Fundamental Aspects of Quantum Dynamics:

II: Topology

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Quantum Few- and Many-Body Physics in Ultracold Atoms
Wuhan
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Topology

Global Properties invariant under Continuous Deformation

$$\int_{\textbf{Surface}} \textbf{Curvature} = 4\pi$$

- **Topological Phase Transition**
  (Kosterlitz and Thouless, 1970s)

- **Topological Band Theory**
  (Thouless, et.al 1980, Haldane 1988)

- **Topological Field Theory**
  (Haldane 1988)

The Nobel Prize in Physics 2016

Photo: A. Mahmoud
David J. Thouless
Prize share: 1/2

Photo: A. Mahmoud
F. Duncan M. Haldane
Prize share: 1/4

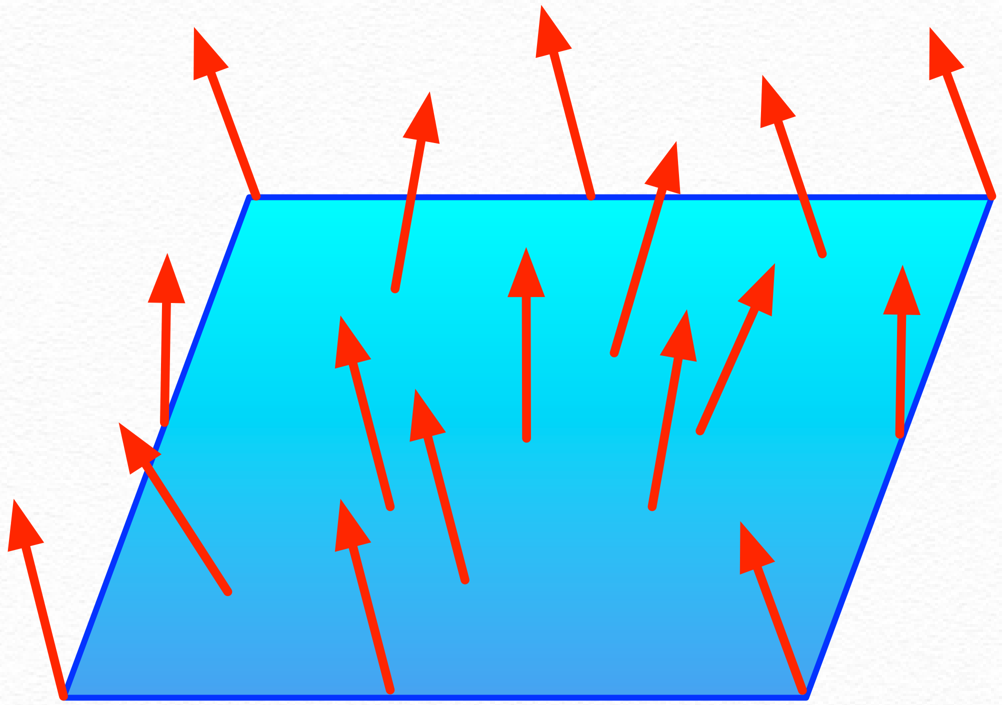
J. Michael Kosterlitz
Prize share: 1/4

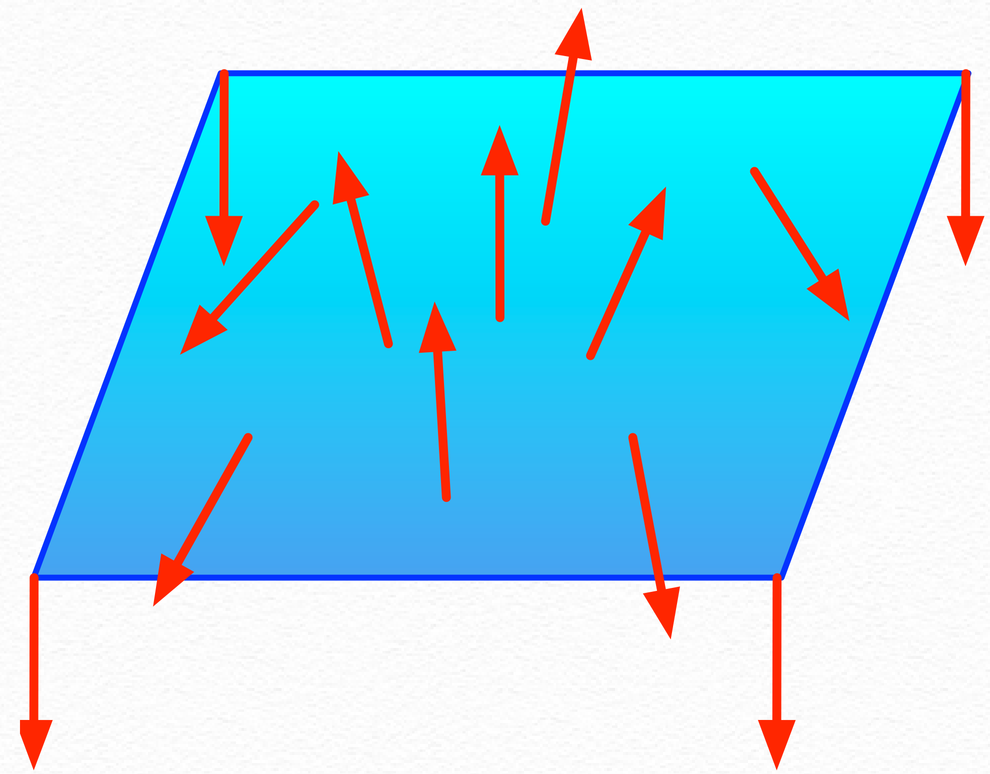
Topological Insulator, Topological Superconductor, Topological Quantum Computing……

Topological Band Theory

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} (\hat{c}^\dagger_{\uparrow,\mathbf{k}}, \hat{c}^\dagger_{\downarrow,\mathbf{k}}) H_{\mathbf{k}} \begin{pmatrix} \hat{c}_{\uparrow,\mathbf{k}} \\ \hat{c}_{\downarrow,\mathbf{k}} \end{pmatrix}$$

$$\mathcal{H}(\mathbf{k}) = \frac{1}{2}\mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$$





Topological Trivial | **Topological Non-trivial**

$$\mathcal{C} = \frac{1}{4\pi}\int_{BZ} d^2k \left(\frac{\partial \hat{\mathbf{h}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{h}}}{\partial k_y} \right) \cdot \hat{\mathbf{h}}$$

$$\Pi_2(S^2) = Z$$

Topology and Dynamics

I. Realization

Dynamics $\Longrightarrow$ Topological Band Theory

II. Detection

Near Equilibrium

Far from Equilibrium

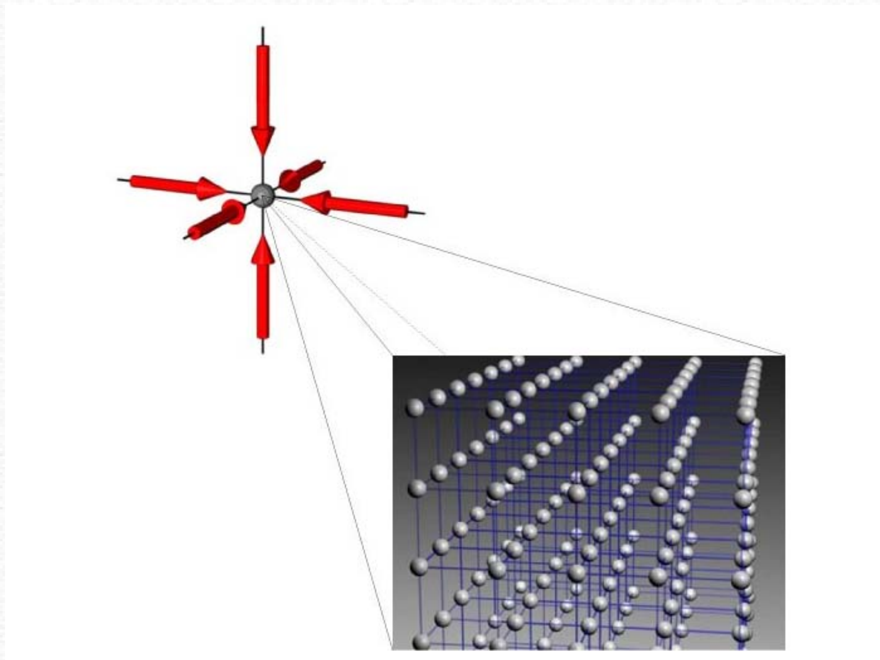
Topology and Dynamics

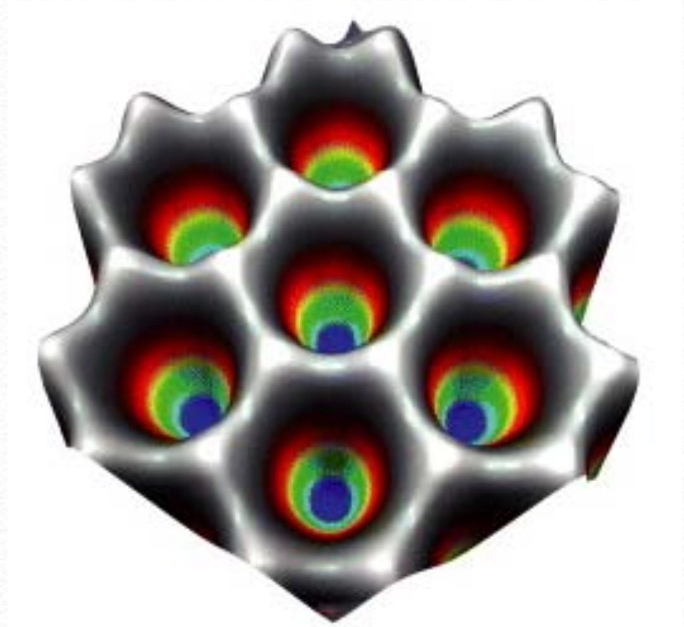
I. Realization

Dynamics $\Rightarrow$ Topological Band Theory

Ref. Wei Zheng and HZ, PRA, 89, 061603(R), 2014

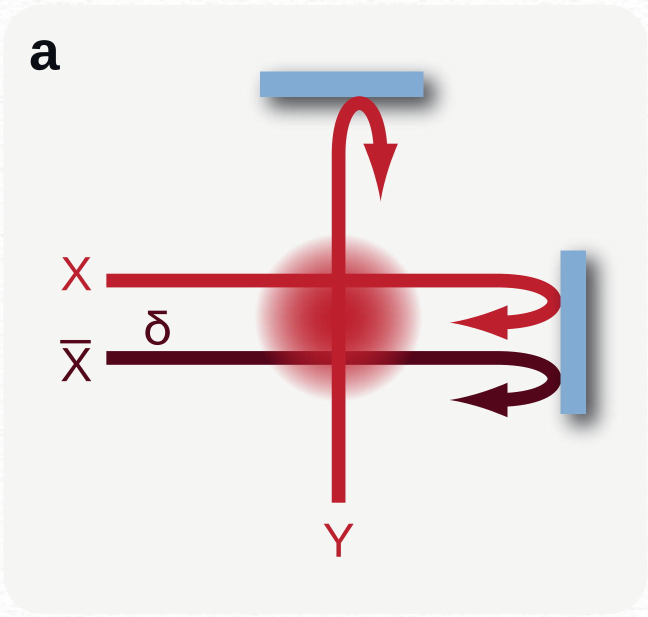
Optical Lattice

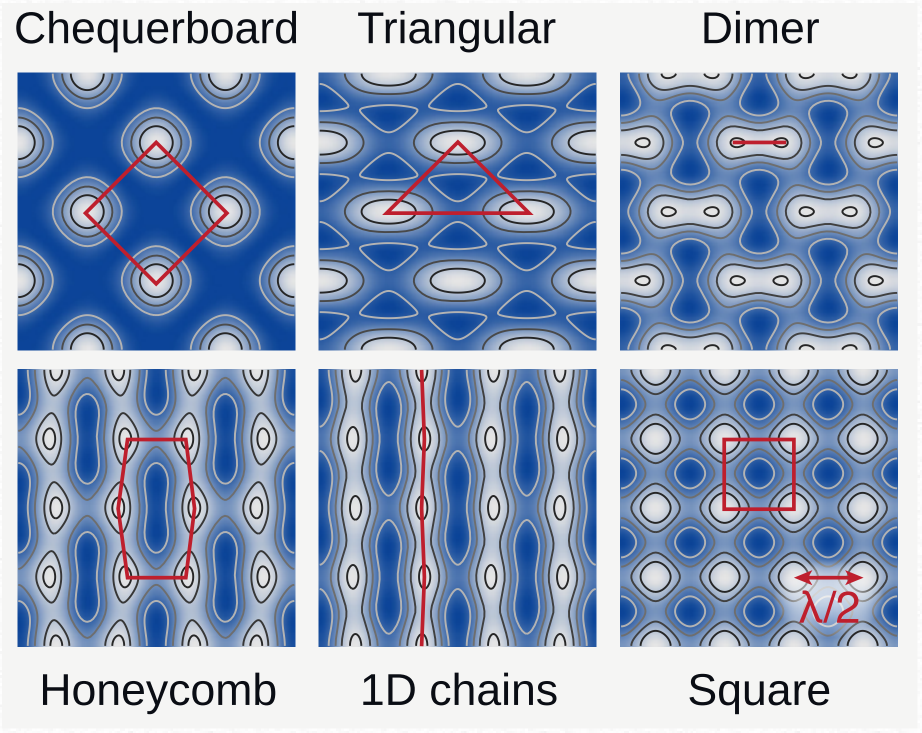




Cubic Lattice

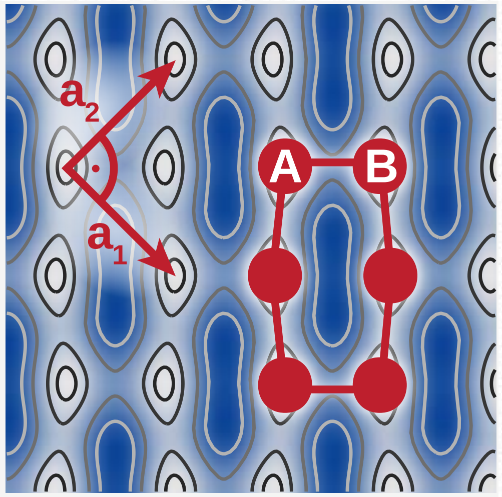
Triangular Lattice

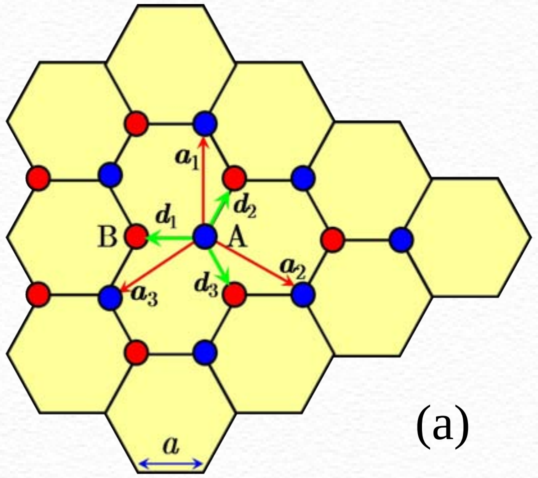




Tunable Geometry (ETH, 2012)

Dirac Point: Gapless



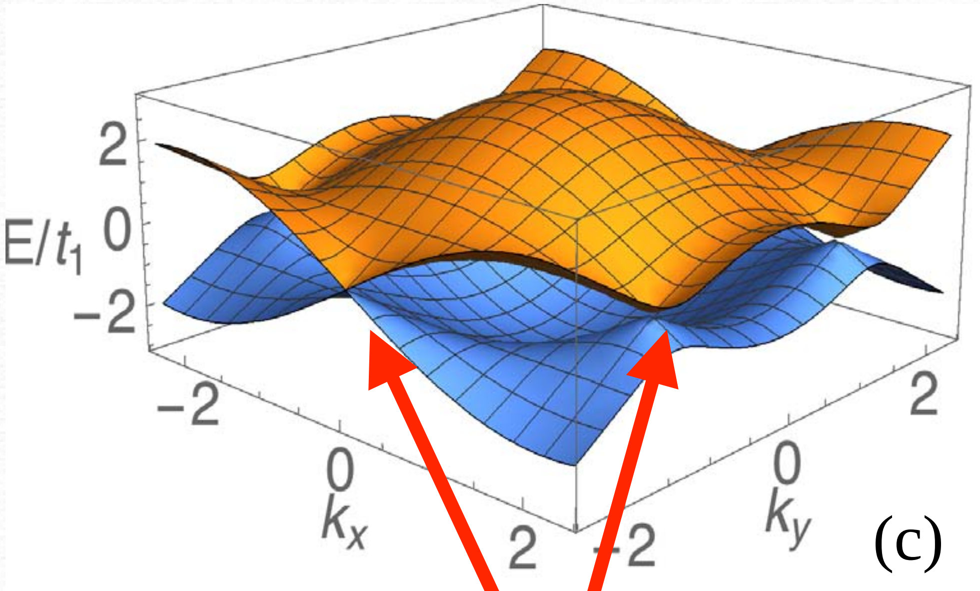


(a)

Honeycomb lattice

$$\hat{H} = -t_1 \sum_{\langle ij \rangle} \left(\hat{c}^{\dagger}_{\mathrm{B},j} \hat{c}_{\mathrm{A},i} + \mathrm{h.c.} \right)$$

Dirac Point



(c)

$$B_x = 0$$
$$B_y = 0$$

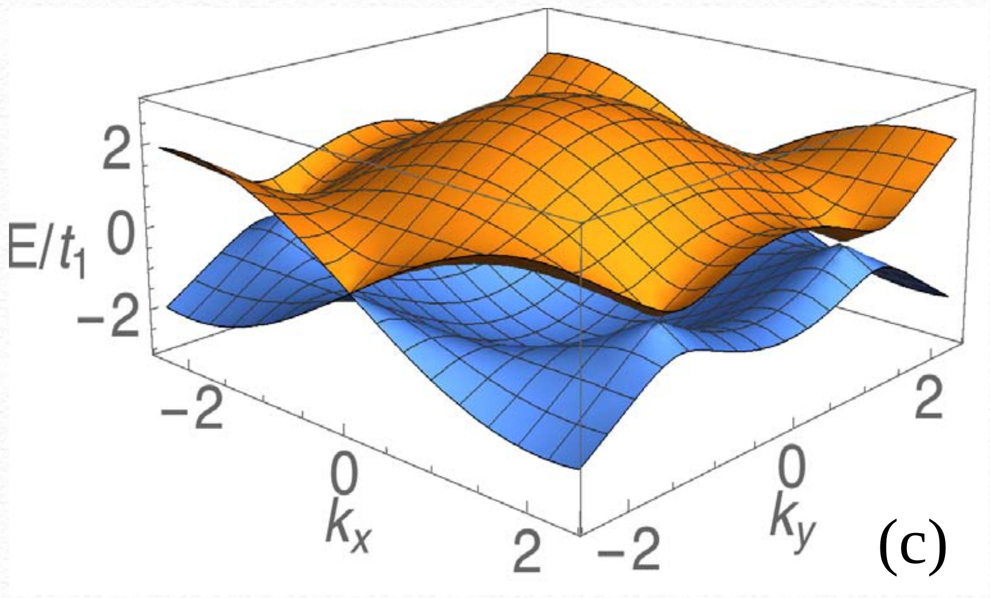
$$\hat{H} = \sum_{\mathbf{k}} \left(\hat{c}^{\dagger}_{\mathrm{A}}(\mathbf{k}), \hat{c}^{\dagger}_{\mathrm{B}}(\mathbf{k}) \right) H(\mathbf{k}) \begin{pmatrix} \hat{c}_{\mathrm{A}}(\mathbf{k}) \\ \hat{c}_{\mathrm{B}}(\mathbf{k}) \end{pmatrix}$$

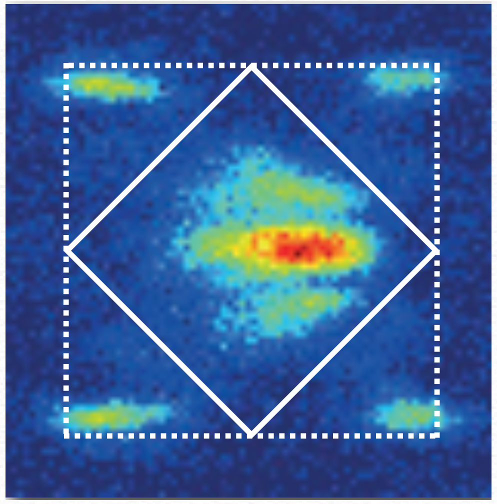
$$H(\mathbf{k}) = \begin{pmatrix} 0 & -t_1 \sum_\alpha e^{-i\mathbf{k}\cdot\mathbf{d}_\alpha} \\ -t_1 \sum_\alpha e^{i\mathbf{k}\cdot\mathbf{d}_\alpha} & 0 \end{pmatrix}$$

$$H(\mathbf{k}) = \mathbf{B}(\mathbf{k}) \cdot \sigma$$

$$B_x(\mathbf{k}) = -t_1 \sum_\alpha \cos(\mathbf{k}\cdot\mathbf{d}_\alpha) ; B_y(\mathbf{k}) = -t_1 \sum_\alpha \sin(\mathbf{k}\cdot\mathbf{d}_\alpha)$$

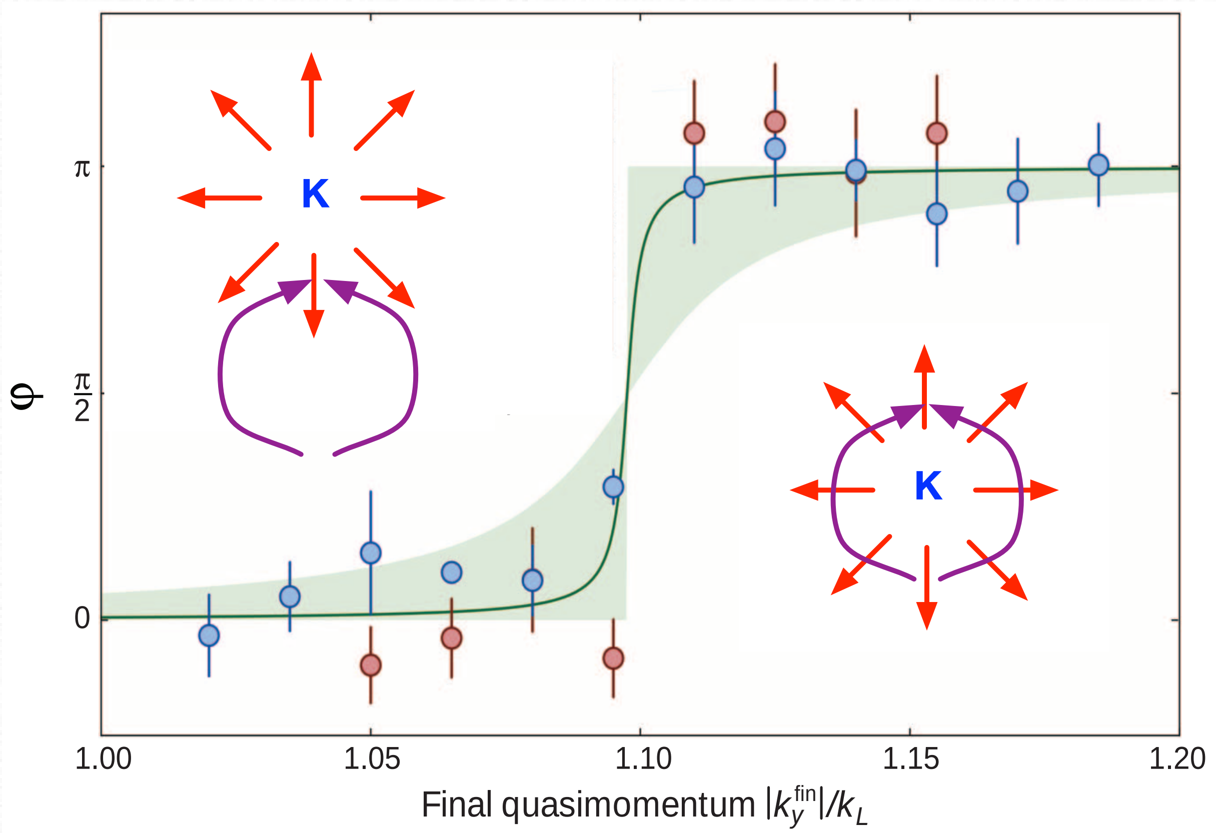
Dirac Point: Berry Curvature





Dirac Point Observed
via Bloch Oscillation, ETH, Nature 2012

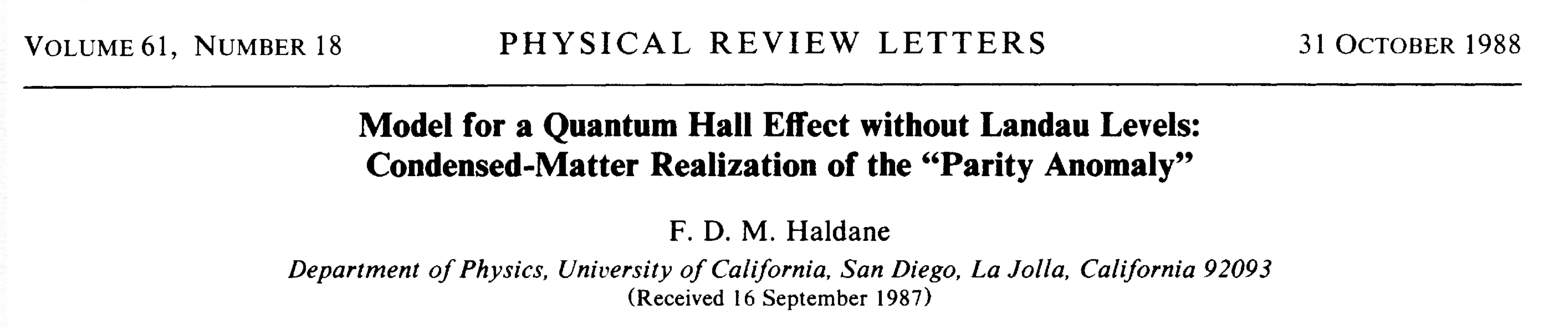
$$\mathbf{H} = \frac{3}{2}\left[\pm q_y \boldsymbol{\sigma}_x + q_x \boldsymbol{\sigma}_y\right]$$



Berry Curvature around Dirac Point Observed
Munich Group, Science, 2015

From Dirac Point to Haldane Model

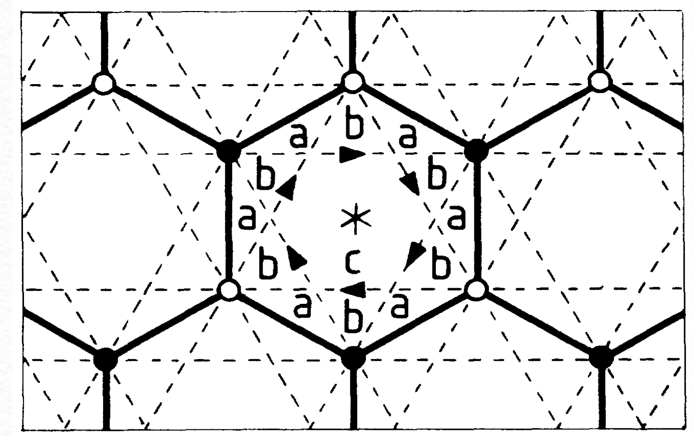
Photo: A. Mahmoud
F. Duncan M. Haldane
Prize share: 1/4

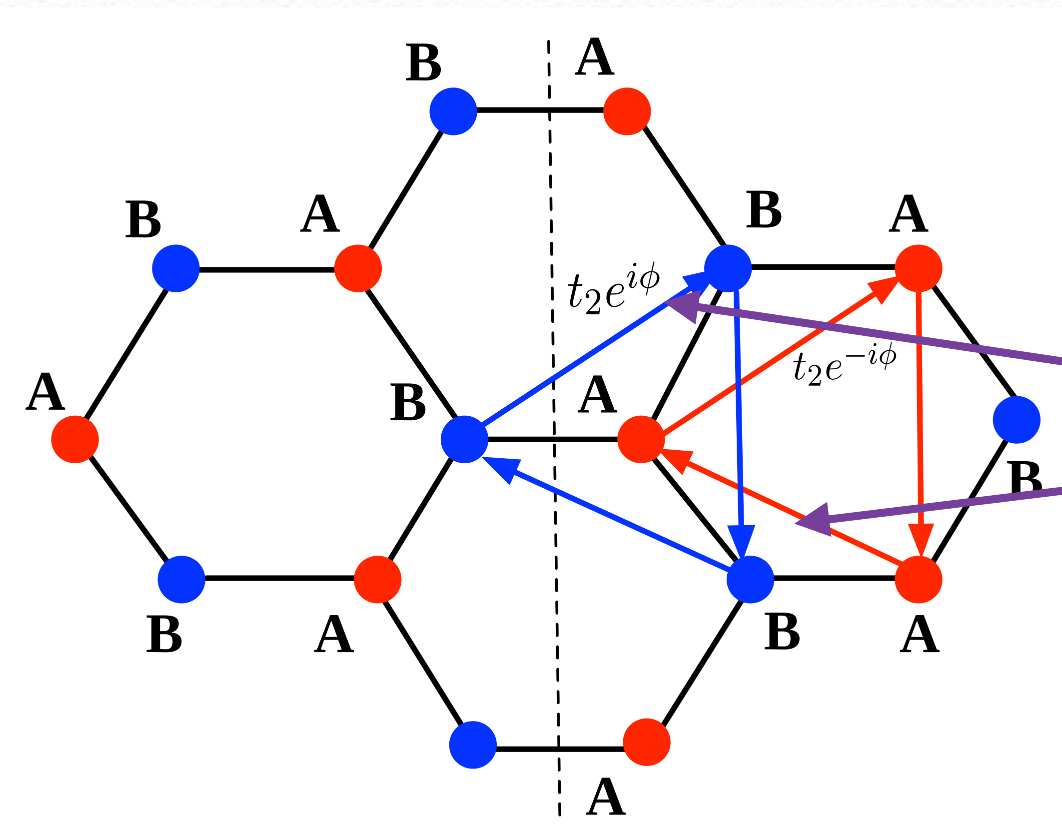


VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane
Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 16 September 1987)





How to realize this nontrivial next-nearest hopping ??

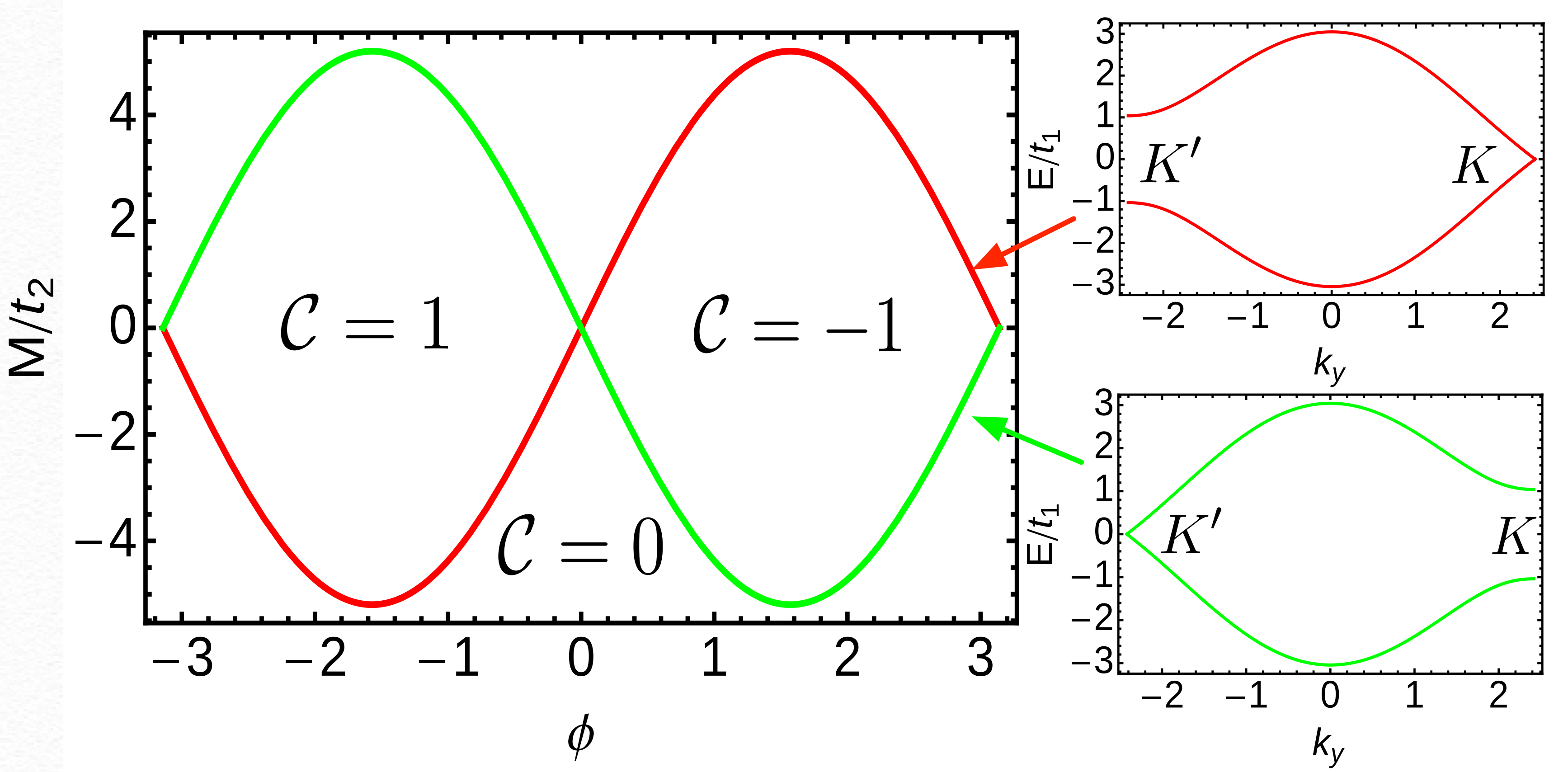
Haldane Model

$$\mathcal{H}(\mathbf{k}) = \frac{1}{2}\mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$$

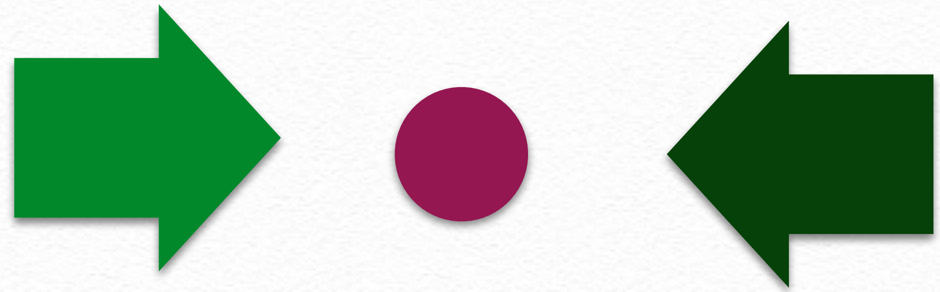
$$h_x = -J_1 \left[\cos k_x + \cos\left(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2}\right) + \cos\left(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2}\right) \right]$$

$$h_y = -J_2 \left[\sin\left(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2}\right) + \sin\left(\frac{k_x}{2} - \frac{\sqrt{3}k_y}{2}\right) \right]$$

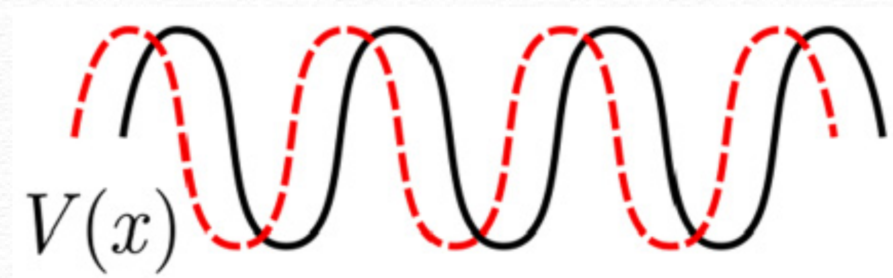
$$h_z = M + 2J_2 \sin\phi \left[\sin\left(\sqrt{3}k_y\right) + \sin\left(\frac{3k_x}{2} - \frac{\sqrt{3}k_y}{2}\right) - \sin\left(\frac{3k_x}{2} + \frac{\sqrt{3}k_y}{2}\right) \right]$$



Shaking Optical Lattice



AOM



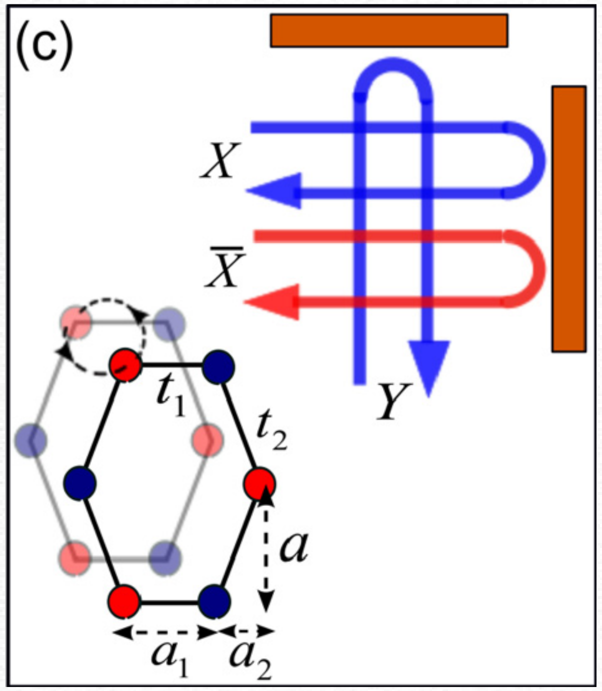
$$\hat{F} = \hat{U}\left(T_i + T, T_i\right) = \hat{\mathcal{T}}\ \exp\left\{-i\int_{T_i}^{T_i+T} dt\ \hat{H}\left(t\right)\right\}$$

For sufficiently fast modulation, if one only concerns the observation at integer period

$$\hat{F} = e^{-i\,\hat{H}_{\text{eff}}\,T}$$

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \sum_{n=1}^{\infty}\left\{\frac{[\hat{H}_n, \hat{H}_{-n}]}{n\omega} - \frac{[\hat{H}_n, \hat{H}_0]}{e^{-2\pi n i\alpha}\,n\omega} + \frac{[\hat{H}_{-n}, \hat{H}_0]}{e^{2\pi n i\alpha}\,n\omega}\right\}$$

Shaking Optical Lattice

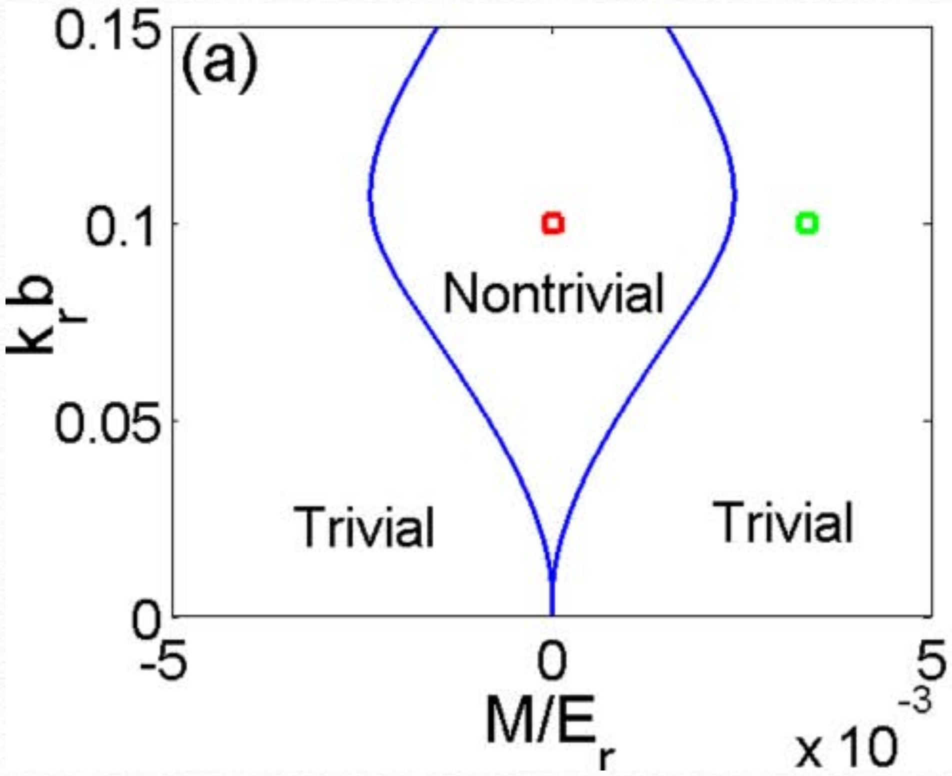


$$H = -\frac{\hbar^2\nabla^2}{2m} + V\left[x + b\sin(\omega t + \varphi), y + b\sin(\omega t)\right]$$

$$x' = x + b\cos(\omega t)$$
$$y' = y + b\sin(\omega t)$$

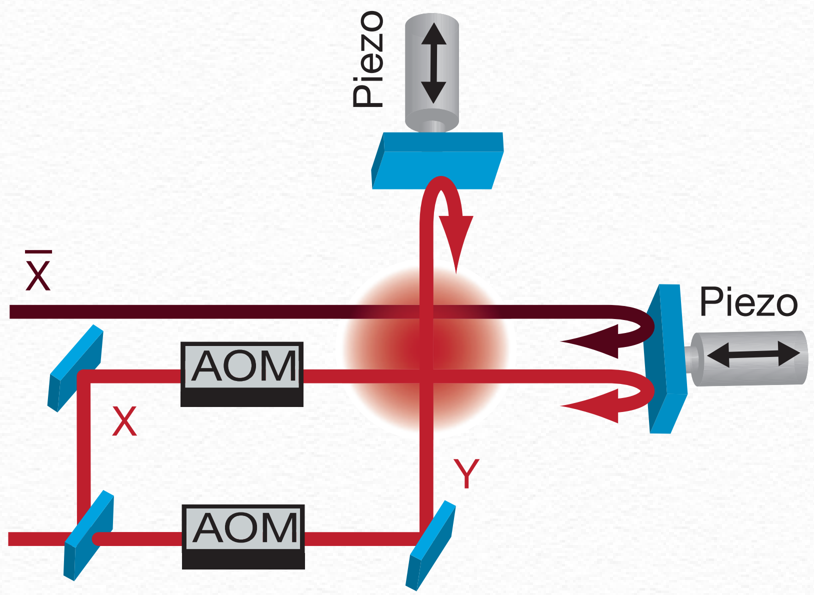
$$H(x,y,t) = \frac{\hbar^2}{2m}\left[-i\partial_x - A_x(t)\right]^2 + \frac{1}{2m}\left[-i\partial_y - A_y(t)\right]^2 + V(x,y)$$

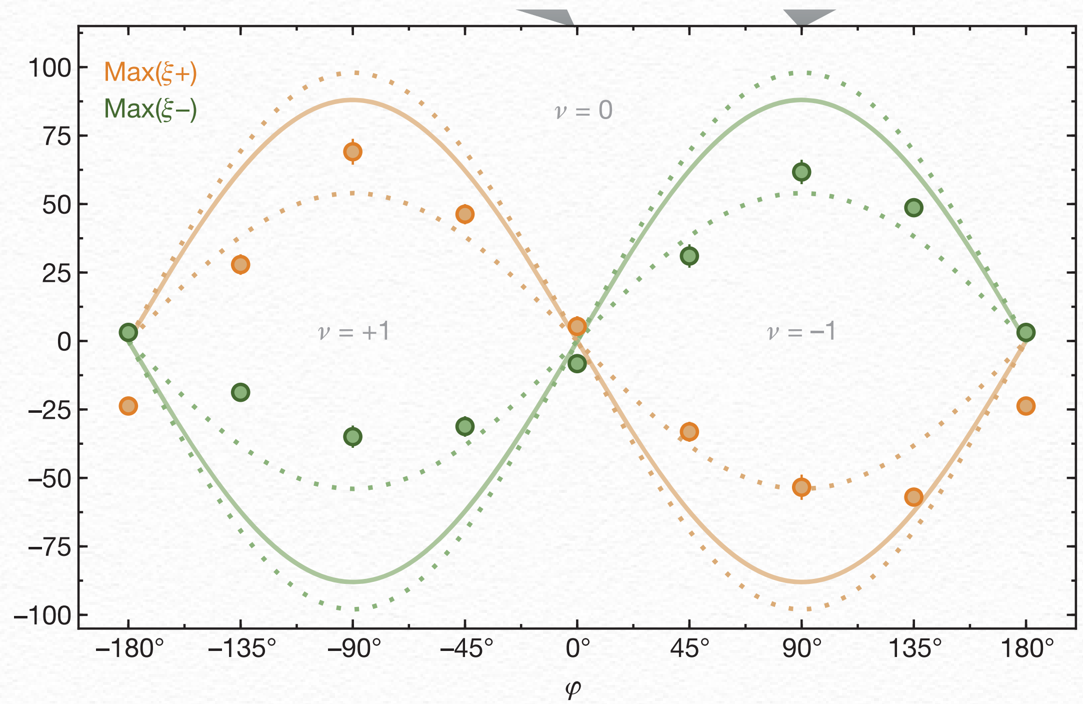
$$A_x(t) = m\omega b\sin(\omega t)/\hbar$$
$$A_y(t) = -m\omega b\cos(\omega t)/\hbar$$



$$H_{\text{eff}}(\mathbf{k}) \approx H_0(\mathbf{k}) + \frac{\left[H_1(\mathbf{k}), H_{-1}(\mathbf{k})\right]}{\omega}$$

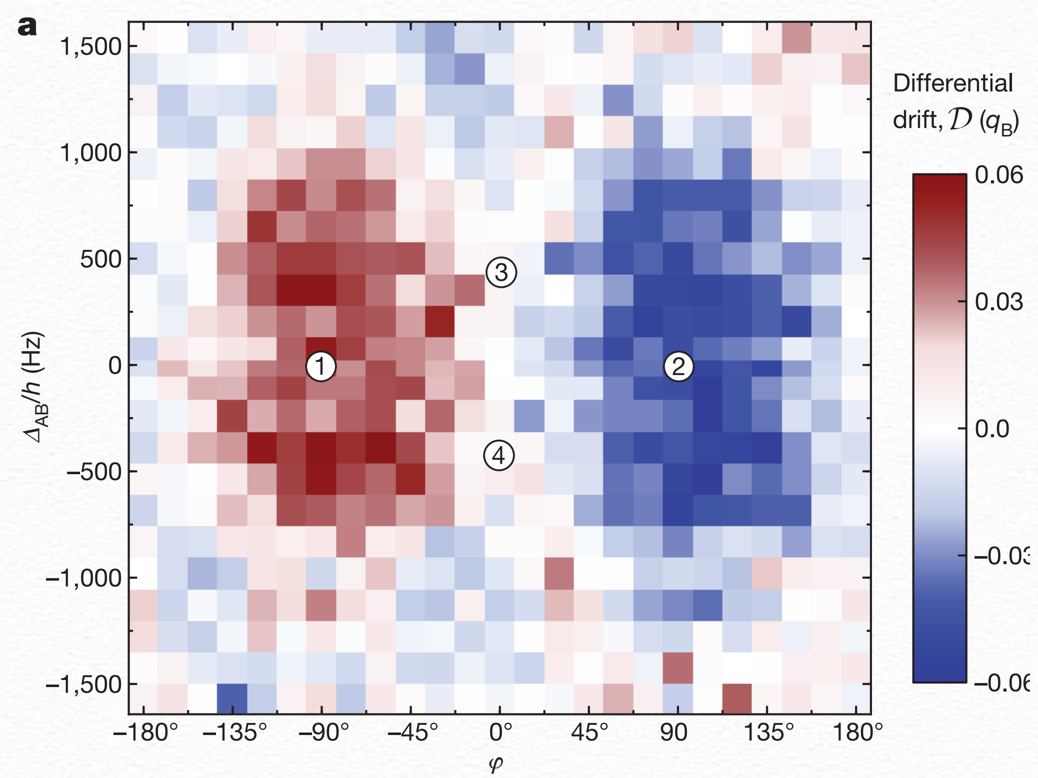
Experimental Realization





In the following we outline the theoretical framework used to obtain effective Hamiltonians for time-modulated optical lattices. In particular, we derive the mapping from an elliptically modulated honeycomb lattice to the Haldane Hamiltonian [S1]. We consider a numerical and analytical approach, compare the results for a wide range of parameters and examine the validity of several approximations for the system studied in the experiment. Some elements of the general framework used there can be found in references [S2–S8], and applications to circularly modulated honeycomb lattices can be found in very recent work [S5, S9, S10].

[S10] Zheng, W. & Zhai, H. Floquet topological states in shaking optical lattices. *Phys. Rev. A* **89**, 061603 (2014).



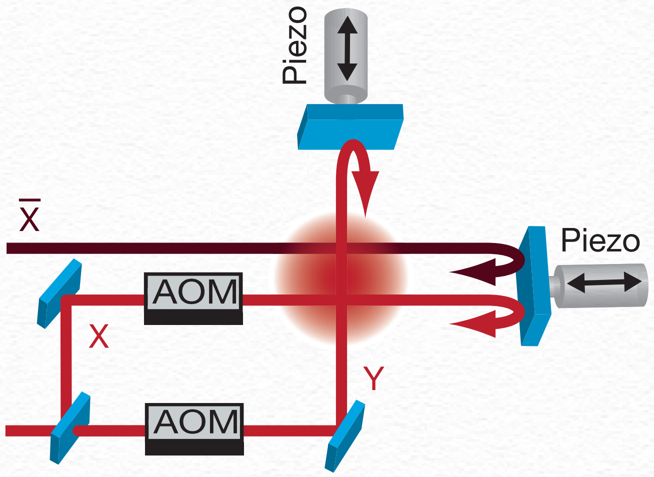
ETH, Nature (2014)

Experimental Realization

Scientific Background on the Nobel Prize in Physics 2016

6.2 Quantum simulations, and artificial states of matter

Also in 2014, the group led by Tilman Esslinger made an experiment with cold ^{40}K atoms in an optical lattice to simulate the precise model proposed by Haldane in 1988 [29]. This shows that reality sometimes surpasses dreams. At the end of his paper Haldane wrote: "While the particular model presented here, is unlikely to be directly physically realizable, it indicates ...". What he could not imagine was that 25 years later, new experimental techniques would make it possible to create an *artificial state of matter* that would indeed provide that "unlikely" realization.



ETH, 2014; Hamburg 2015; USTC 2015

Topology and Dynamics

Dynamics

Topological Band Theory

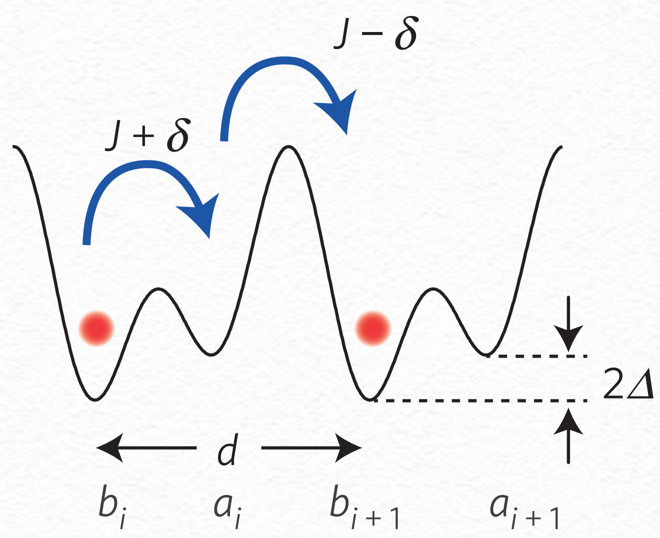
II. Detection

Near Equilibrium

Far from Equilibrium

Ref. Ce Wang, Pengfei Zhang, Xin Chen, Jinlong Yu and Hui Zhai, PRL, 118, 185701 (2017)

Rice-Mele Model

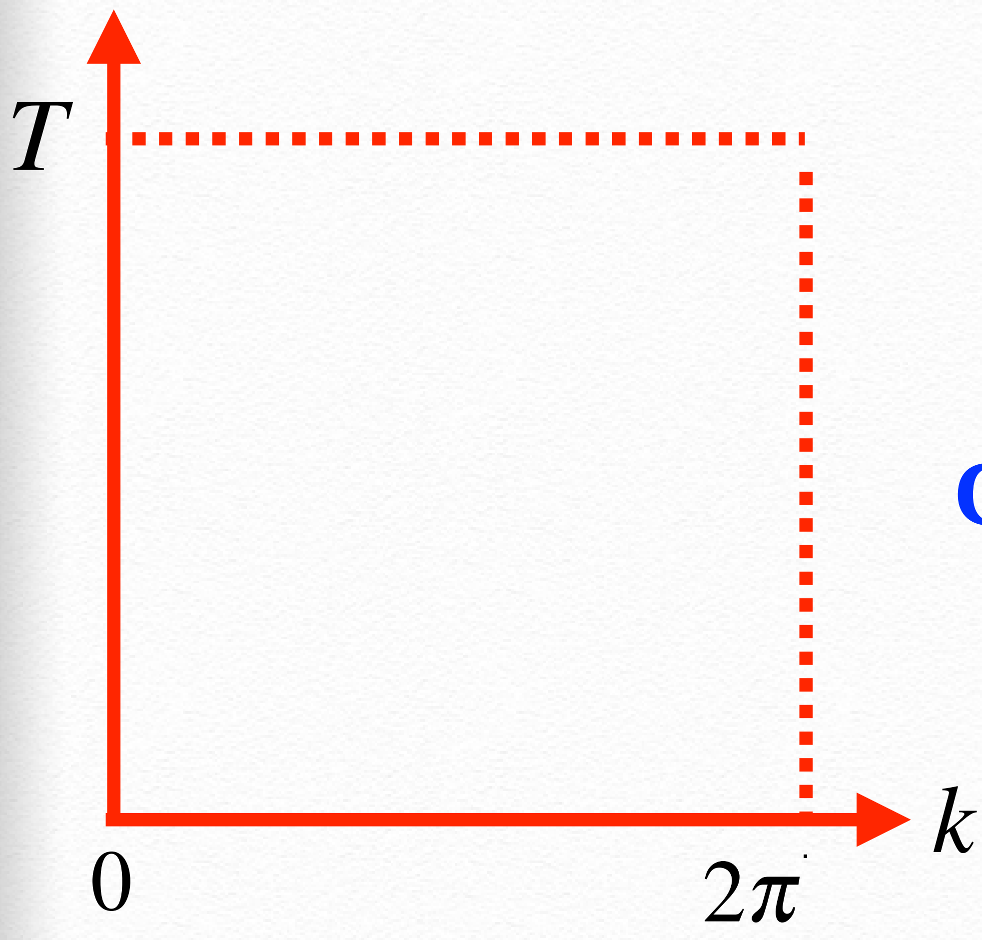


$$\hat{\mathcal{H}} = \sum_i \Big(-(J+\delta)\hat{a}_i^\dagger \hat{b}_i - (J-\delta)\hat{a}_i^\dagger \hat{b}_{i+1} + \text{h.c.} + \Delta(\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i) \Big)$$

$$= \sum_k (\hat{a}_k^\dagger \; \hat{b}_k^\dagger)\, H_k \begin{pmatrix} \hat{a}_k \\ \hat{b}_k \end{pmatrix}$$

$$H_k = \mathbf{d}(k) \cdot \boldsymbol{\sigma}$$

$$\mathbf{d}(k) = (2J_0 \cos\frac{k}{2}, 2\delta \sin\frac{k}{2}, \Delta)$$



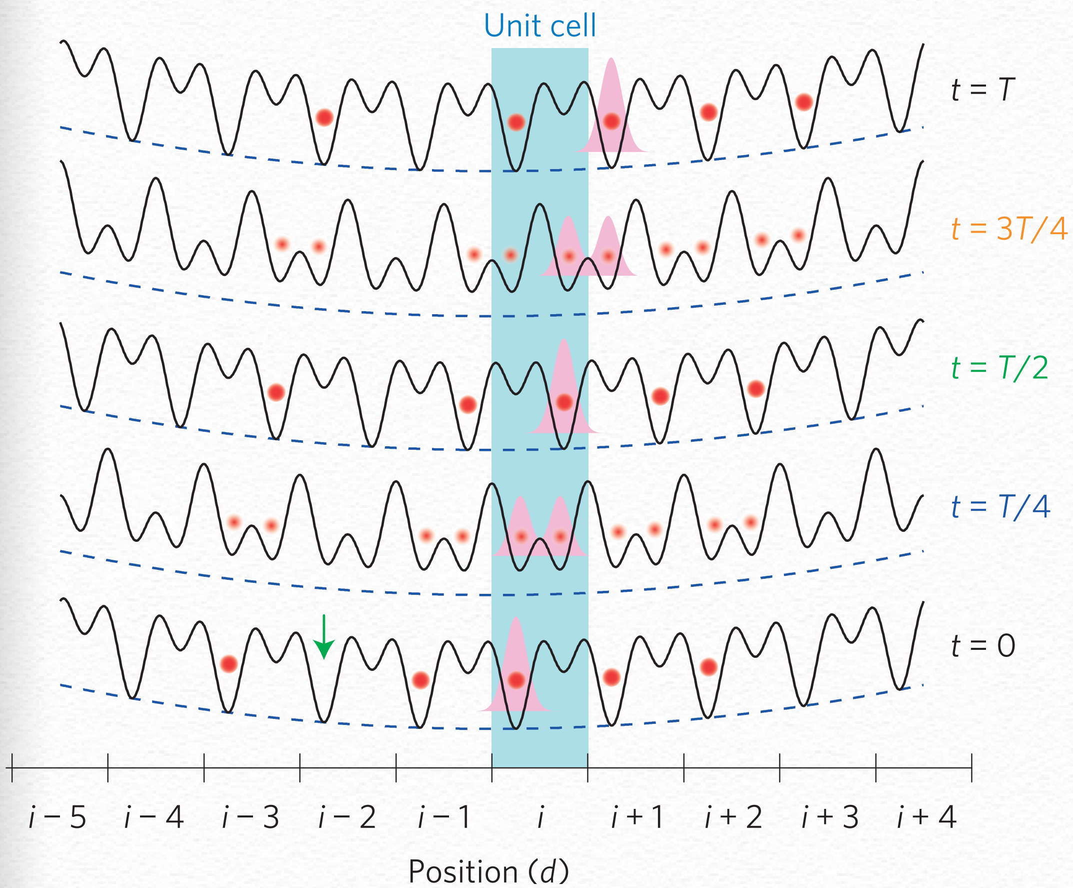
Chern number defined in 1+1 space

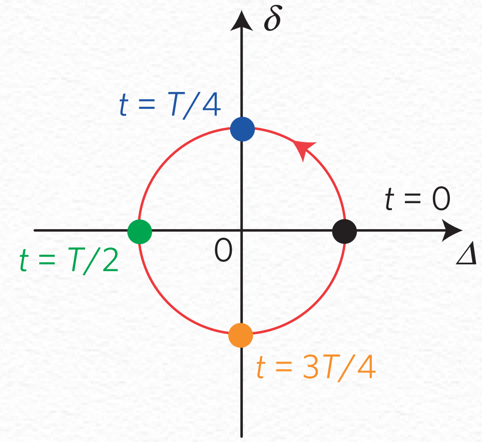
$$\frac{1}{2\pi}\int_{BZ} dk \int_0^T dt \frac{1}{2}\hat{\mathbf{d}}(k,t)\cdot\Big(\partial_k \hat{\mathbf{d}}(k,t) \times \partial_t \hat{\mathbf{d}}(k,t)\Big)$$

Adiabatic Charge Pumping

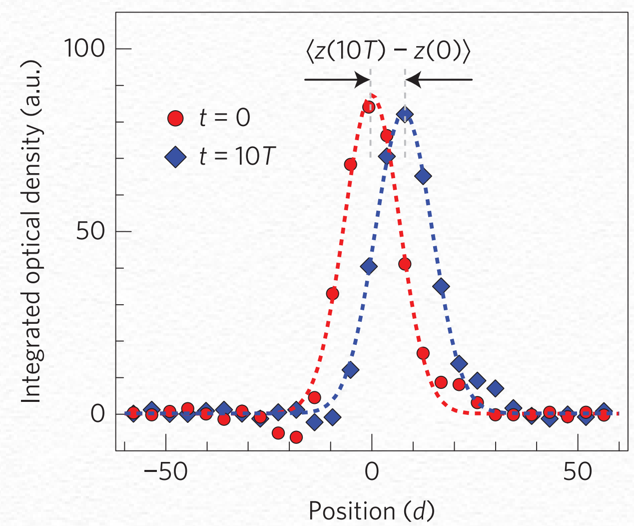
Charge Pumping:

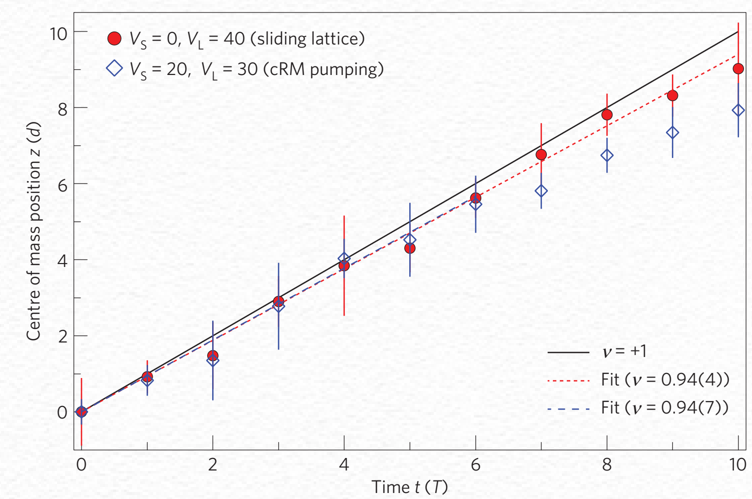
$$\hat{\mathcal{H}} = \sum_i \left(-(J+\delta)\hat{a}_i^\dagger \hat{b}_i - (J-\delta)\hat{a}_i^\dagger \hat{b}_{i+1} + \text{h.c.} + \Delta(\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i) \right)$$





Time Dependent





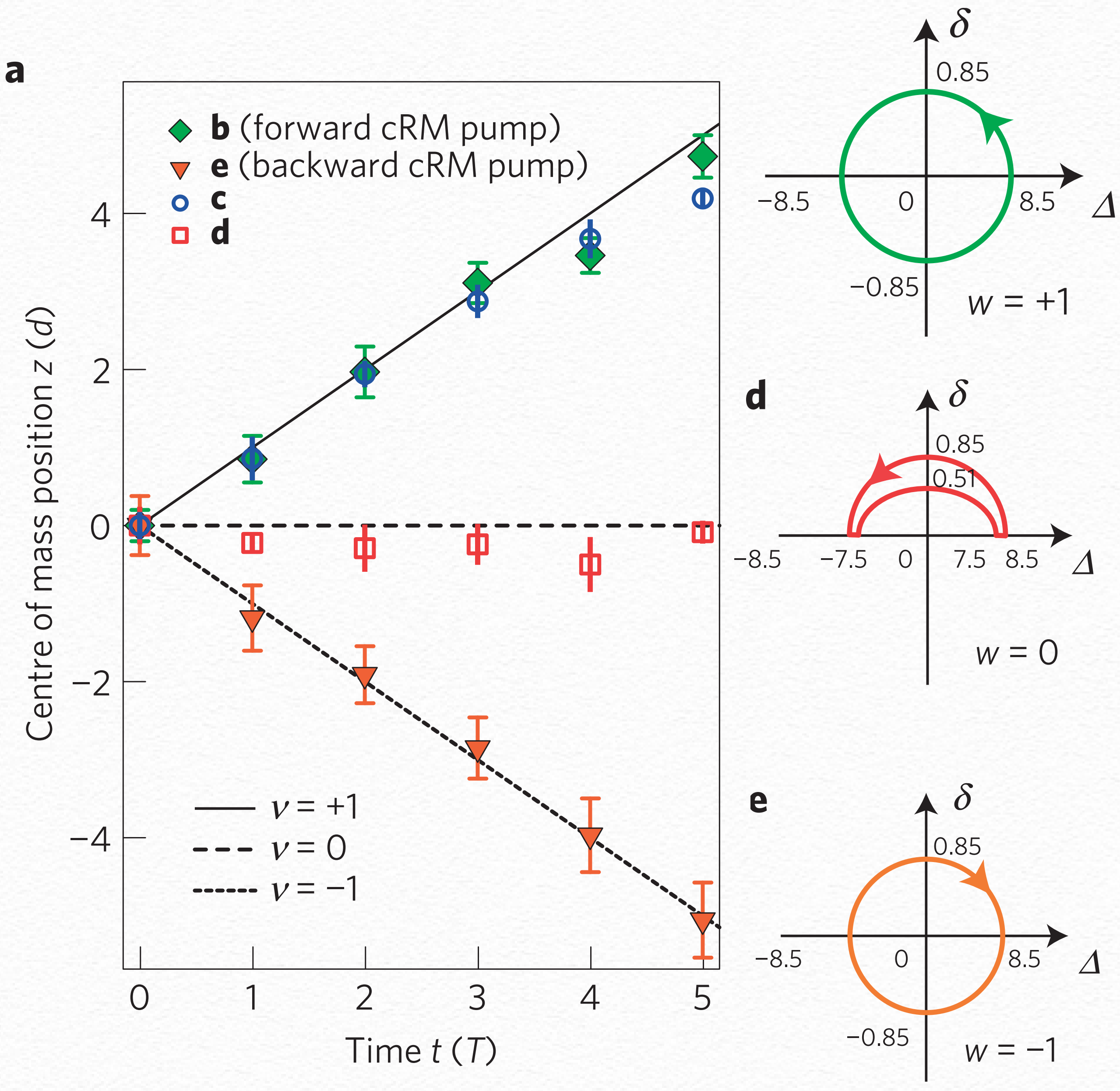
Charge Pumping is quantized to Chern number:

$$\Delta Q = -\frac{1}{2\pi}\int_{BZ} dk \int_0^T dt \, \frac{1}{2}\hat{\mathbf{d}}(k,t)\cdot\left(\partial_k \hat{\mathbf{d}}(k,t) \times \partial_t \hat{\mathbf{d}}(k,t)\right) = -C$$

Quantized Charge Pumping

Charge Pumping is quantized to Chern number:

$$\Delta Q = -\frac{1}{2\pi}\int_{BZ} dk \int_0^T dt \frac{1}{2}\hat{\mathbf{d}}(k,t)\cdot\left(\partial_k\hat{\mathbf{d}}(k,t)\times\partial_t\hat{\mathbf{d}}(k,t)\right) = -C$$



Munich group
Kyoto group
Nat. Phys. 2016

Second Chern Number

1+1 D → 1st Chern Number → 2D Quantum Hall

Spatial Time

2+1+1 D → 2nd Chern Number → 4D Quantum Hall

Spatial Time Parameter

LETTER

doi:10.1038/nature25000

Exploring 4D quantum Hall physics with a 2D topological charge pump

Michael Lohse[1,2], Christian Schweizer[1,2], Hannah M. Price[3,4], Oded Zilberberg[5] & Immanuel Bloch[1,2]

Physical Consequence of 2D Chern Insulator

Physical Consequence of Topological Number

At or Near Equilibrium

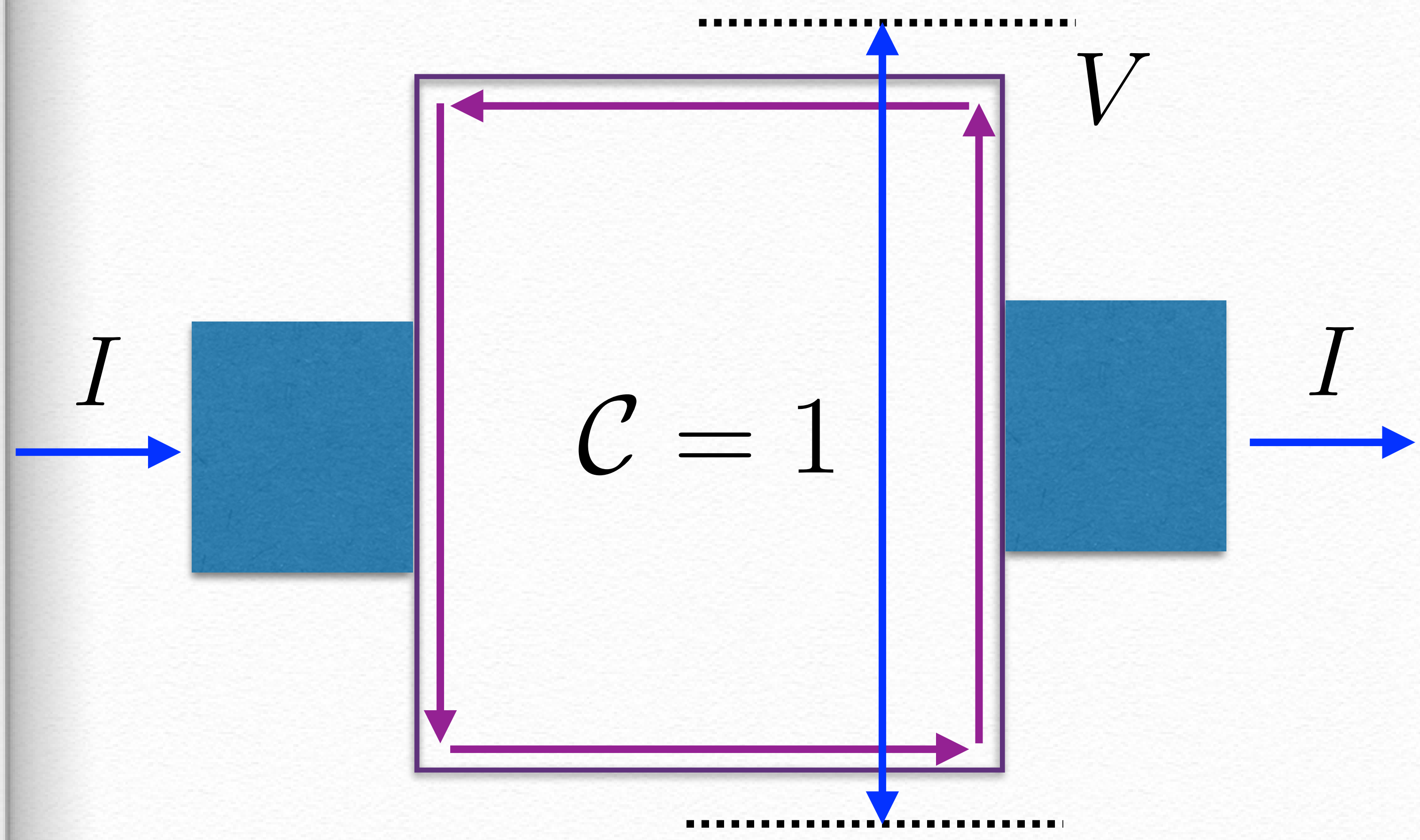
From from Equilibrium

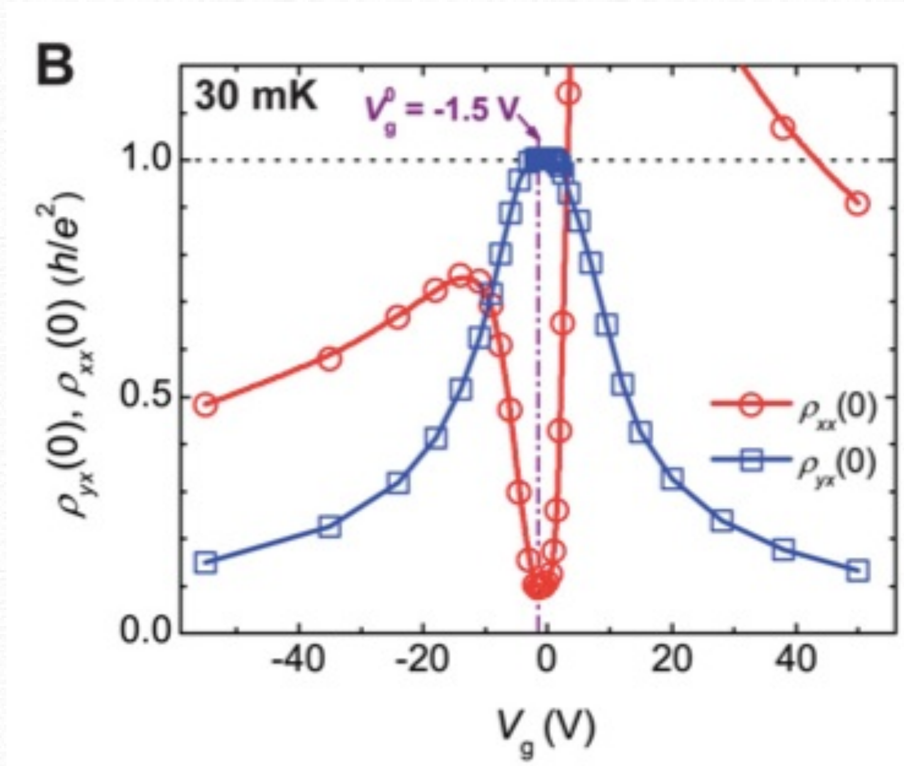
Quantized Edge State
Quantized Hall Conductance

?

Bulk-Edge Correspondence

Quantum Hall Effect





Xue's group
Science 2013

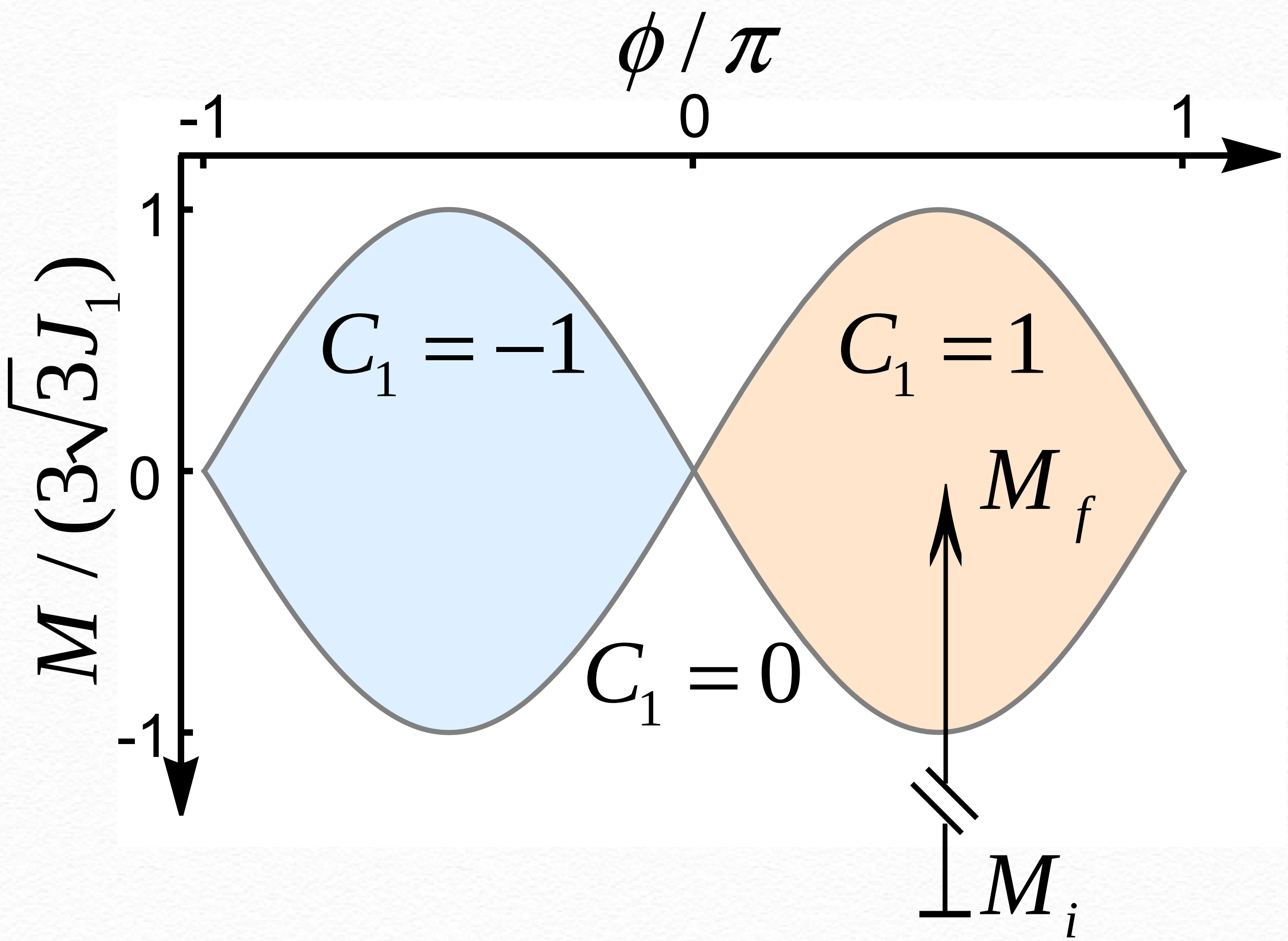
Physical Consequence of 2D Chern Insulator

At Equilibrium:

Number of Edge States	=	Bulk Chern Number	=	Quantized Hall Conductance

Bulk-Edge Correspondence

Quench Dynamics



Physical Consequence of Chern Number

At Equilibrium:

Number of Edge States	=	Bulk Chern Number	=	Quantized Hall Conductance

Bulk-Edge Correspondence

Quench Dynamics (without dissipation)

Number of Edge States	≠	Chern Number of Hamiltonian	≠	Quantized Hall Conductance
		≠		
		Chern Number of Wave Function		

D'Alessio and Rigol (2015)
Caio, Cooper, Bhassen (2015)

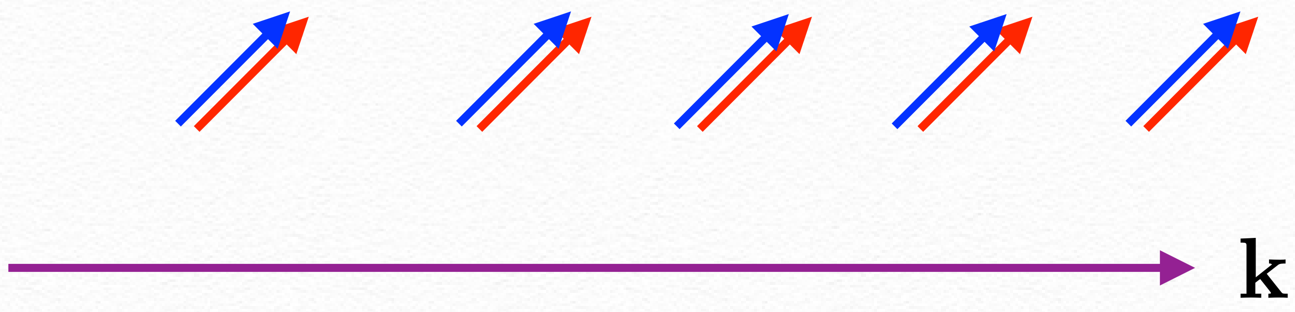
Hu, Zoller, Budike (2016),
Wilson, Song, Refael (2016)
Unal, Mueller, Oktel (2016)

Is there a Quantized Value one can extract from the quench dynamics, and this value equals to the Chern number of the bulk Hamiltonian.

Description of Quench Dynamics

A two-band Chern Insulator $\mathcal{H}(\mathbf{k}) = \frac{1}{2}\mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$

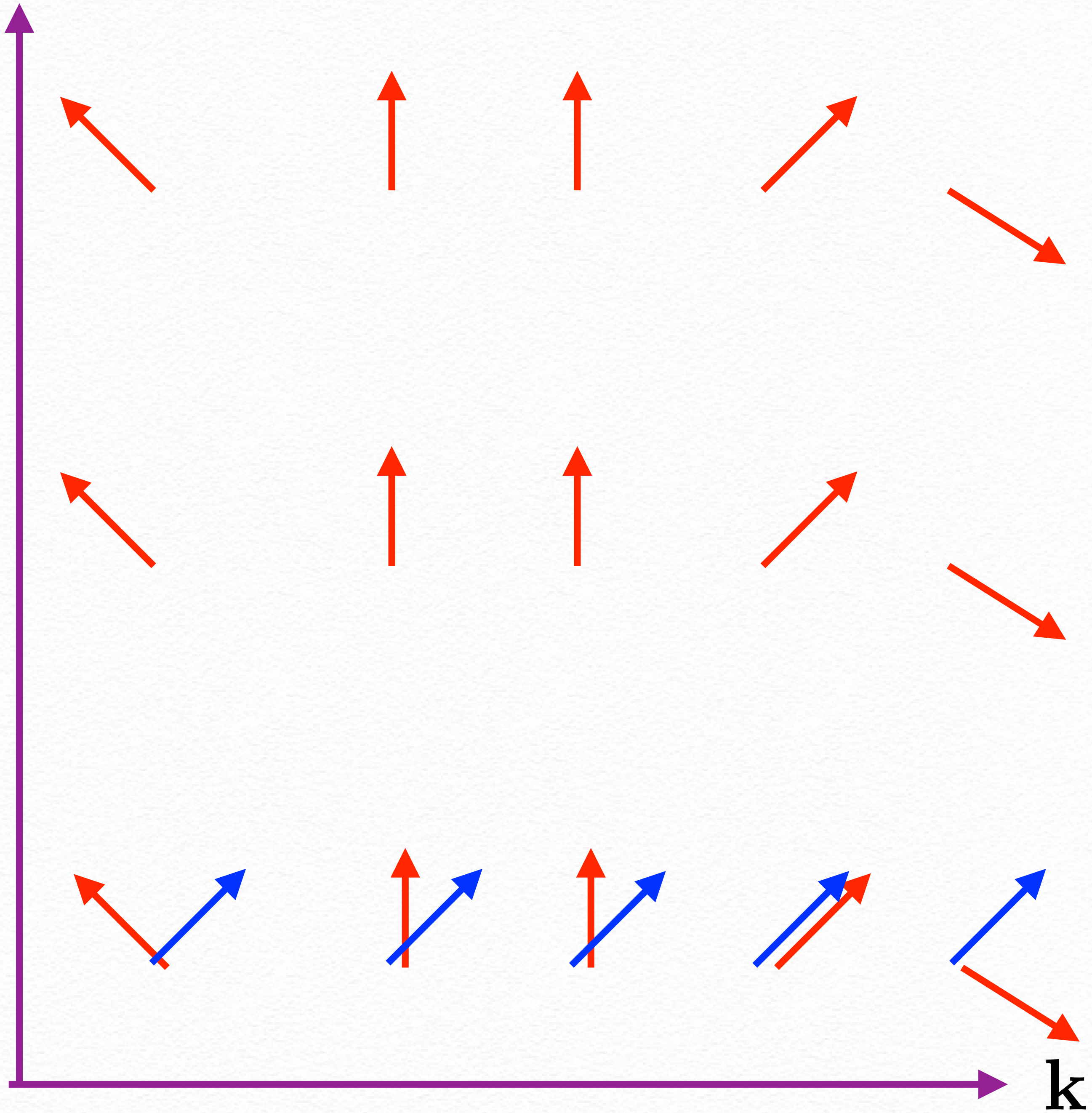
Initial hamiltonian $\mathbf{h}^{\mathrm{i}}(\mathbf{k})$



Description of Quench Dynamics

A two-band Chern Insulator $\mathcal{H}(\mathbf{k}) = \frac{1}{2}\mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$

Quench from $\mathbf{h}^{\mathrm{i}}(\mathbf{k}) \Rightarrow \mathbf{h}^{\mathrm{f}}(\mathbf{k})$



Description of Quench Dynamics

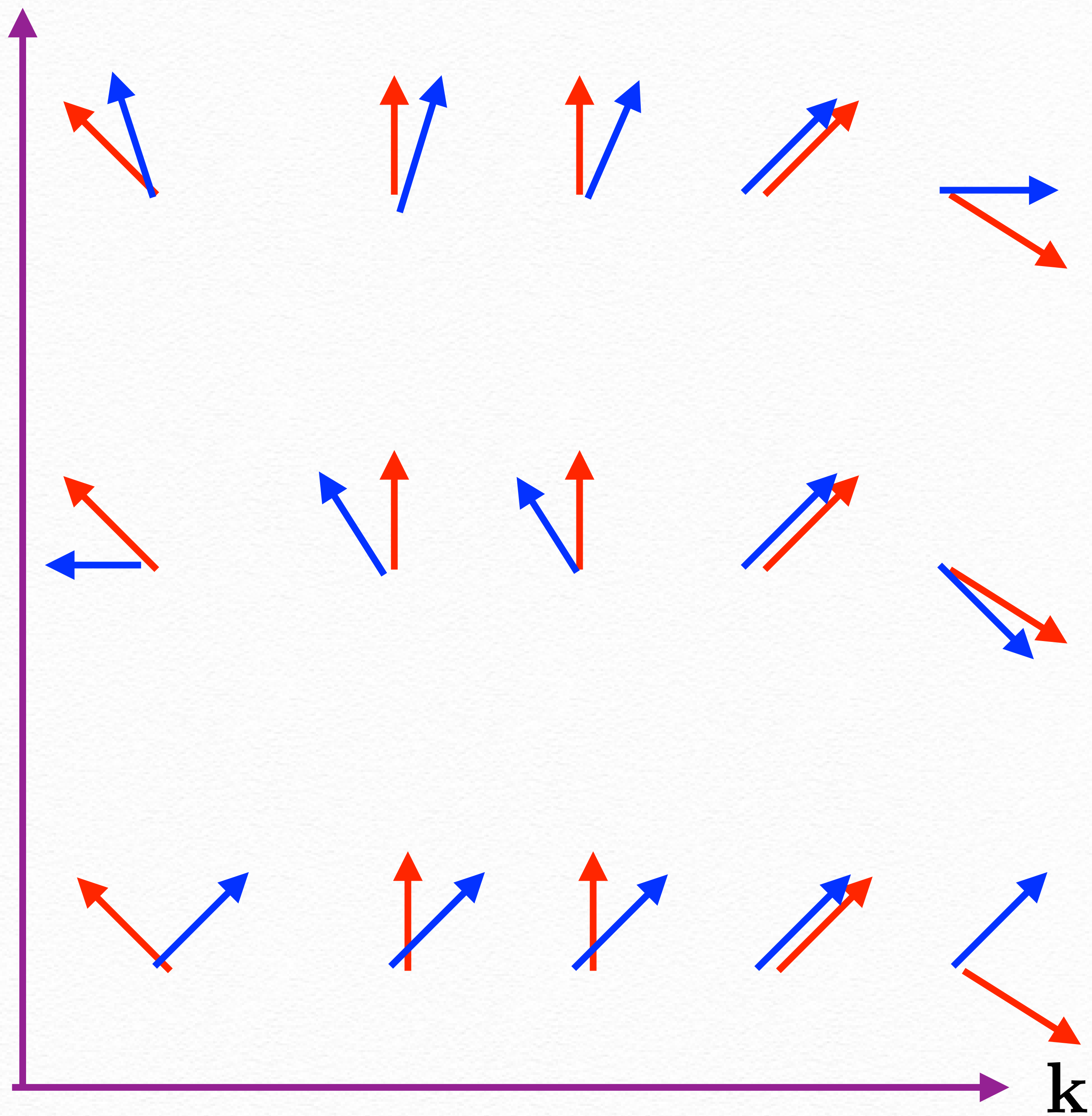
A two-band Chern Insulator $\mathcal{H}(\mathbf{k}) = \frac{1}{2}\mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$

Quench from $\mathbf{h}^{\mathrm{i}}(\mathbf{k}) \Rightarrow \mathbf{h}^{\mathrm{f}}(\mathbf{k})$

$$\zeta(\mathbf{k}, t) = \exp\left\{-\frac{i}{2}\mathbf{h}^{\mathrm{f}}(\mathbf{k}) \cdot \boldsymbol{\sigma} t\right\} \zeta^{\mathrm{i}}(\mathbf{k}),$$

$$\mathbf{n} = \zeta^{\dagger}(\mathbf{k}, t)\boldsymbol{\sigma}\zeta(\mathbf{k}, t),$$

$$[k_x, k_y, t] \Rightarrow \mathbf{n}$$



Theorem: Topology from Dynamics

For a two-band Chern Insulator $\mathcal{H}(\mathbf{k}) = \frac{1}{2}\mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$

Considering the quench dynamics described by:

$$\zeta(\mathbf{k}, t) = \exp\left\{-\frac{i}{2}\mathbf{h}^{\mathrm{f}}(\mathbf{k}) \cdot \boldsymbol{\sigma} t\right\} \zeta^{\mathrm{i}}(\mathbf{k}),$$

$$\mathbf{n} = \zeta^{\dagger}(\mathbf{k}, t)\boldsymbol{\sigma}\zeta(\mathbf{k}, t),$$

this defines a Hopf map f : $[k_x, k_y, t] \rightarrow \mathbf{n}$

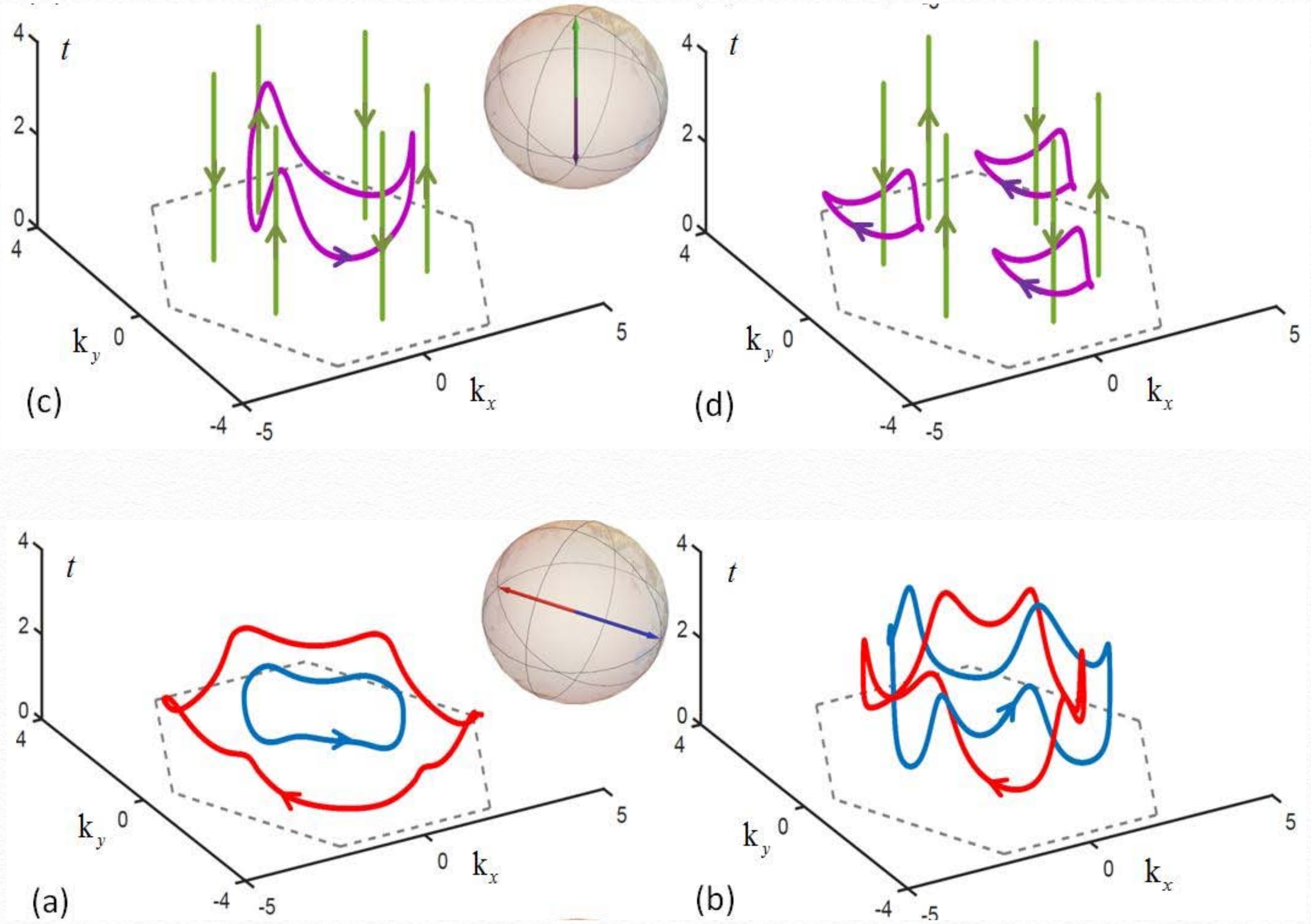
The linking number of $f^{-1}(\mathbf{n}_1)$ and $f^{-1}(\mathbf{n}_2)$

= The chern number of the final Hamiltonian

$$\Pi_3(S^2) = \Pi_2(S^2) = Z$$

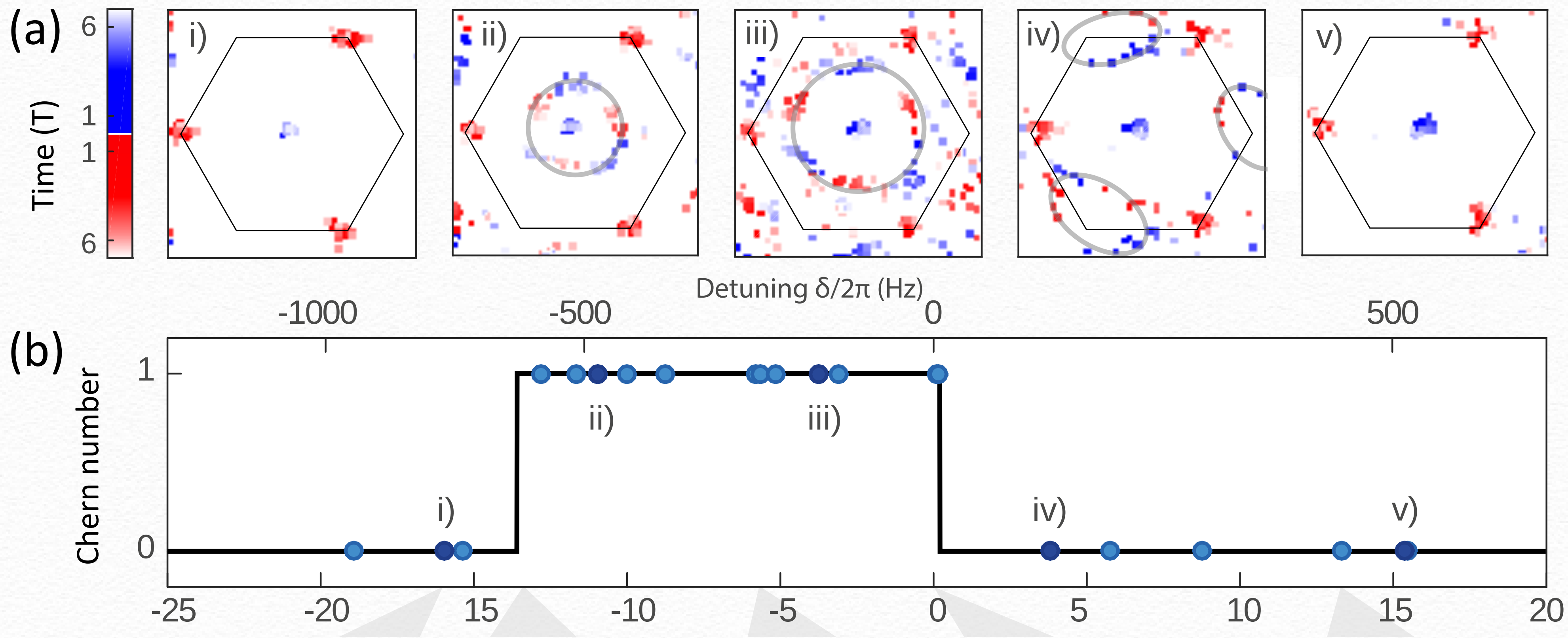
Example of Theorem

Topological Trivial   Topological Non-trivial



Experimental Observations

Haldane Model: Hamburg group



arXiv: 1709.01046

We thereby map out the trivial and non-trivial Chern number areas of the phase diagram. As shown by Wang et al. (ref. [13]), the Chern number of the post quench Hamiltonian maps onto the linking number between this contour and the position of the static vortices [Fig. 1(a)]. We thus demonstrate that the direct mapping between two topological indices – a static and a dynamical one – allows for an unambiguous measurement of the Chern number.

See also USTC group

Summary

I. Using periodic driven to realize 2D band
with non-zero Chern number.

Dynamics ⇉ Topological Band Theory

II. 1st Chern Number

Near Equilibrium:

1+1 D Charge Pumping
2D Transport

Traditional Way

Far from Equilibrium:

2+1 D Linking Number
in quench dynamics

Novel Way

Thank You Very Much for Attention !