right)$



(G Goldstein, NA '15)

$$A_{\pm\pm}(v)=C_{L}\left(\left(1-F_{L}\frac{\exp\left(-2F_{L}/\pi c\right)}{\pi c}\right)\left(\pm\frac{V}{2}\pm k_{F}\right)+\frac{v}{2}\right)\quad F_{L}=4k_{F}A_{L}\quad K_{L}=\left(1-G_{L}\frac{\exp\left(-2G_{L}/\pi c\right)}{\pi c}\right)\quad C_{L}=\frac{2\pi}{4k_{F}A_{L}\left(1+\exp\left(-\frac{2F_{L}}{\pi c}\right)\right)}$$

$$A_{L}=\frac{l}{L}\left(1+\frac{2k_{F}}{\pi c}\left(1-\frac{2l}{L}\right)\right)\quad B_{L}=\frac{l}{L}\frac{1}{\frac{1}{2\pi}+\frac{2N}{\pi cL}}\quad G_{L}=2nB_{L}\quad H_{L}=\frac{G_{L}}{2\pi}\left(1+\exp\left(-\frac{2G_{L}}{\pi c}\right)\right)+2n\left(1-G_{L}\frac{\exp\left(-2G_{L}/\pi c\right)}{\pi c}\right)$$

2. The Heisenberg Chain: Theory and Experiment

The XXZ Hamiltonian

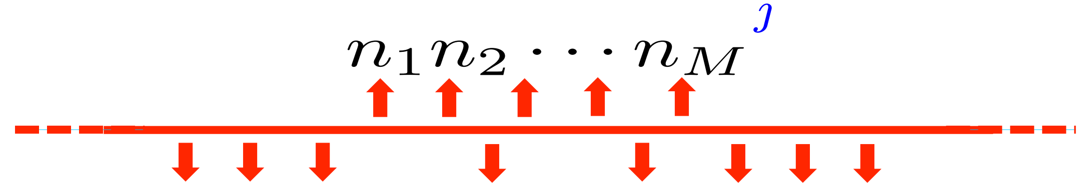
$$H = J\sum_j \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)$$

The phase diagram

Quench from initial states: *Recall GGE fails*

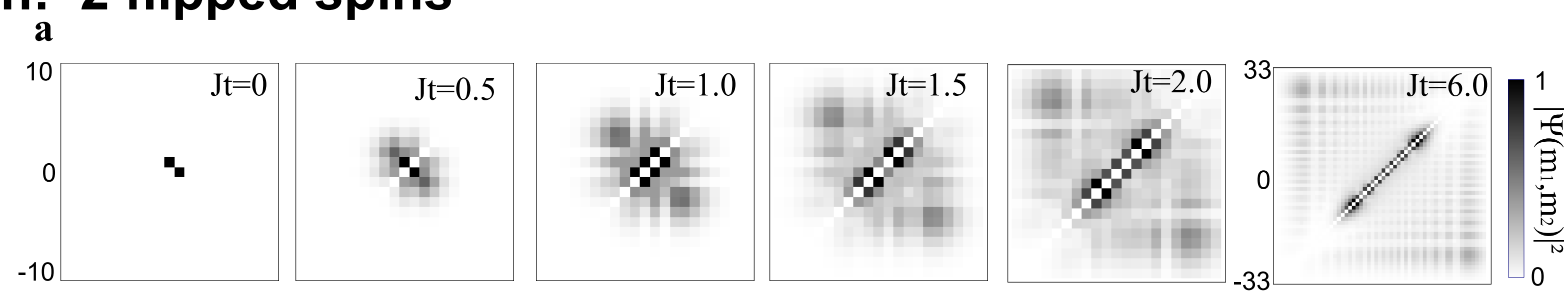
$$|\Psi_0\rangle = \theta(n_1 > n_2 > ... n_N) \prod_j \sigma_{n_j}^+ |0\rangle$$

e. g.



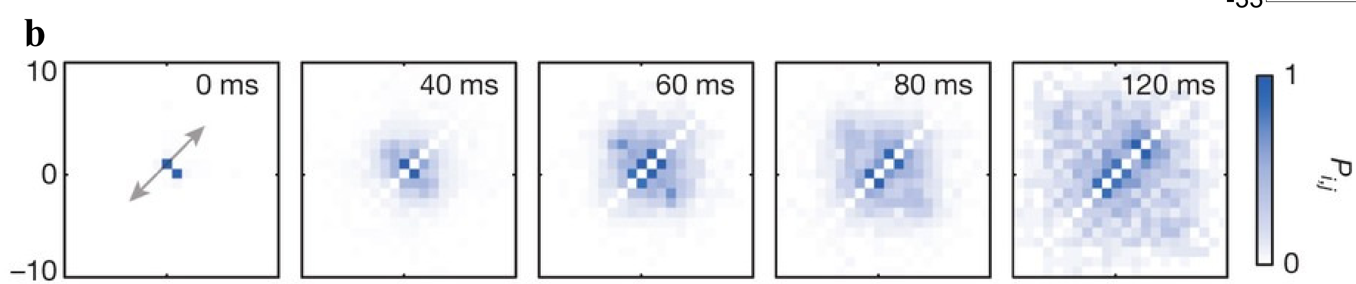
Time evolution: 2 flipped spins

Theory:



Experiment:

Munich group '13



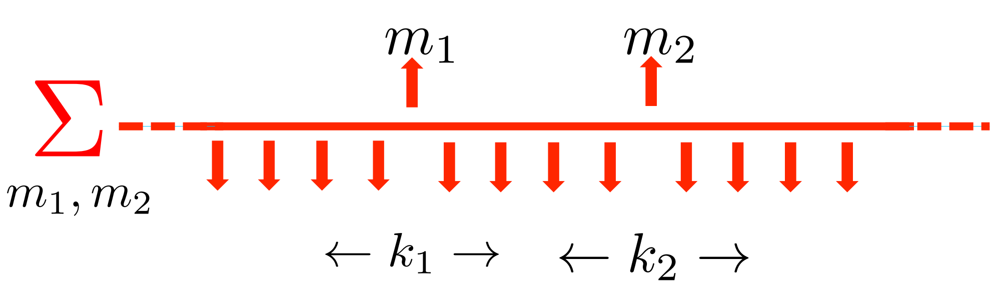
Eigenstates of the Heisenberg Chain

Eigenstates of the XXZ (M flipped spins)

$$|k\rangle = \sum_{\{m_j\}} \mathcal{S} \prod_{i<j} [\theta(m_i - m_j) + s(k_i, k_j)\theta(m_j - m_i)] \prod_j e^{ik_j m_j} \sigma^+_{m_j} |0\rangle$$

$$s(k_i, k_j) = e^{i\phi(k_i,k_j)} = -\frac{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_i}}{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_j}}$$

$$E = 4J \sum_{j=1}^{M} (\Delta - \cos k_j)$$



Time evolution of the XXZ magnet

i. Critical region $-1 < \Delta < 0$ $\qquad \Delta = -\cos\mu \quad (0 < \mu < \frac{\pi}{2})$

Reparametrize: $\Delta \to \mu, \quad k \to \alpha$

$$e^{ik} \to \frac{\sinh\frac{i\mu-\alpha}{2}}{\sinh\frac{i\mu+\alpha}{2}} \longrightarrow$$

$$s(k_1,k_2) \to \frac{\sinh(\frac{\alpha_1-\alpha_2}{2} - i\mu)}{\sinh(\frac{\alpha_1-\alpha_2}{2} + i\mu)}$$

$$E(k) \to E(\alpha) = \frac{4J\sin^2\mu}{\cosh\alpha - \cos\mu}$$

The contour expression of the initial state:

$$|\Psi_0\rangle = \theta(n_1 > n_2 > \ldots n_N)\prod_j \sigma^+_{n_j}|0\rangle$$

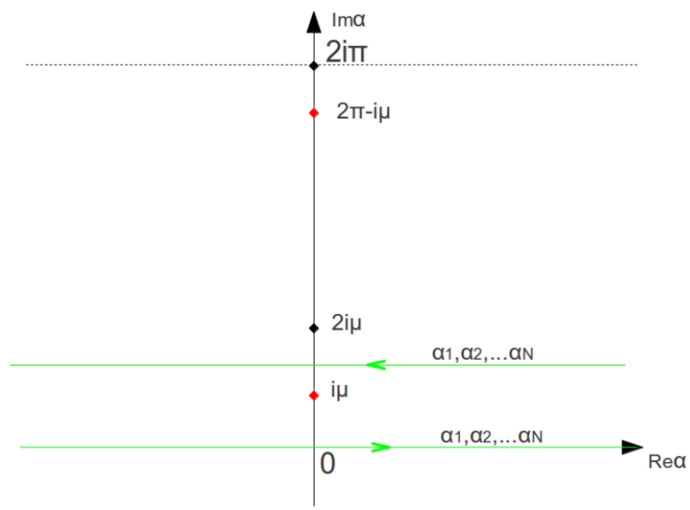
Expanded in terms of eigenstates

$$|\Psi_0\rangle = \sum_{\{m_j\}}\int_{\gamma_j}\prod_j\Big[\frac{d\alpha_j}{2\pi}\frac{\sin\mu}{2\sinh\frac{\alpha_j+i\mu}{2}\sinh\frac{\alpha_j-i\mu}{2}}\Big]\prod_j\Big[\frac{\sinh(\frac{i\mu-\alpha_j}{2})}{\sinh(\frac{i\mu+\alpha_j}{2})}\Big]^{m_j-n_j}$$

$$\times\prod_{i<j}\Big[\theta(m_i-m_j) + \frac{\sinh(\frac{\alpha_i-\alpha_j}{2} - i\mu)}{\sinh(\frac{\alpha_i-\alpha_j}{2} + i\mu)}\theta(m_j-m_i)\Big]\prod_j \sigma^+_{m_j}|0\rangle$$

Evolution of the XXZ magnet

The contour:



The time evolved state:

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j [\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}] \prod_{i<j} [\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)]$$

$$\times \prod_j [\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}]^{m_j-n_j} e^{-iE(\alpha_j)t} \sigma^+_{m_j} |0\rangle$$

Evolution of the XXZ magnet

ii. $\Delta < -1$ *Ferromagnetic regime*

$$\Delta = -\cosh\lambda \rightarrow \lambda > 0$$

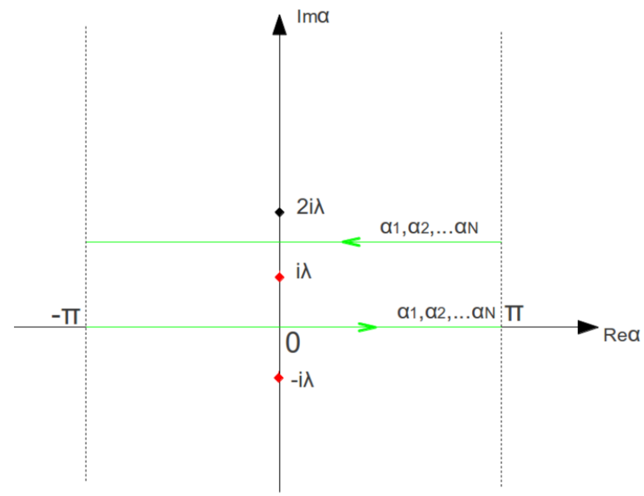
Reparametrize:

$$e^{ik} \rightarrow \frac{\sin\frac{i\lambda-\alpha}{2}}{\sin\frac{i\lambda+\alpha}{2}}$$

$$s(k_1,k_2) \rightarrow \frac{\sin(\frac{\alpha_1-\alpha_2}{2}-i\lambda)}{\sin(\frac{\alpha_1-\alpha_2}{2}+i\lambda)}$$

$$E(k) \rightarrow E(\alpha) = -\frac{4J\sinh^2\lambda}{\cos\alpha-\cosh\lambda}$$

The contour:



The time evolved state

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j [\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}] \prod_{i<j}[\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)]$$

$$\times \prod_j [\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}]^{m_j-n_j} e^{-iE(\alpha_j)t}\sigma^+_{m_j}|0\rangle$$

Evolution of the XXZ magnet

Some results - *local magnetization and bound states*

- *Spin currents*

Start from

$$|\Psi_0\rangle = \sigma^-_{-1}\sigma^-_0\sigma^-_{+1}|\Uparrow\rangle$$

Calculate:

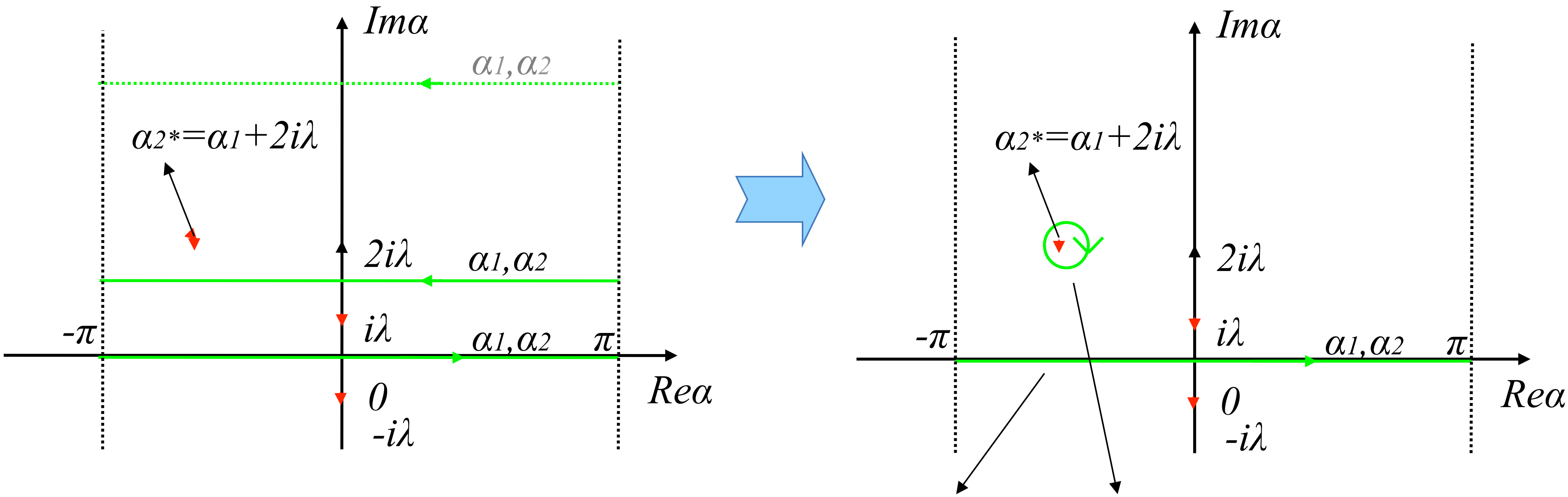
$$M(n,t) = \langle\Psi(t)|\sigma^z_n|\Psi(t)\rangle$$

$$I(n,t) = i\langle\Psi(t)|(\sigma^+_n\sigma^-_{n+1} - \sigma^-_n\sigma^+_{n+1})|\Psi(t)\rangle$$

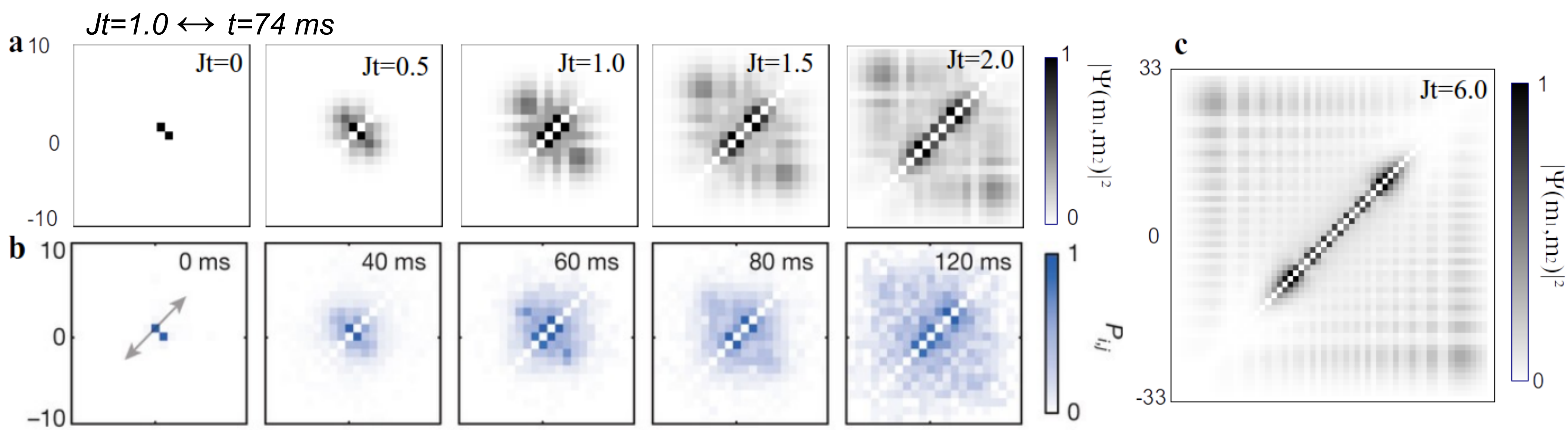
For different values of anisotropy Δ

- as the anisotropy increases the weight of the bound states increases

Contour Shift and Bound States



$$\Psi^{1,0}(m_1, m_2; t) = \Psi_{magn}(m_1, m_2; t) + \Psi_{bound}(m_1, m_2; t)$$

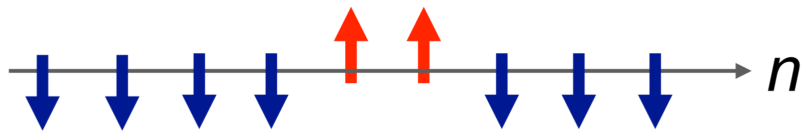


b. T. Fukuhara et al, Nature **502**, 76 (2013)

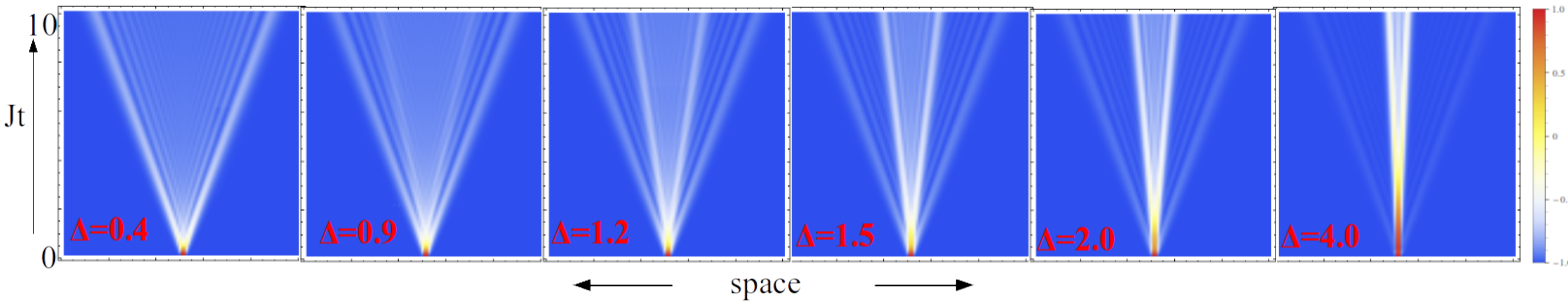
Observables

- **Local Magnetization** $M(n,t) \equiv \langle \Psi(t) | \sigma_n^z | \Psi(t) \rangle$

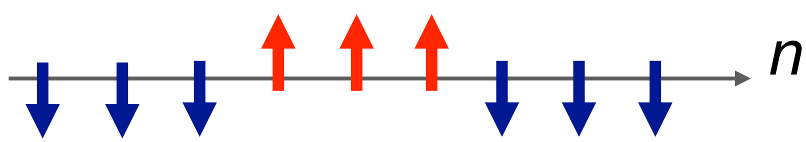
$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ | \Downarrow \rangle =$

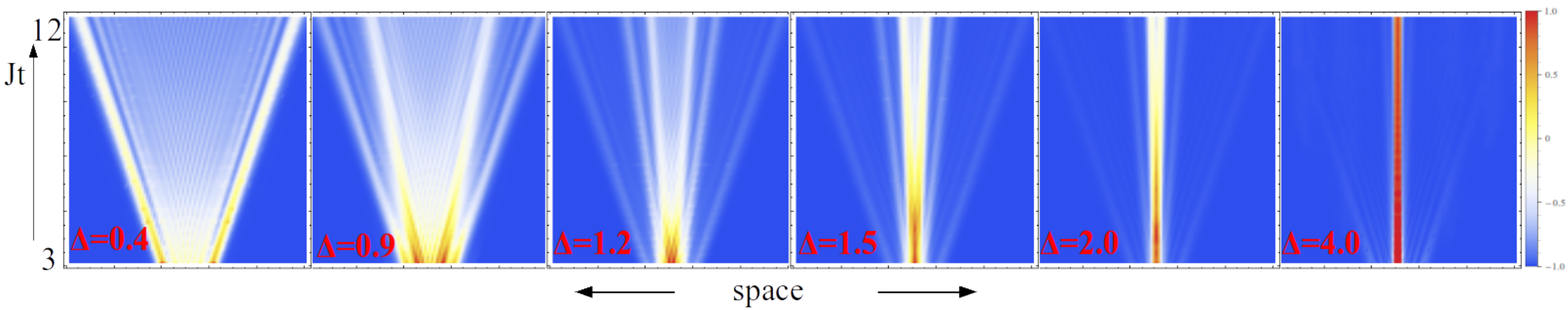


(cf. Ganahl et al. '12)



$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ \sigma_{-1}^+ | \Downarrow \rangle =$





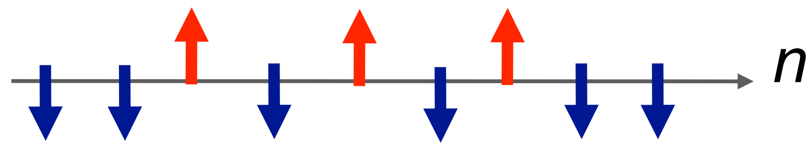
$$\frac{v_{bound}}{v_{magn}} = \frac{\sin \mu}{\sin(n\mu)} (|\Delta| = \cos \mu)$$

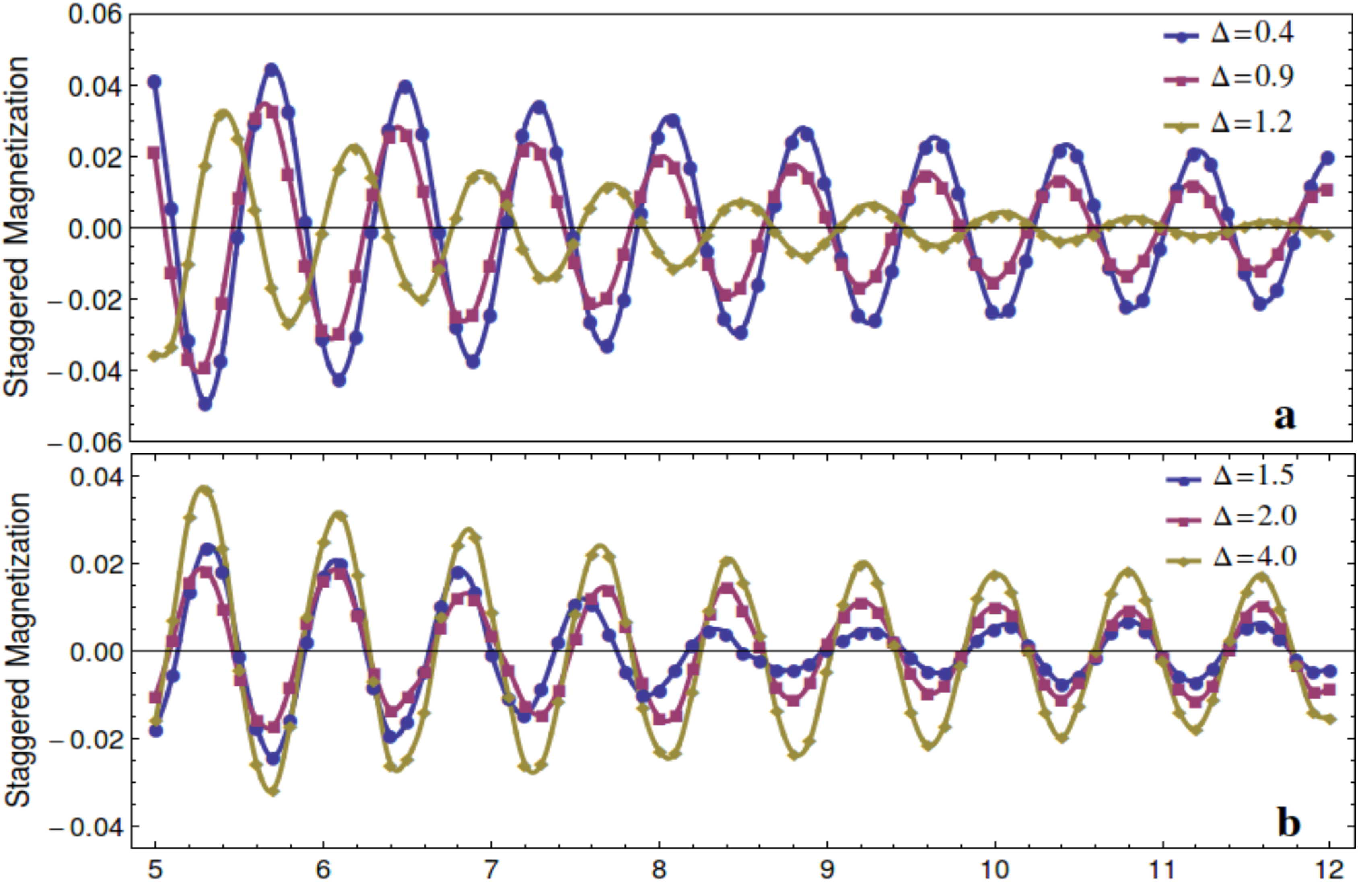
$$\frac{v_{bound}}{v_{magn}} = \frac{\sinh \lambda}{\sinh(n\lambda)} (|\Delta| = \cosh \lambda)$$

- *Staggerd Magnetization (Order Parameter)* $$M_s(t) = \frac{1}{N}\sum_n (-1)^n \langle \sigma_n^z \rangle (t)$$

Quench across a QCP $\Delta = \infty \rightarrow |\Delta| < 1$

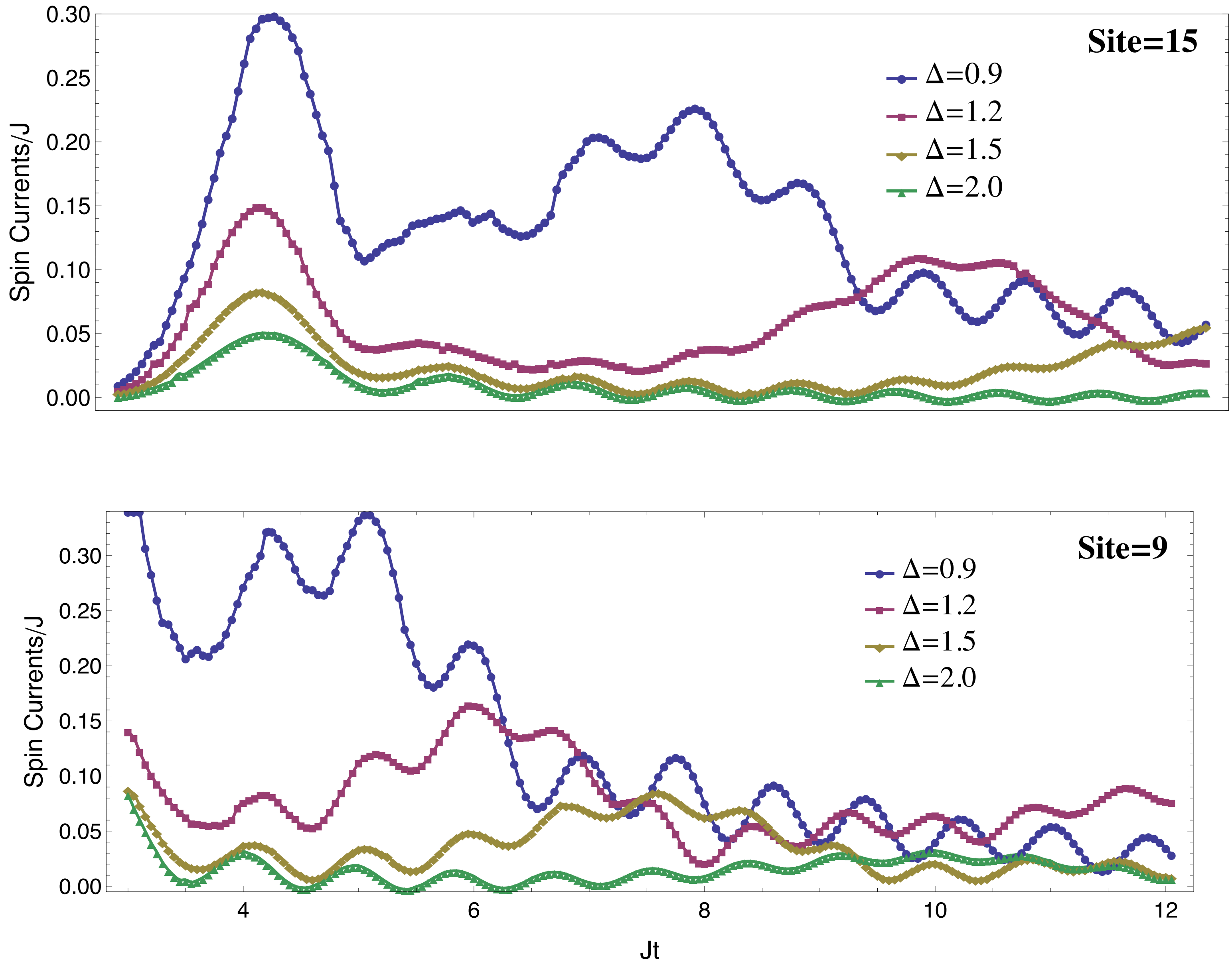
$$|\Psi_0\rangle = \sigma \quad \sigma_0^+ \sigma \quad {}_2 | \Downarrow \rangle$$





Evolution of the XXZ magnet

Spin currents - evolution



The Gaudin-Yang Model

- The Gaudin-Yang Hamiltonian (Gaudin '67 Yang '67)

$$H_{GY} = \sum_{\sigma=\uparrow,\downarrow} \int_x \partial_x \Psi^\dagger_\sigma(x) \partial_x \Psi_\sigma(x) + c \int_x \Psi^\dagger_\uparrow(x) \Psi^\dagger_\downarrow(x) \Psi_\downarrow(x) \Psi_\uparrow(x)$$

Describes *spin-1/2* fermions/bosons with short range interactions, e.g. ^{6}Li or ^{40}K

- The *N*-particle eigenstates with *M* spins down and *N-M* spins up

$$|\mu, k\rangle = \int F^{\mu,k}_{a_1 \dots a_N}(x_1 \dots x_N) \underbrace{\Psi^\dagger_{a_1}(x_1) \dots \Psi^\dagger_{a_N}(x_N)|0\rangle}_{|x,\alpha\rangle}$$

labels of the down spins

$a_i = \downarrow$ if $i \in \{\alpha\}$, $a_i = \uparrow$ if $i \notin \{\alpha\}$

takes the form

$$= \sum_{P \in S_N, R \in S_M} \int_x \sum_\alpha (-1)^P e^{i(Pk)_i x_i} \prod_{i<j} S^{\alpha,R}(\mu_i - \mu_j) \prod_{i=1}^{M} I(\mu_i, Pk, \alpha_{R^{-1}i}) \theta(\alpha) \theta(x) |x, \alpha\rangle$$

where:

$$I(\mu, k, \alpha) = \frac{ic}{\mu - k_\alpha + ic/2} \prod_{n<\alpha} \frac{\mu - k_n - ic/2}{\mu - k_n + ic/2}$$

with

$$\theta(\alpha) = \theta(\alpha_1 < \dots < \alpha_M)$$

$$\theta(x) = \theta(x_1 < \dots < x_N)$$

$$S^{\alpha,R}(\mu_i - \mu_j) = \frac{\mu_i - \mu_j + ic\,\mathrm{sgn}(\alpha_{R^{-1}i} - \alpha_{R^{-1}j})}{\mu_i - \mu_j - ic}$$

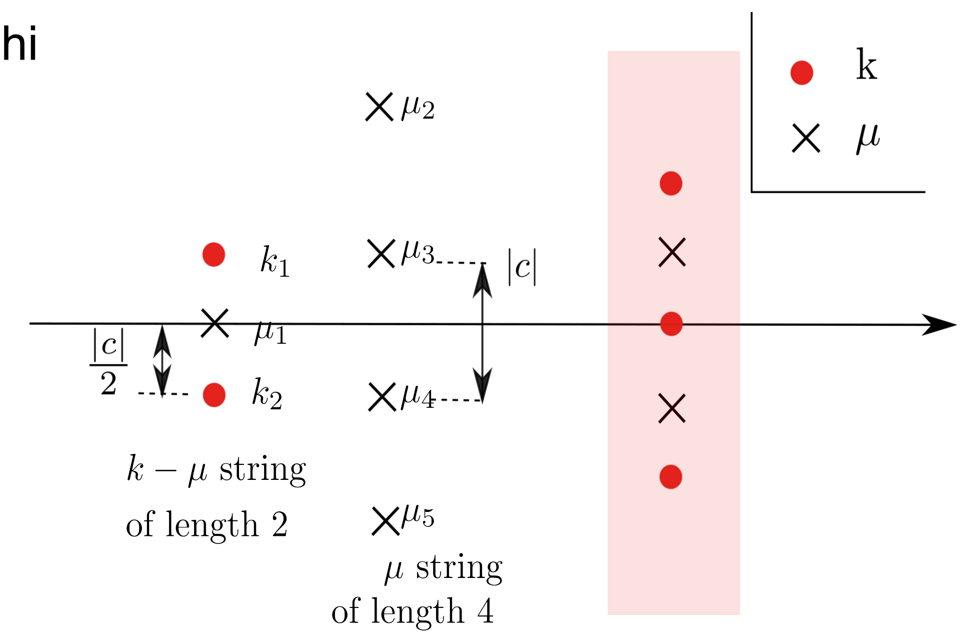
The Gaudin-Yang Model

Imposing periodic boundary conditions leads to the BAE:

$$e^{ik_n L} = \prod_{j=1}^{M} \frac{k_n - \mu_j + ic/2}{k_n - \mu_j - ic/2}$$

$$\prod_{j\neq i} \frac{\mu_i - \mu_j + ic}{\mu_i - \mu_j - ic} = \prod_{n} \frac{\mu_i - k_n + ic/2}{\mu_i - k_n - ic/2}$$

Solutions - *String Hypothesis:* M. Takahashi



- $c > 0:$

μ-string $\quad \mu = \bar{\mu} + ic(n/2 + 1/2 - j)$ for $j = 1, ..., n$

- $c < 0$

μ string

$k - \mu$ string of length 2 $k_{1,2} = \mu \pm ic/2$

- Strings as bound states

$k_1 = k_2 - ic$ Wavefunction consists of $\exp[-|c|(x_1 - x_2)]$

$\mu_1 = \mu_2 + ic$ Wavefunction consists of $\prod_{\alpha_2 < j < \alpha_1} \frac{\mu - k_j}{\mu - k_j + ic}$

Yudson Approach

$$S_R(\mu_m - \mu_n) = \frac{\mu_m - \mu_n - ic\epsilon \mathrm{sgn}(\alpha_{R^{-1}m} - \alpha_{R^{-1}n})}{\mu_m - \mu_n + ic\epsilon}$$

$$I(\mu, \alpha, \vec{k}\rangle = \frac{\epsilon ic}{\mu - k_\alpha - ic\epsilon} \prod_{j<\alpha} \frac{\mu - k_j + \epsilon ic}{\mu - k_j - \epsilon ic}$$

- Identity resolution

$$I = \int_C dk \int_{C'} d\mu \, |k, \mu\rangle(k, \mu|$$

- Bethe Ansatz state

$$|\mu, k\rangle = \sum_{\substack{P\in S_N \\ R\in S_M}} \int_x \sum_\alpha e^{i\sum_i k_{Pi} y_i} (-1)^P \prod_{m<n} S_R(\mu_m - \mu_n) \prod_m I(\mu_m, \alpha_{R^{-1}m}, Pk)\theta(x)\theta(\alpha)|x, \alpha\rangle$$

- Yudson State $(P = I \; R = I)$

$$|\mu, k) = \int_x \sum_\alpha e^{i\sum_j k_{Pj} y_j} \prod_m I(\mu_m, \alpha_m, k)\theta(x)\theta(\alpha)|x, \alpha\rangle$$

- Equivalent to eigenstate expansion

$$\sum_{k,\mu} |\mu, k\rangle(\mu, k|\theta(\mu)\theta(k) = \sum_{\mu,k} |\mu, k\rangle(\mu, k|$$

$$|\mu, k\rangle = \sum_{P,R} S_{k,\mu}(P, R)|R\mu, Pk)$$

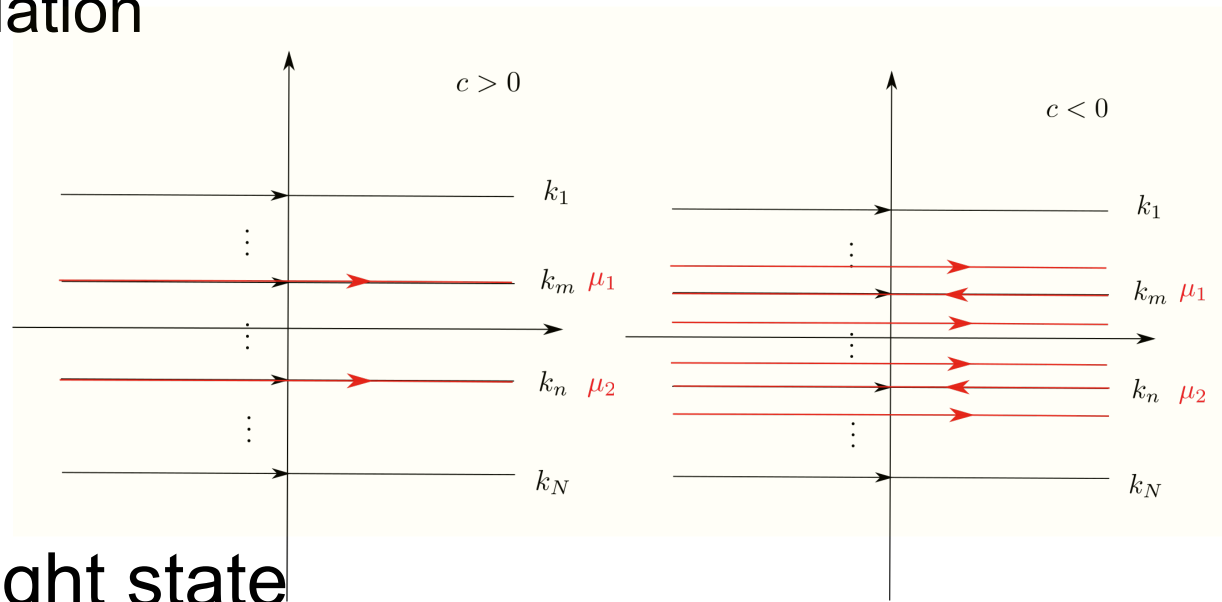
$$S_{k,\mu}(P, R) = (-1)^P \prod_{i<j} S_R(\mu_m - \mu_n)$$

$$\sum_{R,P} |\mu, k\rangle(\mu, k|\theta(\mu)\theta(k) = \sum_{R,P} |R\mu, Pk\rangle(R\mu, Pk|\theta(\mu)\theta(k)$$

The Contour Integral

- Infinite system
  - No need to solve Bethe equation
  - Normalization simplified

Illustration of integration contour for k and μ α_1=m $\alpha_2 = n$



- Includes non-highest weight state
  - States with finite parameter are highest weight *(Braak Andrei 98)*
  - Spin lowering operator

- Includes string solutions contribution in contour integral
- How to obtain the contour? Guess and prove

Central theorem I

$$Const \int_C dk \int_{C'} d\mu \, \langle y, \alpha | \underbrace{k, \mu \rangle (k, \mu}_{= I?} | x \beta \rangle \, \theta(\alpha)\theta(\beta) = \prod_i^N \delta(x_i - y_i) \prod_m^M \delta(\alpha_m - \beta_m)$$

- Single impurity (one down spin) $c > 0$, explicitly:

$$Const \int_C dk \sum_P (-1)^P e^{i \sum_j k_j (y_{P^{-1}j} - x_j)} \int_{C'} d\mu J(\mu) \, \theta(y)\theta(x)\theta(\alpha)\theta(\beta) = \prod_i \delta(y_i - x_i)\delta(\alpha - \beta)$$

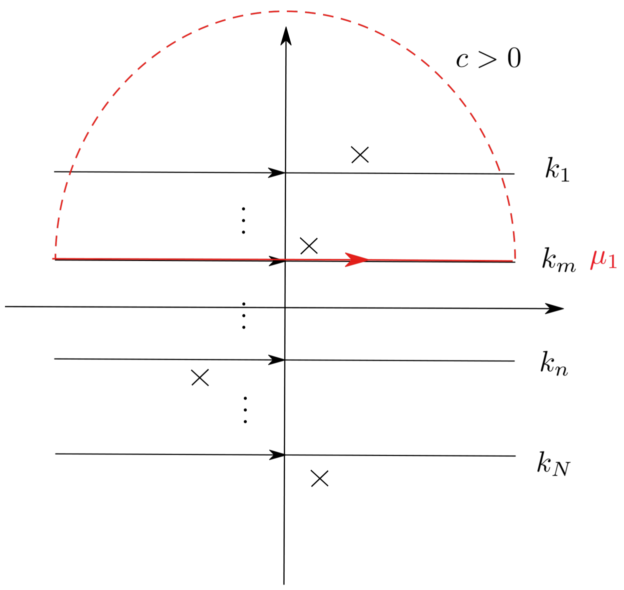
$$J(\mu) = \frac{-ic}{\mu - k_{P\alpha} + ic/2} \prod_{\substack{m<\alpha \\ Pm \geq \beta}} \frac{\mu - k_{Pm} - ic/2}{\mu - k_{Pm} + ic/2} \frac{ic}{\mu - k_\beta - ic/2} \prod_{\substack{n<\beta \\ P^{-1}n \geq \alpha}} \frac{\mu - k_n + ic/2}{\mu - k_n - ic/2}$$

- Properties of $J(\mu)$:

$J(\mu) = O(1/\mu^2)$ *for large μ*

$$\int_C d\mu J(\mu) = 2\pi i \sum_{\substack{o \leq \beta \\ P^{-1}o \geq \alpha}} R(k_o + ic/2)$$

$$R(k_o + ic/2) = \begin{cases} -ic + O(\frac{1}{k_o}) & \text{if } k_o = k_{P\alpha} = k_\beta \\ O(\frac{1}{k_o}) & \text{if } k_o = k_{P\alpha} \neq k_\beta \\ & \text{ or } k_o = k_\beta \neq k_{P\alpha} \\ O(\frac{1}{k_o^2}) & \text{In other cases} \end{cases}$$



Central Theorem II

$$J(\mu) = \frac{-ic}{\mu - k_{P\alpha} + ic/2} \prod_{\substack{m<\alpha \\ Pm\geq\beta}} \frac{\mu - k_{Pm} - ic/2}{\mu - k_{Pm} + ic/2} \frac{ic}{\mu - k_\beta - ic/2} \prod_{\substack{n<\beta \\ P^{-1}n\geq\alpha}} \frac{\mu - k_n + ic/2}{\mu - k_n - ic/2}$$

$$\int_C d\mu J(\mu) = 2\pi i \sum_{\substack{o\leq\beta \\ P^{-1}o\geq\alpha}} R(k_o + ic/2)$$

$$R(k_o + ic/2) = \begin{cases} -ic + O(\frac{1}{k_o}) & \text{if } k_o = k_{P\alpha} = k_\beta \\ O(\frac{1}{k_o}) & \text{if } k_o = k_{P\alpha} \neq k_\beta \\ & \text{ or } k_o = k_\beta \neq k_{P\alpha} \\ O(\frac{1}{k_o^2}) & \text{In other cases} \end{cases}$$

Now need carry out *k*-integrals

$$Const \int_C \sum_P (-1)^P e^{i\sum_j k_j (y^{-1}_P j - x_j)} \sum_{o\leq\beta; P^{-1}o\geq\alpha} R(k_o + ic/2)\ \theta(y)\theta(x)\theta(\alpha)\theta(\beta)$$

- k_j contour closed in the upper plane if $y_{P^{-1}i} > x_j$
  k_j contour closed in the lower plane if $y_{P^{-1}i} < x_j$

- Partial fraction decomposite into polynomial terms $E(k_1, \ldots, k_N)$

$$E(k) \equiv E(k_1, \ldots, k_N) = \prod_{\{m\}} \frac{-ic}{k_o - k_{Pm} + ic} \prod_{\{n\}} \frac{ic}{k_o - k_n}$$

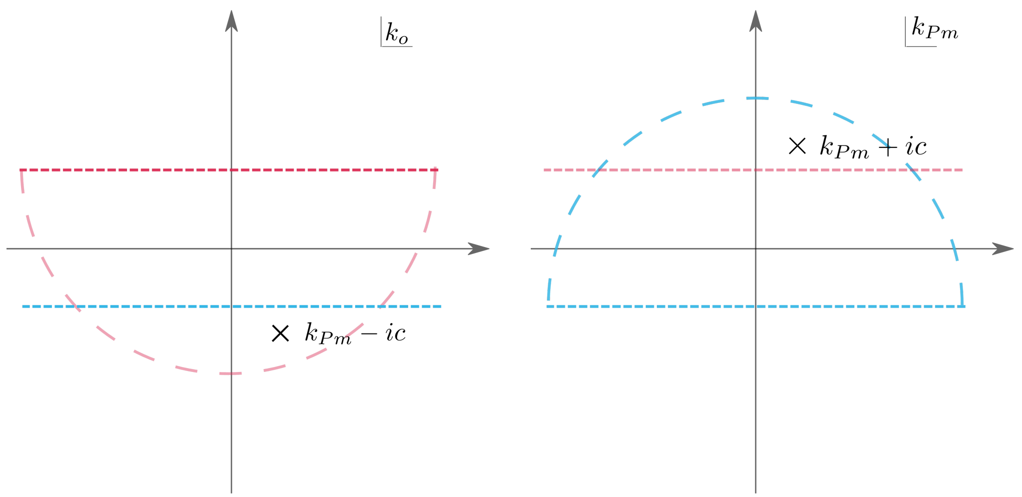
Two type of poles

1. $k_o = k_n$ ✕ Wavefunction is antisymmetry in ks
2. $k_o = k_{Pm} - ic$ ✕ Pole lies outside enclosed contour

$Pm > \beta > o$
$m < \alpha < P^{-1}o$

Central Theorem III

$m < \alpha < P^{-1}o \qquad Pm > \beta > o$



If $k_{Pm} - ic$ appears in the k_o integration,
then, $y_{P^{-1}o} < x_o$
If $k_o + ic$ appears in the k_{Pm} integration,
then, $y_m > x_{Pm}$

Since $x_o < x_{Pm}$, this means $y_{P^{-1}o} < y_m$
This conflicts with $P^{-1}o > m$
No pole contributes to the k integral

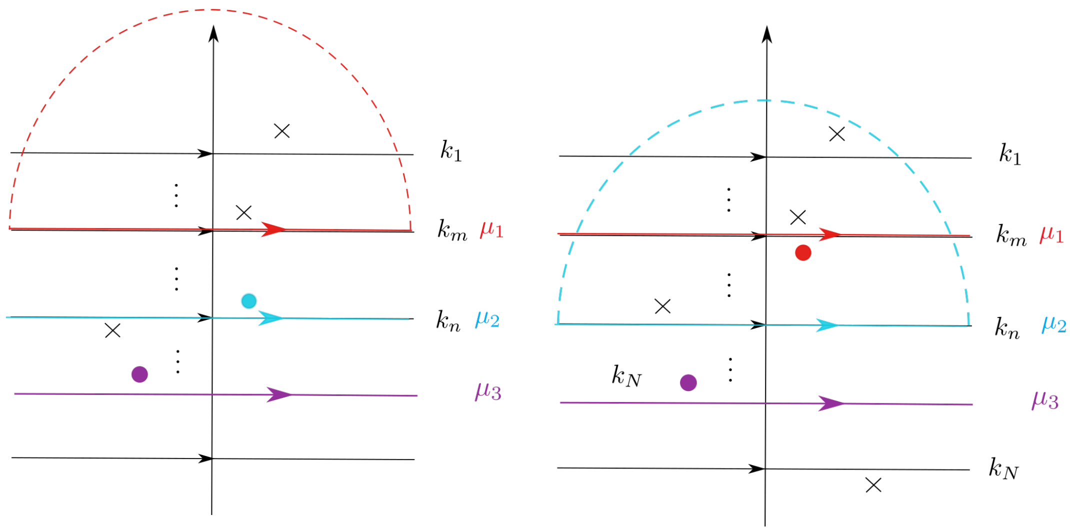
Central Theorem Proved

$$Const \int_C dk \sum_P (-1)^P e^{i\sum_j k_j (y_{P\text{-}1j} - x_j)} \int_{C'} d\mu J(\mu)\ \theta(y)\theta(x)\theta(\alpha)\theta(\beta) = \prod_i \delta(y_i - x_i)\delta(\alpha - \beta)$$

Central Theorem IV

- Multiple impurities with $c > 0$

$$\int_C dk \int_{C'} d\mu \sum_{P,R} (-1)^P e^{i\sum_i k_i (y_{P^{-1}i} - x_i)} \prod_{m<n} S(\mu_m - \mu_n) \prod_{m=1}^{M} J(\mu_m)\theta(x)\theta(y)\theta(\alpha)\theta(\beta) = \prod_i \delta(y_i - x_i) \prod_m \delta_{\beta_m \alpha_m}$$



Close μ_1 contour in the upper plane

Pole at $\mu_1 = k_o + ic/2,\ o < \beta_1,\ P^{-1}o > \alpha$

Close μ_2 contour in the upper plane

Pole at $\mu_2 = k_u + ic/2,\ u < \beta_2,\ P^{-1}u > \alpha_{R^{-1}2}$

✕ $\mu_2 = \mu_1 - ic = k_o - ic/2$ if $\alpha_{R^{-1}1} > \alpha_{R^{-1}2}$

$o < \beta_1 < \beta_2$

$P^{-1}o > \alpha_{R^{-1}1} > \alpha_{R^{-1}2}$

$\frac{\mu_2 - k_o + ic/2}{\mu_2 - k_o - ic/2}$ is factor of $J(\mu_2)$.

This cancels the pole @ $\mu_2 - k_o + ic/2$

All poles of μ come from $J(\mu)$ not $S(\mu_m - \mu_n)$

Only $k_o = k_{Pm} - ic$ type of pole for k integration

- Central theorem follows

Central Theorem V

- Attractive Interaction

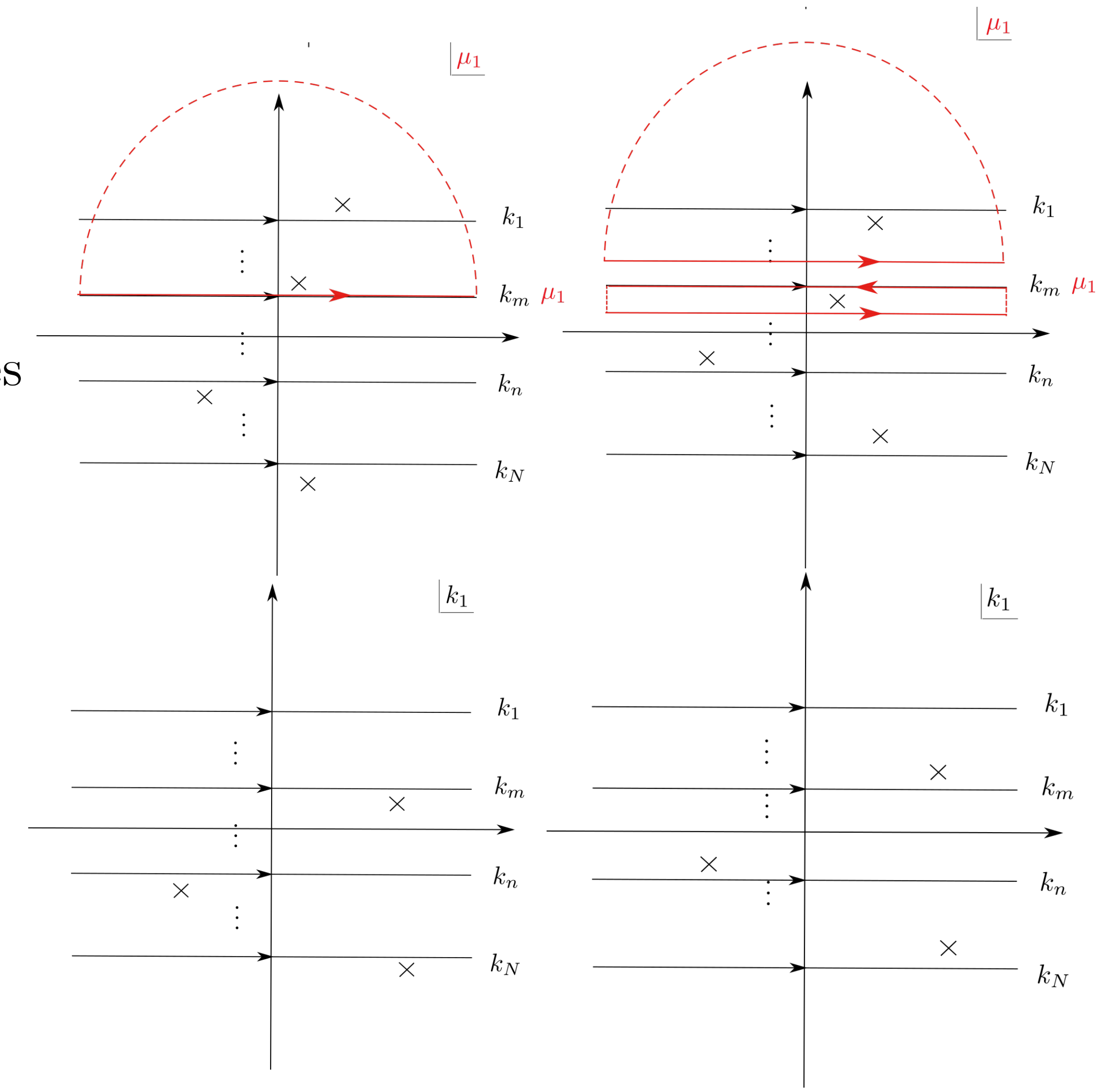
$$\mu = k_o + ic/2$$

$$k_o = k_{Pm} - ic$$

Contribution from same set of poles

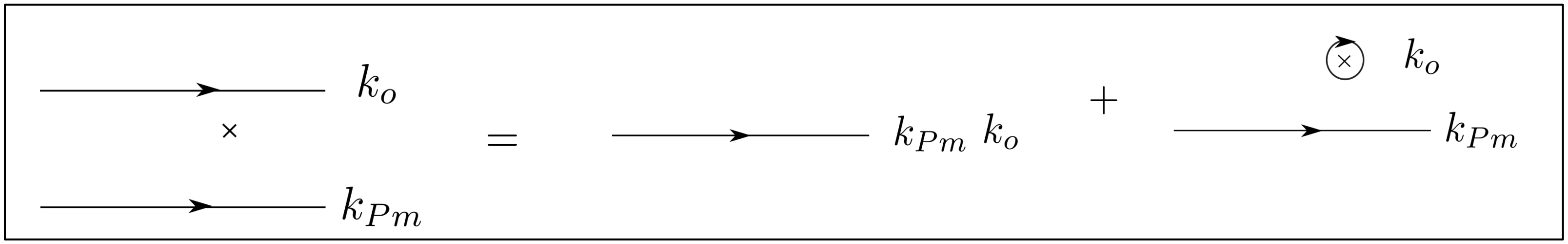
Not true for different contours

- *Central Theorem Proven as before*



$c > 0$ $c < 0$

Bound States I



$k_o = k_{Pm} - ic$

$\mu = k_o + ic/2$

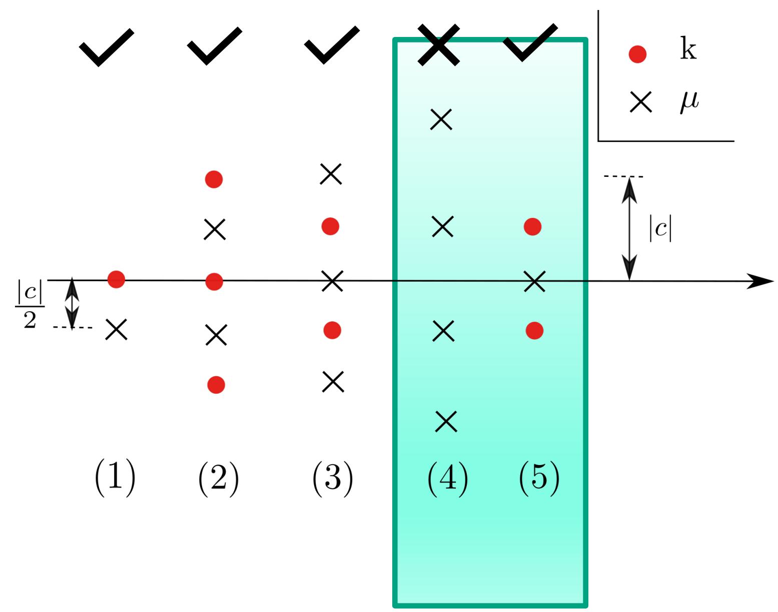
$$\int_C dk f(k_o, k_{Pm}) = \int dk f(k_o, k_{Pm}) - 2\pi i \int dk_{Pm} \text{Res}(f(k_o, k_{Pm}), k_o = k_{Pm} - ic)$$

- Extract string solution

- Integrate out μ first, $\mu = k_o + ic/2$
- Shift all k contours to real axis
- A pole = a string solution

  $k_o + ic/2 = \mu = k_{Pm} - ic/2$
- Each μ connects at most two ks
- Short string may connect into longer strings with more μs
- For $c > 0$, $k - \mu$ pair

  For $c < 0$, $k - \mu$ pair, $k - \mu$ string and $\mu - k$ string of any length



String solution prediction

Bound States II

- Interpretation as bound states

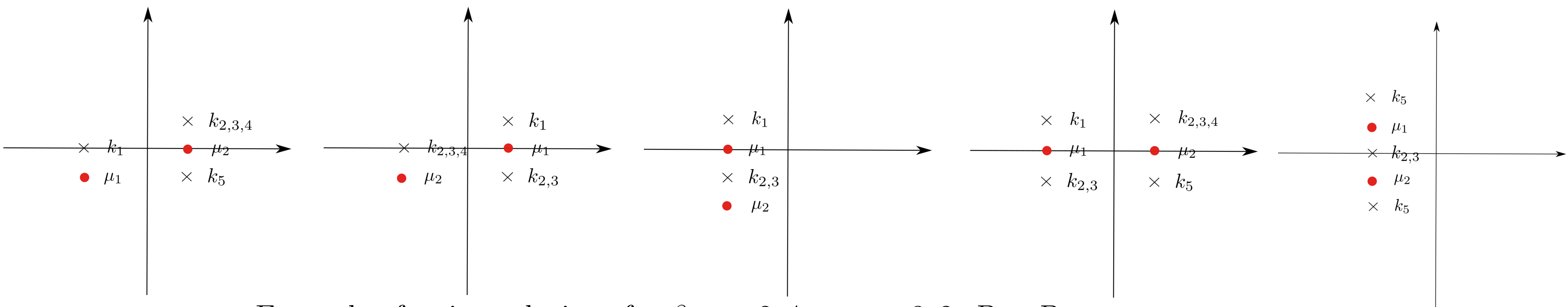
$k-\mu$ pair: no physical significance
depends on how contour is closed

$k-\mu$ strings of length n: n-particles and (n-1)-magnon bound state

$\mu-k$ pair of length m: Composite of $k-\mu$ pairs of length $m-1$ and $k-\mu$ pairs

- Complete Basis

All $k-\mu$ pair, $k-\mu$ string and composite of the two

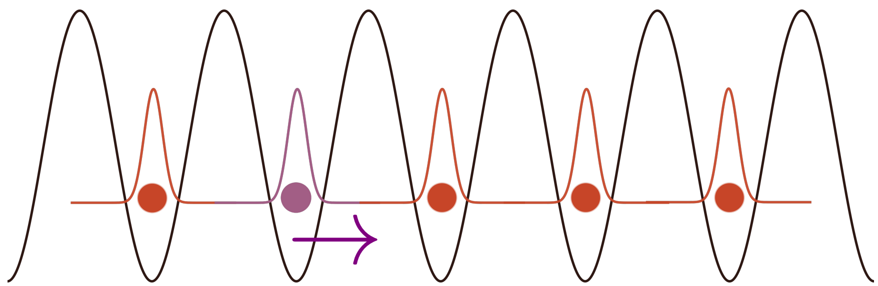


Example of string solutions for $\beta_{1,2}=2,4$, $\alpha_{1,2}=3,2$, $P=P_{15}$.

Time Evolution I

Initial state $$|\phi_0\rangle = \left(\frac{1}{\pi\sigma^2}\right)^{N/4} \int dx' e^{\left(-\sum_j^N \frac{(x'_j - ja)^2}{2\sigma^2} + ik_0 x'_\beta\right)} |x', \beta\rangle$$

$|\phi_0\rangle$



$|x'_i - x'_j| > 3\sigma$

$$|\phi(t) = e^{-iH_{GY}t}|\phi_0\rangle = \int_C dk \int_{C'} d\mu \, e^{-iE(k,\mu)t} \, |k, \mu\rangle (k, \mu|\phi_0\rangle$$

$$= \frac{(4\pi\sigma^2)^{\frac{N}{4}}}{(2\pi)^{M+N}c^M} \int dy \sum_\alpha \int_C dk \int_{C'} d\mu \sum_{P,R} (-1)^P e^{-i\sum_i k_i^2 t}$$

$$e^{-\sum_i k_i^2\sigma^2/2 + i\sum_i k_i (y_{P^{-1}i} - x_i)} \prod_{m<n} S^{\alpha,R}(\mu_m - \mu_n) \prod_m J(\mu_m)\theta(y)\theta(\alpha)|y, \alpha\rangle + O(e^{-\frac{a^2}{4\sigma^2}})$$

with

$$S^{\alpha,R}(\mu_m - \mu_n) = \frac{\mu_m - \mu_n + ic\,\mathrm{sgn}(\alpha_{R^{-1}m} - \alpha_{R^{-1}n})}{\mu_m - \mu_n - ic}$$

$$J(\mu) = \frac{-ic}{\mu - k_{P\alpha} + ic/2} \prod_{\substack{m<\alpha \\ Pm\geq\beta}} \frac{\mu - k_{Pm} - ic/2}{\mu - k_{Pm} + ic/2} \frac{ic}{\mu - k_\beta - ic/2} \prod_{\substack{n<\beta \\ P^{-1}n\geq\alpha}} \frac{\mu - k_n + ic/2}{\mu - k_n - ic/2}$$

Time Evolution II

- Two particle dynamics

- Closed form wave function (same expression for *c>0* and *c<0*)

$$f_{\uparrow,\downarrow}(y_1,y_2,t) = \frac{\sigma}{2\sqrt{\pi}i(t+\sigma^2/2i)}\left(e^{\frac{i(y_2-x_1)^2}{4(t+\sigma^2/2i)}+\frac{i(y_1-x_2)^2}{4(t+\sigma^2/2i)}}(1-\frac{c}{2}(1+i)\theta(y_2-y_1)\sqrt{\pi(t+\sigma^2/2i)}\mathrm{erfc}(\alpha_1(t))e^{\alpha_1(t)^2}\right)$$

$$-e^{\frac{i(y_1-x_1)^2}{4(t+\sigma^2/2i)}+\frac{i(y_2-x_2)^2}{4(t+\sigma^2/2i)}}c/2(1+i)\theta(y_1-y_2)\sqrt{\pi}\sqrt{(t+\sigma^2/2i)}\,\mathrm{erfc}(\alpha_2(t))e^{\alpha_2(t)^2}$$

with

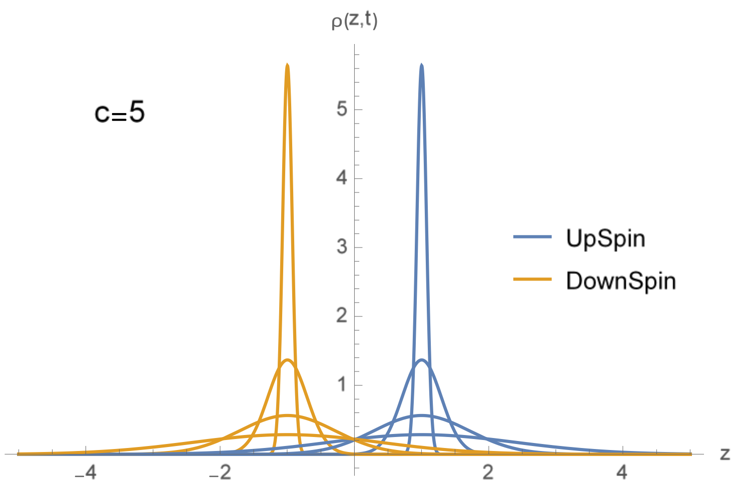
$$\alpha_1(t) = \frac{(1-i)(y_2-x_1-y_1+x_2+2ic(t+\sigma^2/2i)}{4\sqrt{(t+\sigma^2/2i)}} \qquad \alpha_2(t) = \frac{(1-i)(y_1-x_1-y_2+x_2+2ic(t+\sigma^2/2i)}{4\sqrt{(t+\sigma^2/2i)}}$$

- Density $\quad \langle\rho_\uparrow(z)\rangle = \int dy|f_{\uparrow,\downarrow}(z,y)|^2 \quad \langle\rho_\downarrow(z)\rangle = \int dy|f_{\uparrow,\downarrow}(y,z)|^2$

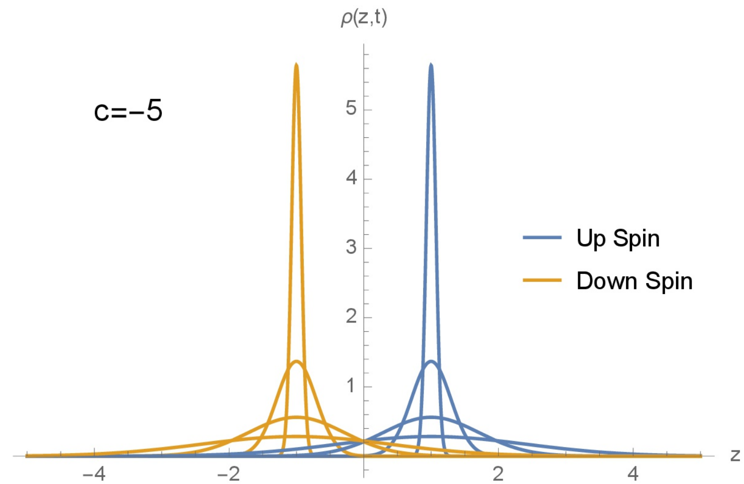
- Normalized noise function $\quad \langle\rho_\uparrow(z)\rho_\downarrow(z')\rangle = |f_{\uparrow\downarrow}(z,z')|^2$

$$C(z/t,-z/t,t) = \frac{\langle\rho_\uparrow(z/t)\rho_\downarrow(-z/t)\rangle}{\langle\rho_\uparrow(z/t)\rangle\langle\rho_\downarrow(-z/t)\rangle} - 1 = \langle\frac{\delta\rho_\uparrow(z/t)\delta\rho_\downarrow(-z/t)\rangle}{\langle\rho_\uparrow(z/t)\rangle\langle\rho_\downarrow(-z/t)\rangle}$$

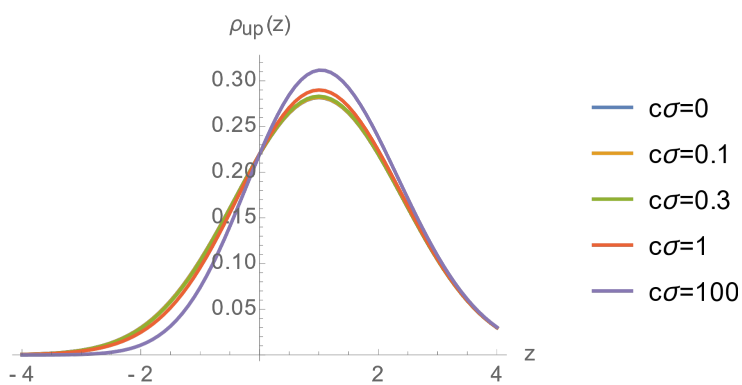
- Density Evolution



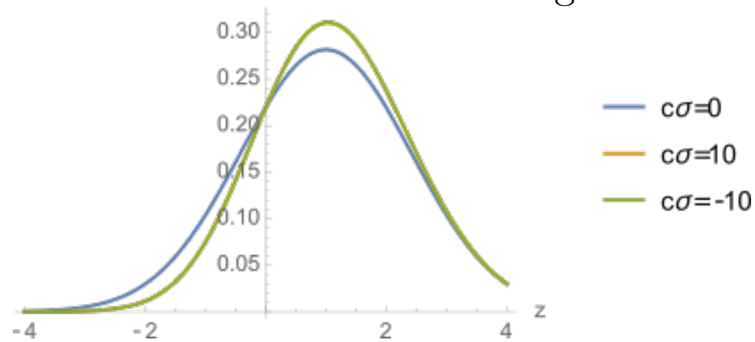
(a) Time evolution for repulsive gases



(b) Time evolution for attractive gases



(c) Comparison of the density of the upspin for repulsive gases with different interaction strength



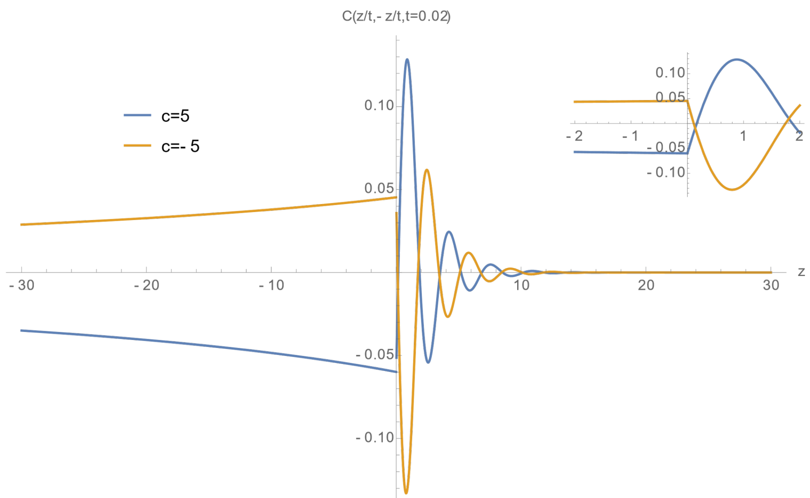
(d) Comparison of the density of the upspin for repulsive, attractive and free models

Density profile depends weakly on the sign of interaction. This indicates the missing of bound states.

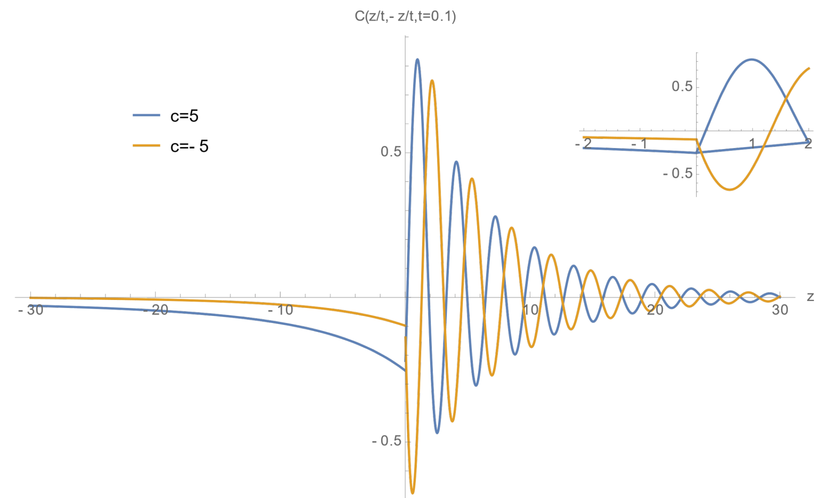
Noise I

- Normalized noise function

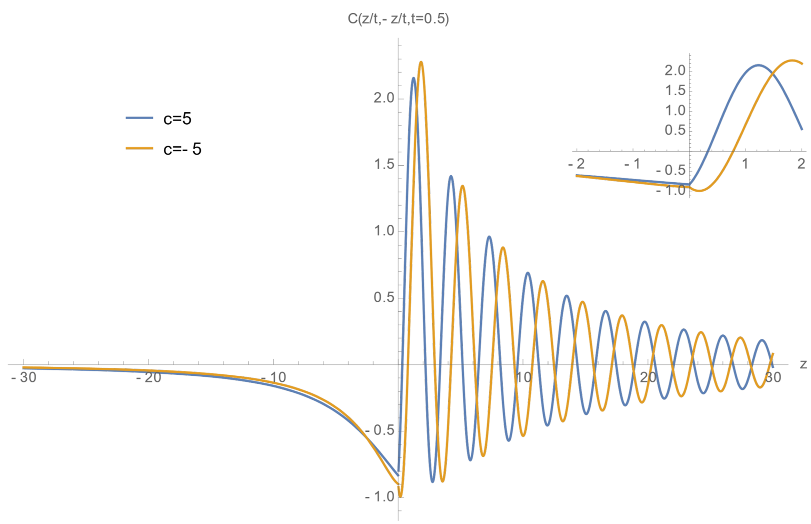
$$C(z/t,-z/t,t)=\frac{\langle\rho_{\uparrow}(z/t)\rho_{\downarrow}(-z/t)\rangle}{\langle\rho_{\uparrow}(z/t)\rangle\langle\rho_{\downarrow}(-z/t)\rangle}-1=\langle\frac{\delta\rho_{\uparrow}(z/t)\delta\rho_{\downarrow}(-z/t)\rangle}{\langle\rho_{\uparrow}(z/t)\rangle\langle\rho_{\downarrow}(-z/t)\rangle}$$



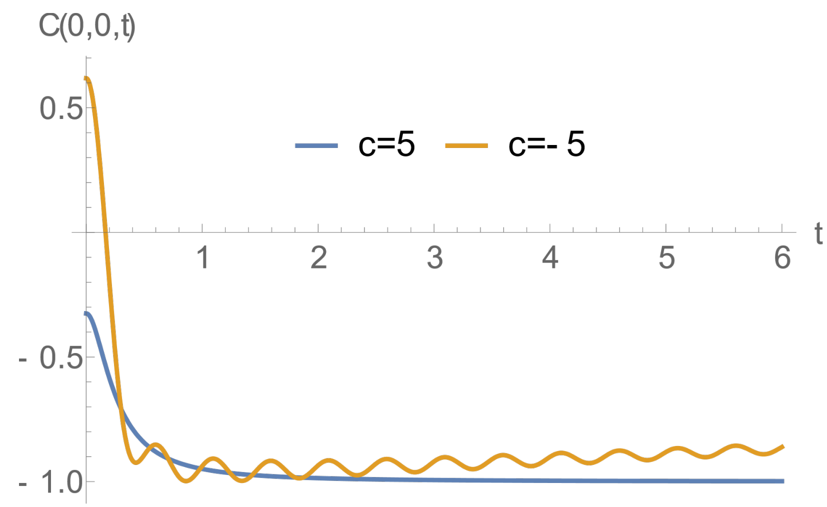
(a)Normalized noise function at $t=0.02$



(b)Normalized noise function at $t=0.1$



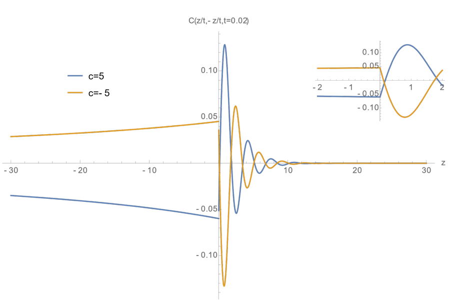
(c)Normalized noise function at $t=0.5$



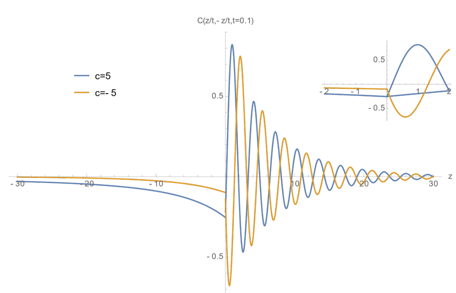
Comparison of normalized noise function at the origin $C(0,0,t)$

Normalized noise function

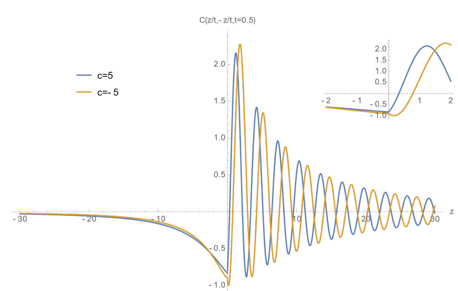
- Noise function spread faster than linearly shortly after the quench then stablize at linear rate
- Noise function near the origin shows different signs for repulsive and attractive models shortly after the quench
- Noise function near the origin quickly approaches -1 indicating no overlaps among particles
- Noise function at the origin increases gradually for attractive models while remain -1 for repulsive ones. Bound state effects in asympototic limit.



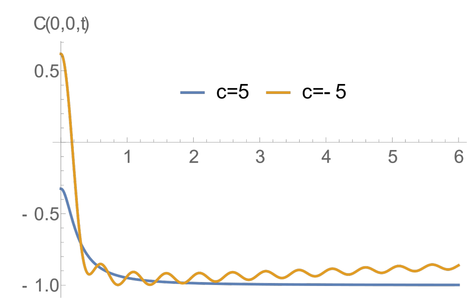
(a)Normalized noise function at $t = 0.02$



(b)Normalized noise function at $t = 0.1$



(c)Normalized noise function at $t = 0.5$



(d)Comparison of normalized noise fu
at the origin $C(0, 0, t)$

Time Evolution

- N-particle dynamics, long time asymptotics

Saddle point approximation

$$\int_C g(z)e^{tf(z)}dz \overset{s \gg 1}{\rightarrow} g(z_0)e^{tf(z_0)}\sqrt{\frac{2\pi}{tf''(z_0)}}e^{i\alpha}$$

Saddle point z_0: $f'(z_0) = 0$

Asymptotic wavefunction

Integrate out μ

Shift k contour C to C' to include saddle points

If no pole is crossed (repulsive Gaudin-Yang model)
Original integral replaced by saddle point approximated value

If pole is crossed (attractive Gaudin-Yang model)
Original integral transforms into integral over C' and residue at $k_o = k_{Pm} - ic$
Then apply saddle point approximation for both terms

Time Evolution

- Asymptotic dynamics for the repulsive case

Saddle Point Approximated Wavefunction ($N \times N!$ terms) $\qquad \xi_i = y_i/t$

$$f(\xi,\alpha,t) = \frac{i\sigma^{\frac{N}{2}}}{2^{\frac{N}{2}}\pi^{\frac{N}{4}}c(it)^{\frac{N}{2}}} e^{\sum_i it\xi_i^2 - \xi_i^2\sigma^2/2 - i\xi_i x_{Pi} - \sigma^2\xi_{P^{-1}\beta}k_0}(-1)^P \sum_o R(\xi_o + ic/2)\theta(\xi_1 < \dots \xi_N)$$

- The densities

$$\rho_\uparrow(z) = \frac{(N-1)\sigma}{2\sqrt{\pi}t} e^{-\frac{\sigma^2 z^2}{4t^2}} + O((c\sigma)^4)$$

$$\rho_\downarrow(z) = \frac{\sigma}{2\sqrt{\pi}t} e^{-\frac{\sigma^2(z-2k_0t)^2}{4t^2}} + O((c\sigma)^4)$$

Corrections larger thab Lieb-Liniger gases $O(exp(-|c|a))$ *indicating stronger interaction effects*

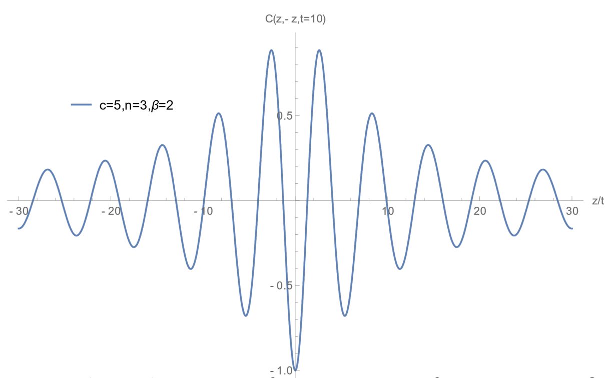
- Normalized noise function

$$C(z,z',t) = \frac{1}{N-1}\times$$

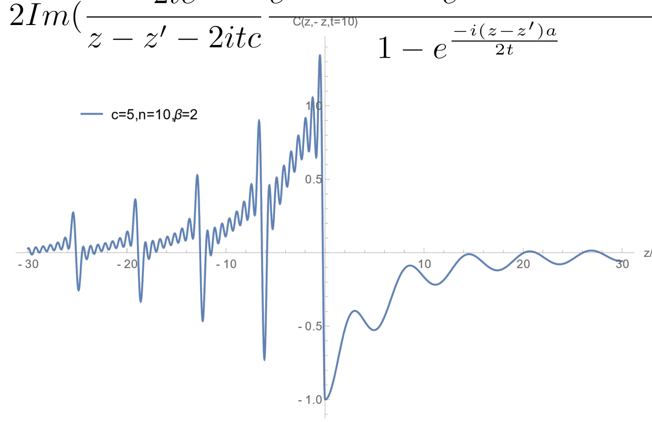
$$\Bigg(\theta(z-z')\big((1+\frac{4t^2c^2}{(z-z')^2+4t^2c^2})(\beta-1)+(N-\beta)\frac{(z-z')^2}{(z-z')^2+4t^2c^2} - 2Im(\frac{2tc}{z-z'-2itc}\frac{e^{-i\frac{i(z-z')a}{2t}} - e^{-\frac{i(z-z')\beta a}{2t}}}{1-e^{-\frac{i(z-z')a}{2t}}})$$

$$+\theta(z'-z)\big((1+\frac{4t^2c^2}{(z-z')^2+4t^2c^2})(N-\beta)+(\beta-1)\frac{(z-z')^2}{(z-z')^2+4t^2c^2} - 2Im(\frac{2tc}{z-z'-2itc}\frac{e^{-\frac{i(z-z')a}{2t}} - e^{-\frac{i(z-z')(N-\beta+1)a}{2t}}}{1-e^{\frac{-i(z-z')a}{2t}}})\Bigg) - 1$$

$C(z,-z,t=10)$



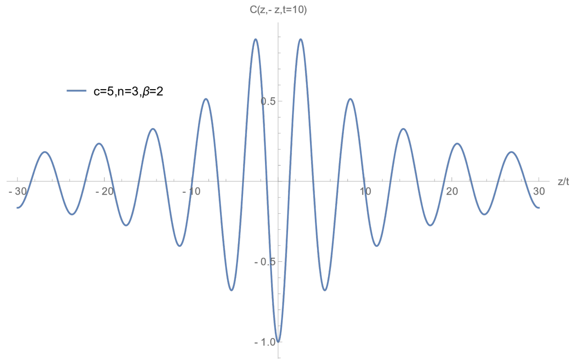
Normalised noise function of $N=3$, $\beta=2$



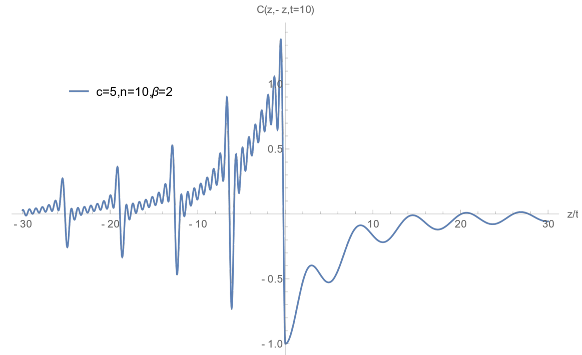
Normalised noise function of $N=10$, $\beta=2$

Time evolution

- Leading order normalized correlation function



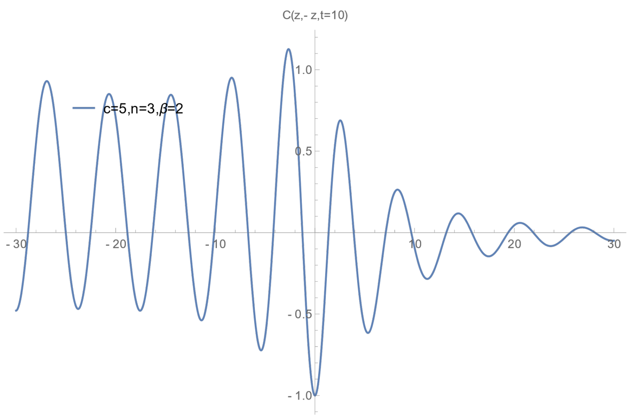
Normalised noise function of $N = 3,\ \beta = 2$



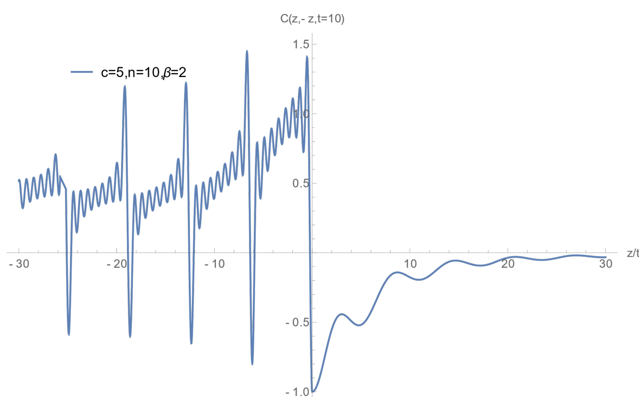
Normalised noise function of $N = 10,\ \beta = 2$

- Leading order normalized correlation function with kinetic impurity

$$
\begin{aligned}
& C(z,z',t) \\
=& \frac{\theta(z-z')}{N-1}\Big(1+\frac{4t^2c^2}{(z-z')^2+4t^2c^2}e^{\frac{\sigma^2 k_0(z'-z)}{t}}\Big)(\beta-1)+(N-\beta)\frac{(z-z')^2}{(z-z')^2+4t^2c^2}-2e^{-\frac{\sigma^2 k_0(z-z')}{2t}}\Im\Big(\frac{2tc}{z-z'-2ict}\frac{e^{-\frac{i(z-z')a}{2t}}-e^{-i\frac{(z-z')\beta a}{2t}}}{1-e^{-\frac{i(z-z')a}{2t}}}\Big) \\
+& \frac{\theta(z-z')}{N-1}\Big(1+\frac{4t^2c^2}{(z-z')^2+4t^2c^2}e^{\frac{\sigma^2 k_0(z'-z)}{t}}\Big)(N-\beta)+(\beta-1)\frac{(z-z')^2}{(z-z')^2+4t^2c^2}-2e^{-\frac{\sigma^2 k_0(z-z')}{2t}}\Im\Big(\frac{2tc}{z-z'-2ict}\frac{e^{-\frac{i(z-z')a}{2t}}-e^{-i\frac{(z-z')(N-\beta+1)a}{2t}}}{1-e^{-\frac{i(z-z')a}{2t}}}\Big)-1
\end{aligned}
$$



Normalised noise function of $N = 3,\ \beta = 2,\ k_0 = 5.$



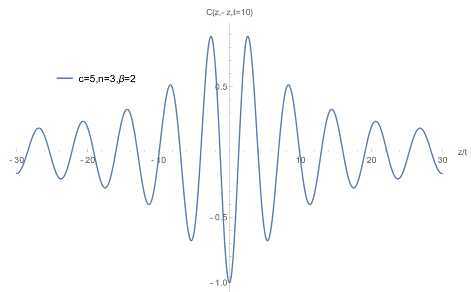
Normalised noise function of $N = 10,\ \beta = 2,\ k_0 = 5$

Time evolution

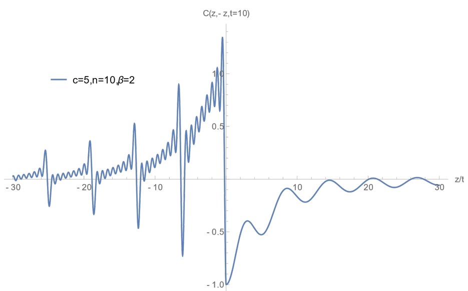
$$C(z/t,-z/t,t)=\frac{\langle\rho_\uparrow(z/t)\rho_\downarrow(-z/t)\rangle}{\langle\rho_\uparrow(z/t)\rangle\langle\rho_\downarrow(-z/t)\rangle}-1=\langle\frac{\delta\rho_\uparrow(z/t)\delta\rho_\downarrow(-z/t)\rangle}{\langle\rho_\uparrow(z/t)\rangle\langle\rho_\downarrow(-z/t)\rangle}$$

- Noise function vanishes with vanishing interaction
- Envelpe of noise function depends on interaction
- Fine interference fringe depends on the number of particles to the side of impurity in the inital state

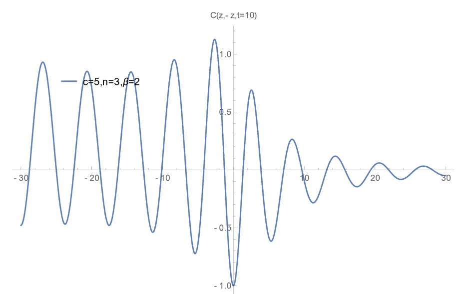
Fine fringes relate to interference among terms with permutated identical particle



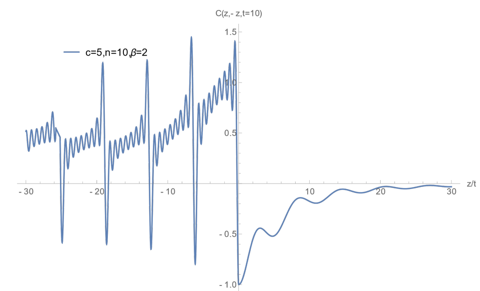
Normalised noise function of $N=3$, $\beta=2$



Normalised noise function of $N=10$, $\beta=2$



Normalised noise function of $N=3$, $\beta=2$, $k_0=5$.



Normalised noise function of $N=10$, $\beta=2$, $k_0=5$

Time evolution

- Asymptotic dynamics for attractive model

Saddle Point Approximated Wavefunction ($N^2 \times N!$ terms)

$$f(\xi,\alpha,t) = f^{\text{free}}(\xi,\alpha,t) + \sum_{o,Pm} f^{\text{bound}}(\xi,\alpha,t,o,Pm)$$

$$\begin{aligned} & f(\xi,x,t) \\ = & \frac{\sigma^{N/2} i}{2^{N/2}\pi^{N/4} c(it)^{N/2}} \; e^{\sum_j i(t-\sigma^2/2i)\xi_{P^{-1}j} - i\xi_{P^{-1}j}(x_j + ik_o\sigma^2\delta_{j\beta}} e^{ik_0x_0 - k_0^2\sigma^2/2} \sum_{o\le\beta P^{-1}0\ge\alpha} R(k_o + ic/2) \\ - & \frac{\sigma^{N/2} i}{2^{(N+1)/2}\pi^{N/4+1/2}(it)^{(N-1)/2}} \sum_{\substack{o\le\beta\le Pm \\ m\le\alpha\le P^{-1}o}} e^{\sum_j i(t-\sigma^2/2i)\xi_{P^{-1}j}} e^{-i\xi_{P^{-1}j}(x_j+ik_o\sigma^2\delta_{j\beta}) + ik_0x_0 - \frac{k_0^2\sigma^2}{2} + \frac{i(\xi_{P^{-1}o}+\xi_m)^2(t-\sigma^2/2i)}{2} - ct(\xi_m - \xi_{Po})} \\ & e^{\frac{i(\xi_{P^{-1}0}+\xi_m)(x_o + x_{Pm} + ik_0\sigma^2\delta_{o\beta} + ik_0\sigma^2\delta_{Pm,\beta})}{2} + \frac{ick_0\sigma^2}{2}(\delta_{Pm\beta}} e^{-\frac{ick_0\sigma^2}{2}\delta_{o\beta} + \frac{c(x_{Pm}-x_o)}{2} + \frac{ic^2(t+\sigma^2/2i)}{2}} R(R(\mu = k_0 + ic/2), k_o = k_{Pm} - ic)) \end{aligned} \tag{1}$$

- Leading order density function

$$\begin{aligned} & \rho_\downarrow(z) \\ = & \frac{\sigma}{2\sqrt{\pi}t}(e^{-\sigma^2(z-2tk_0)^2/4t^2} + 2\sqrt{2}\pi t c^2 (f(\frac{z}{2t}) + g(\frac{z}{2t})) \times (h(N-\beta) + h(\beta-1)) \end{aligned}$$

$$f(\xi) = \text{erfc}(-\frac{\sigma}{\sqrt{2}}(2\xi + \frac{2ct}{\sigma^2} - k_o\delta))e^{\frac{(k_0\sigma^2-2ct)^2}{2\sigma^2} + 4ct\xi + c^2\sigma^2/2} \quad g(\xi) = \text{erfc}(\frac{\sigma}{\sqrt{2}}(2\xi - \frac{2ct}{\sigma^2} - k_o))e^{\frac{(k_0\sigma^2+2ct)^2}{2\sigma^2} - 4ct\xi + c^2\sigma^2/2}$$

$$h(m) = \frac{\exp(-|c|a) - \exp(-(m+1)|c|a)}{1 - \exp(-|c|a)}$$

Bound state contribution is supressed by factor $e^{-|c|a}$

To be continued

Many questions remain:

- Study multi impurity problems to see effects of longer strings
- Beyond the saddle point approximation
- Study dynamics of condensates to see stronger interaction effects
- Dynamics of FFLO states
- Quantum Brownian motion
- Dynamics of bosons systems

Concluions and Outlook

Conclusions:

- Evolution calculable for all coupling regimes – in thermodynamic regime (or non-therm)
- Evolution for all initial states (asymptotic equilibrium or not)

To do list:

- Quench dynamics in other integrable models:
Anderson Dynamics, Kondo dynamics, Hubbard Dynamics
- Floquet dynamics in Lieb-Liniger model
- nonequilibrium transport across quantum dots – quench to steady state
- Scattering cross sections of an electron off a Kondo system

Big Questions:

- What drives thermalization of pure states? Canonical typicality, entanglement entropy (Lebowitz, Tasaki, Short…)
- General principles, variational? F-D theorem out-of-equilibrium? Heating? Entanglement?
- What is universal? RG Classification?

Central theorem II

$$Const \int_C dk \int_{C'} d\mu \, \langle y, \alpha | \underbrace{k, \mu)(k, \mu}_{= I?} | x\beta \rangle \, \theta(\alpha)\theta(\beta) = \prod_i^N \delta(x_i - y_i) \prod_m^M \delta(\alpha_m - \beta_m)$$

- Single impurity (one down spin) $c > 0$, explicitly:

$$Const \int_C dk \sum_P (-1)^P e^{i \sum_j k_j (y_{P^{-1}j} - x_j)} \boxed{\int_{C'} d\mu J(\mu)} \, \theta(y)\theta(x)\theta(\alpha)\theta(\beta) = \prod_i \delta(y_i - x_i)\delta(\alpha - \beta)$$

$$= 2\pi c$$

$$J(\mu) = \frac{-ic}{\mu - k_{P\alpha} + ic/2} \prod_{\substack{m<\alpha \\ Pm \geq \beta}} \frac{\mu - k_{Pm} - ic/2}{\mu - k_{Pm} + ic/2} \frac{ic}{\mu - k_\beta - ic/2} \prod_{\substack{n<\beta \\ P^{-1}n \geq \alpha}} \frac{\mu - k_n + ic/2}{\mu - k_n - ic/2}$$

- Properties of $J(\mu)$:

$J(\mu) = O(1/\mu^2)$ *for large μ*

$$\int_C d\mu J(\mu) = 2\pi i \sum_{\substack{o \leq \beta \\ P^{-1}o \geq \alpha}} R(k_o + ic/2)$$

$$R(k_o + ic/2) = \begin{cases} -ic + O(\frac{1}{k_o}) & \text{if } k_o = k_{P\alpha} = k_\beta \\ O(\frac{1}{k_o}) & \text{if } k_o = k_{P\alpha} \neq k_\beta \\ & \text{ or } k_o = k_\beta \neq k_{P\alpha} \\ O(\frac{1}{k_o^2}) & \text{In other cases} \end{cases}$$