Fundamental Aspects of Quantum Dynamics:

I: Symmetry

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Quantum Few- and Many-Body Physics in Ultracold Atoms
Wuhan
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Fundamental Aspects of Quantum Dynamics

- Related to Fundamental Concepts
- Universal Phenomenon
- Mathematically Rigorous
- Directly Relevant to Experiments

Symmetry and Dynamics

I. Probe

Dynamics $\rightleftarrows$ Symmetry of a Hamiltonian

II. Constraint

Symmetry of a Hamiltonian

The Hamiltonian is invariant under certain operation.

- **Translation** $\mathbf{r} \to \mathbf{r} + \mathbf{R}$
- **Discrete Translation** $\mathbf{r} \to \mathbf{r} + n\mathbf{R}_0$
- **Reflection** $\mathbf{r} \to -\mathbf{r}$
- **Scaling** $\mathbf{r} \to \lambda \mathbf{r} \qquad t \to \lambda^2 t$

For example, non-interacting system:

$$i\hbar \frac{\partial}{\partial t} \Psi = -\sum_i \frac{\hbar^2}{2m} \nabla_i^2 \Psi$$

I. Probe

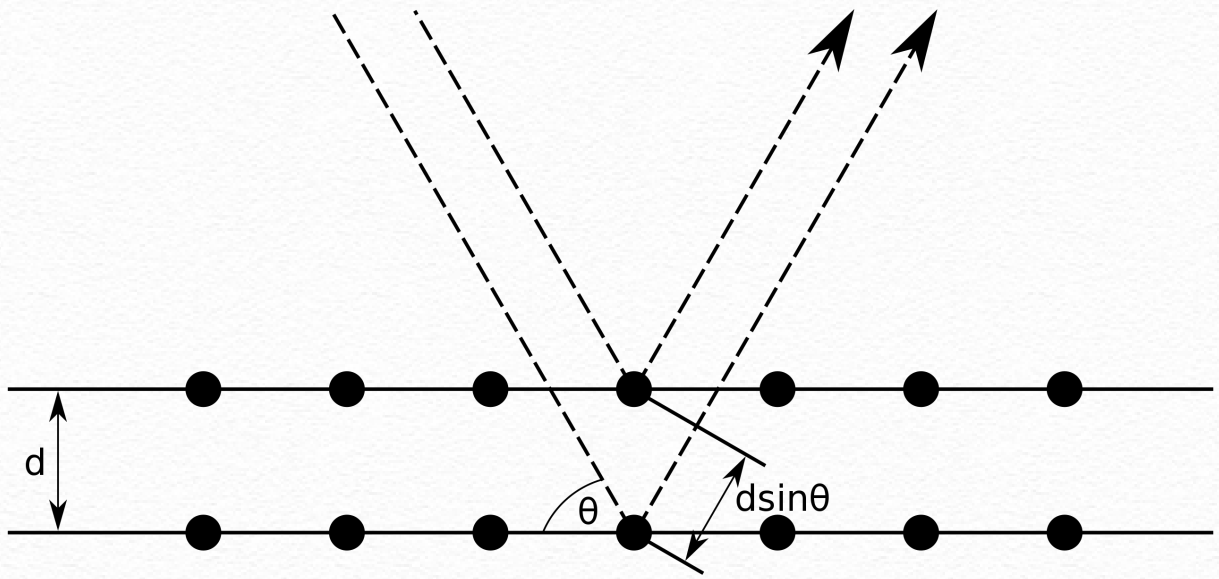
Dynamics $\longrightarrow$ Symmetry of a Hamiltonian

Ref. Shijin Deng, Zhe-Yu Shi, Pengpeng Diao, Qianli Yu, HZ, Ran Qi and Haibin Wu, Science, 371, 353, 2016

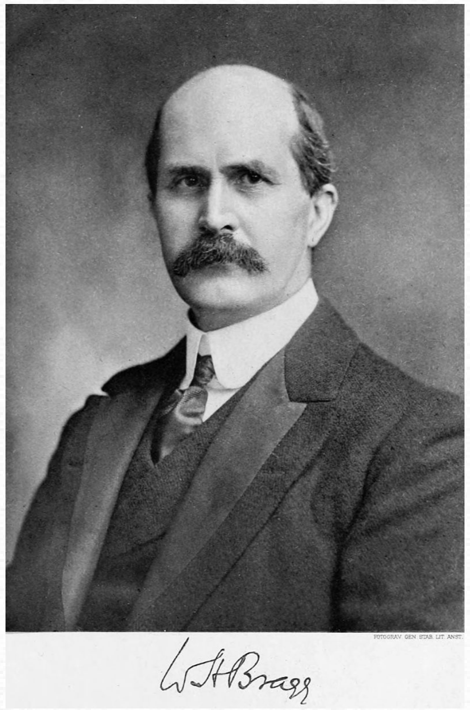
Detection Symmetry via Dynamics

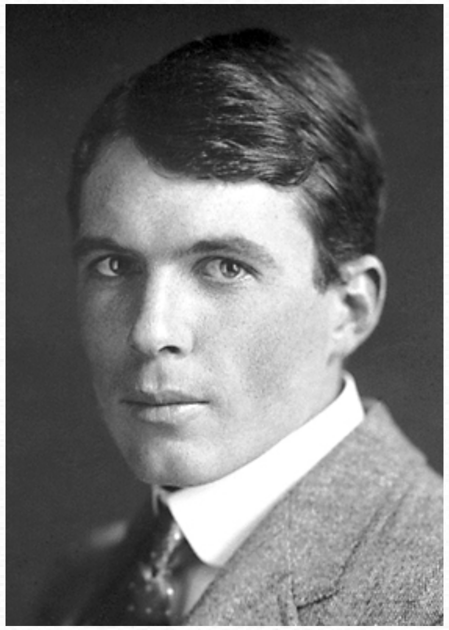
Classical example: Bragg scattering

- **Discrete Translation** $\quad \mathbf{r} \to \mathbf{r} + n\mathbf{R}_0$



$$2d\sin\theta = n\lambda$$

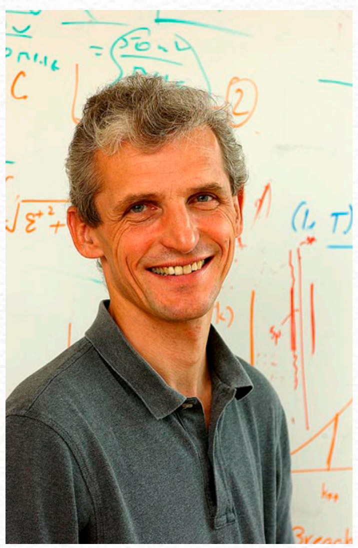




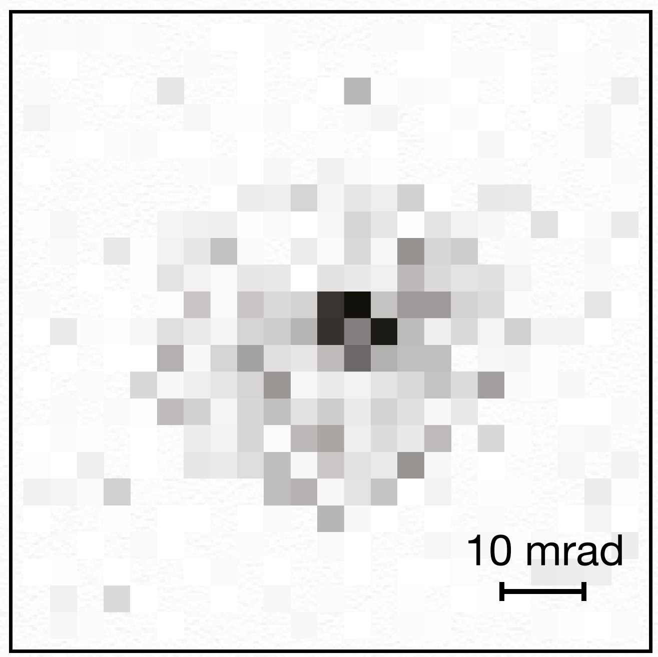
Detection Symmetry via Dynamics

Application for detecting Stripe Phase in BEC

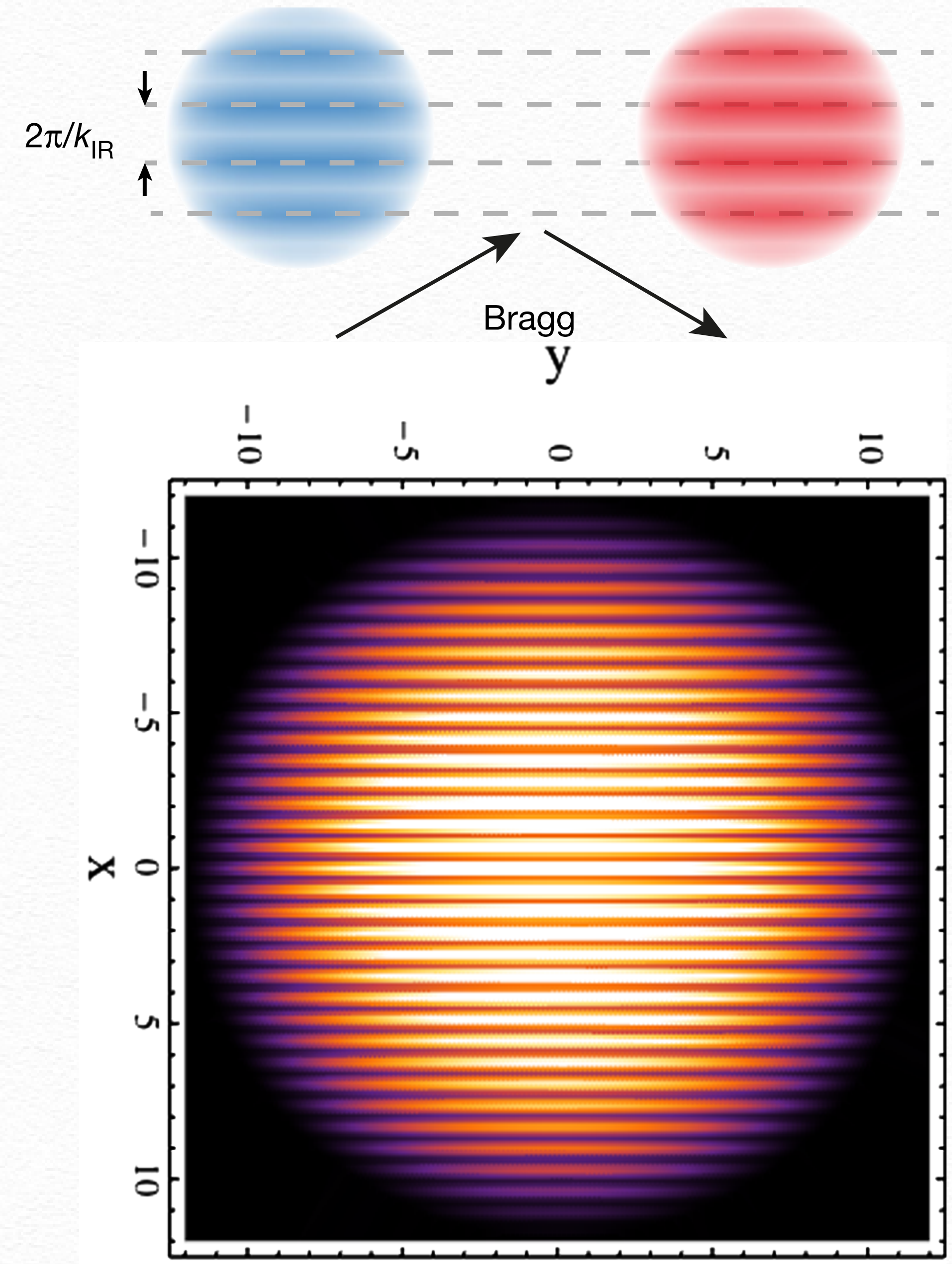
A stripe phase with supersolid properties in spin–orbit-coupled Bose–Einstein condensates



MIT, Nature 2017



Bragg + Rayleigh



Predicted:

Chunji Wang, Chao Gao, Chao-Ming Jian and HZ, PRL, 105, 160403 (2010)

Scale Invariant Quantum Systems

$$i\hbar \frac{\partial}{\partial t}\Psi = -\sum_i \frac{\hbar^2}{2m}\nabla_i^2 \Psi$$

Scale Transformation

$$\mathbf{r}_i \longrightarrow \Lambda \mathbf{r}_i$$
$$t \longrightarrow \Lambda^2 t$$

$$\frac{1}{\Lambda^2} \qquad\qquad \frac{1}{\Lambda^2}$$

No other energy scale except for the kinetic energy

Zoo of Scale Invariant Quantum Gases

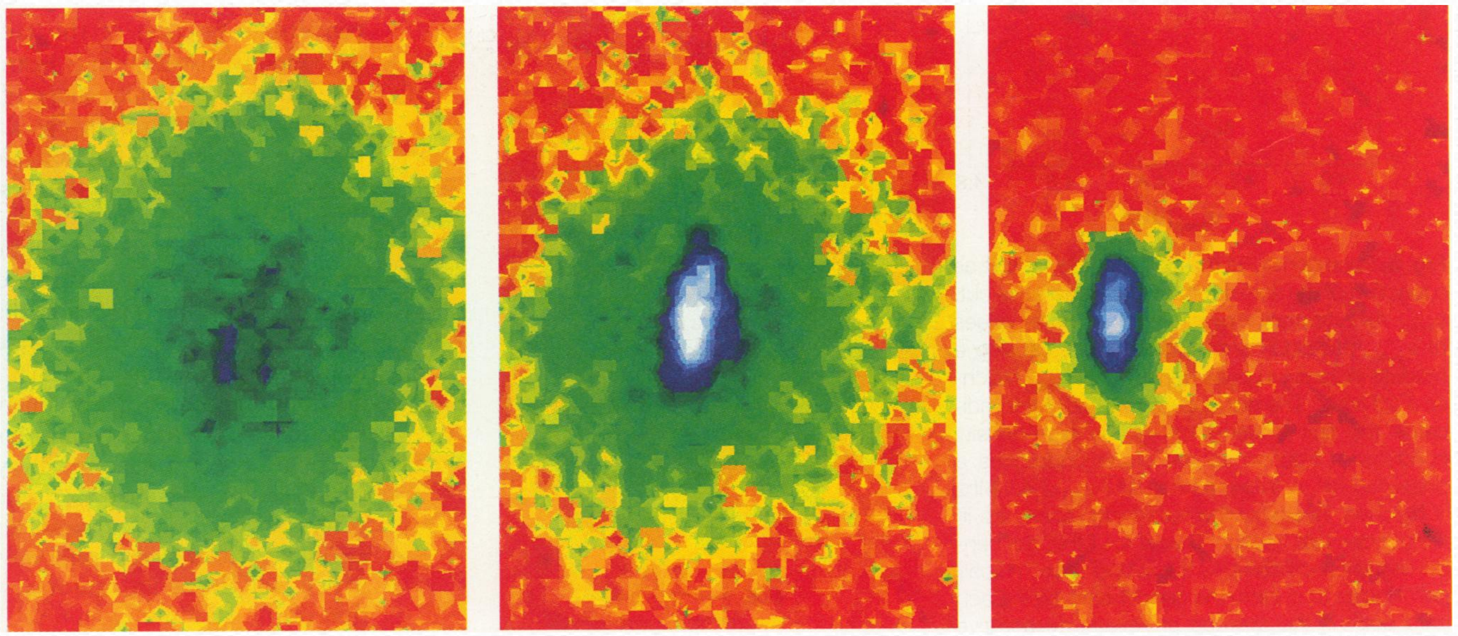
Non-interacting bosons/ fermions at any dimension	**No other length scale except for density**
Unitary Fermi gas at three dimension	**Density and a_s** $a_s = \infty$
Tonks gas of bosons/ fermions at one dimension	**Density and g_{1D}** $g_{1D} = \infty$

Universal behavior: $\boxed{\langle V \rangle = \alpha \langle T \rangle}$

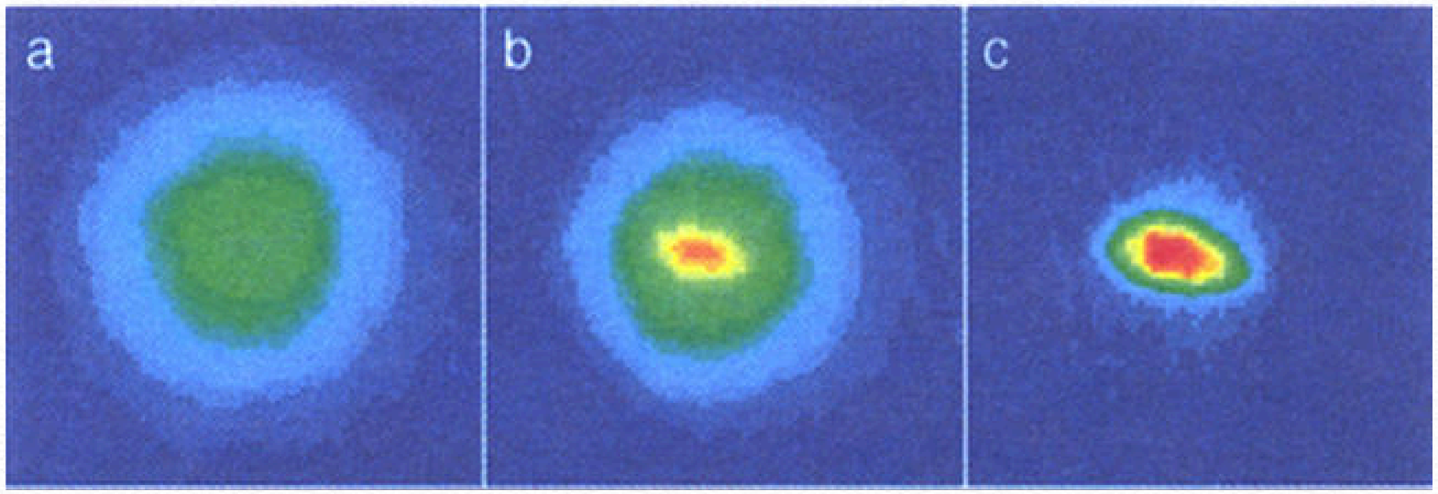
How to detect the scaling symmetry ?

The First BEC Experiments

Expansion dynamics has been used to probe BEC

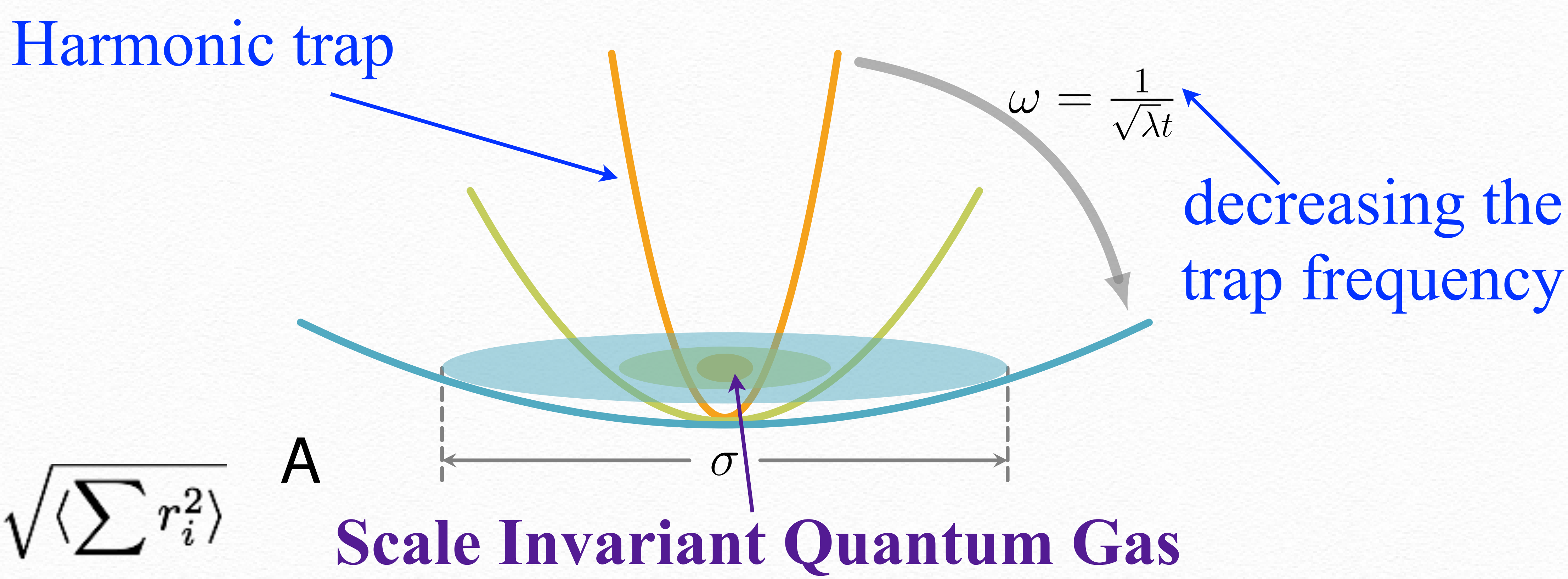


JILA Group



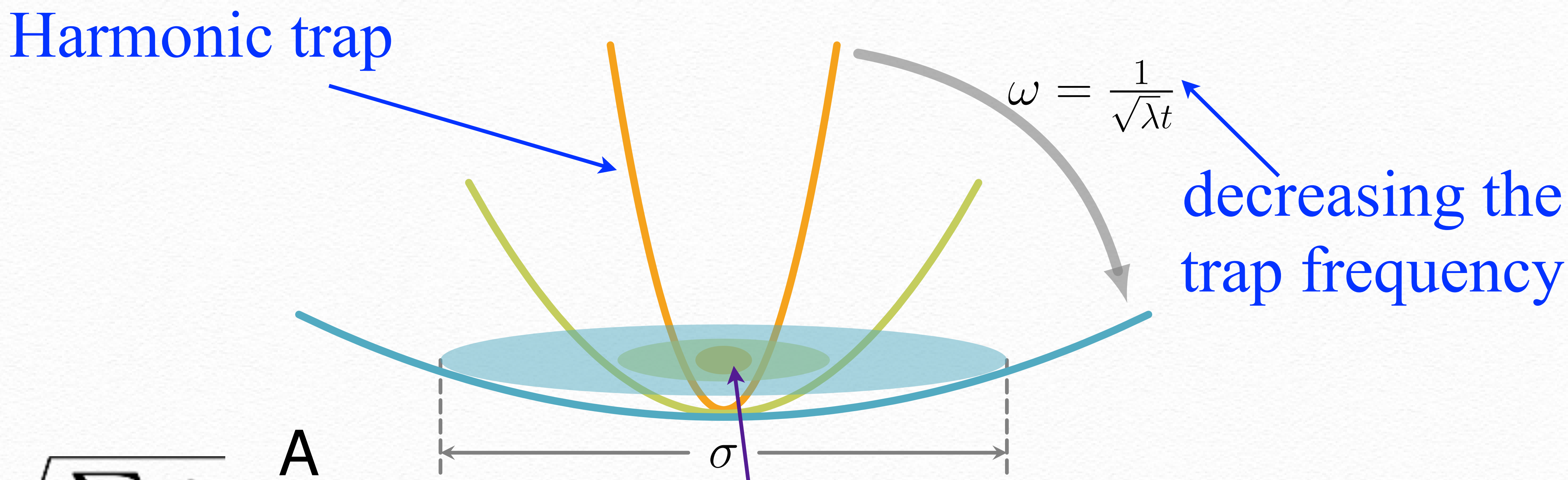
MIT Group

Efimovian Expansion



$$R = \sqrt{\langle \sum r_i^2 \rangle}$$

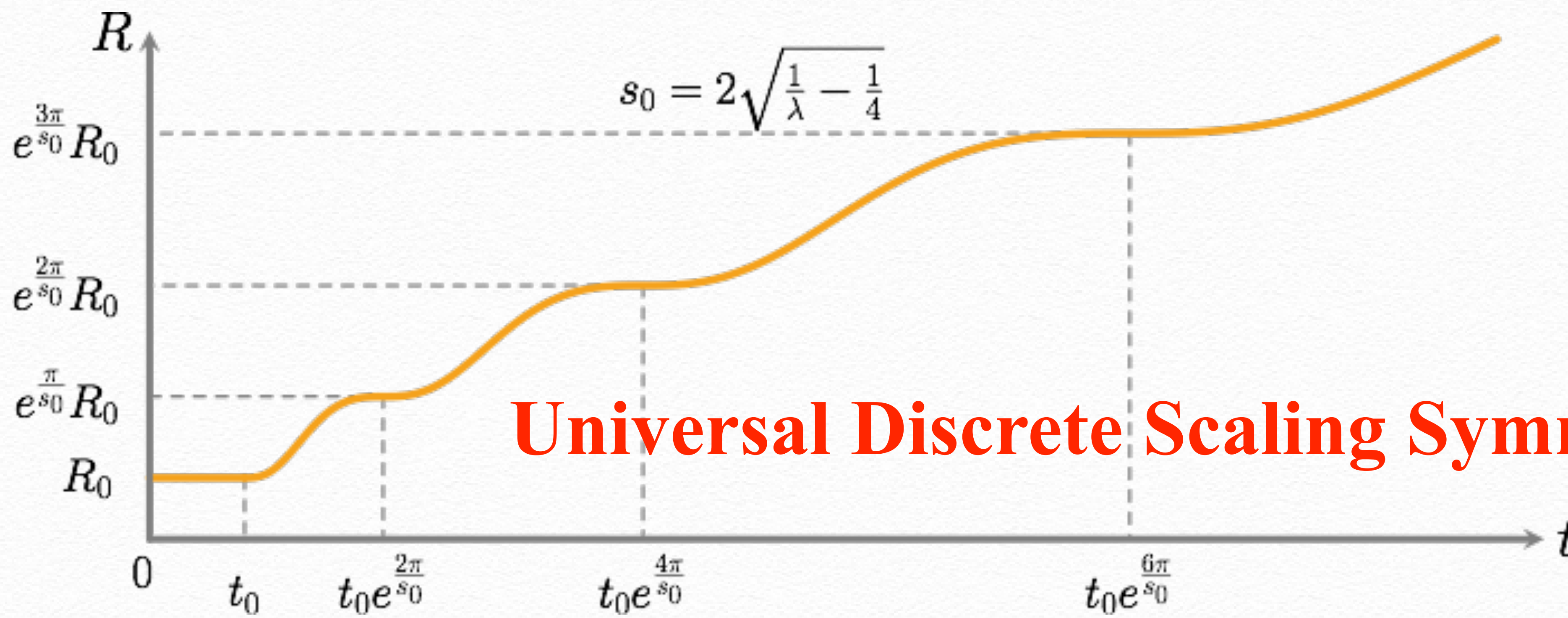
Efimovian Expansion



A

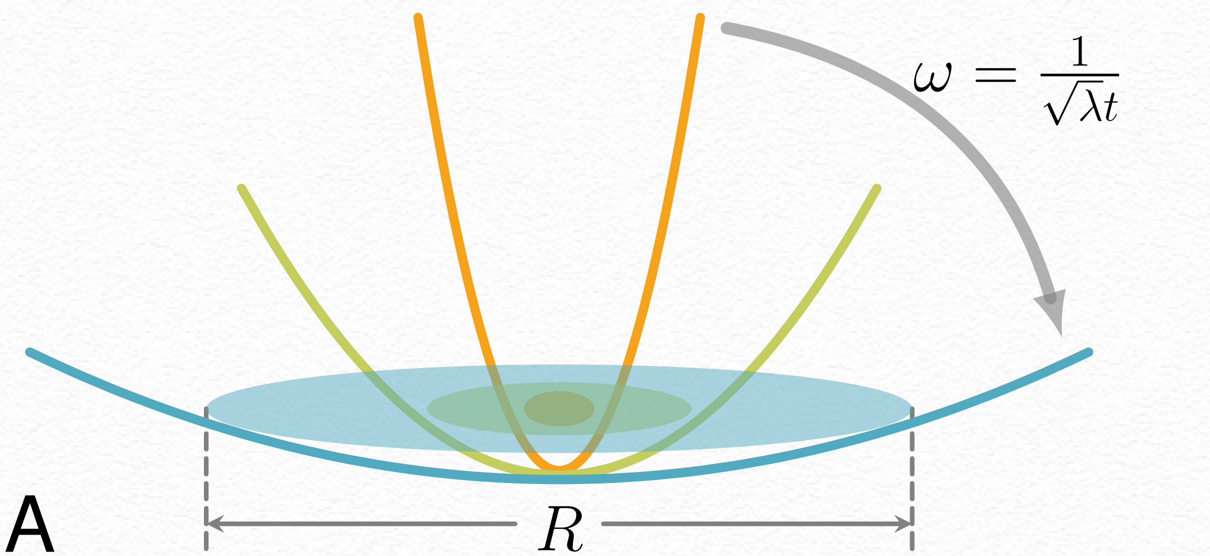
$$R = \sqrt{\langle \sum r_i^2 \rangle}$$

Scale Invariant Quantum Gas



Universal Discrete Scaling Symmetry

Scaling Symmetry in a Harmonic Trap



A

$$i\hbar \frac{\partial}{\partial t}\Psi = \left[H + \sum_i \frac{1}{2} m \omega^2 r_i^2 \right] \Psi$$

$$\frac{1}{\Lambda^2} \qquad \frac{1}{\Lambda^2} \qquad \frac{1}{\Lambda^2}$$

Scale Transformation

$$\mathbf{r}_i \to \Lambda \mathbf{r}_i$$
$$t \to \Lambda^2 t$$

This scaling symmetry exists only if

$$\boxed{\omega = \frac{1}{\sqrt{\lambda} t}}$$

Expansion Dynamics

$$i\frac{d}{dt}R^2 = \sum_i \langle [r_i^2, H] \rangle = 2i\langle \hat{D} \rangle$$

$$i\frac{d}{dt}\langle \hat{D} \rangle = \langle [\hat{D}, H] \rangle = 2i\Big(\langle H \rangle - \omega^2 R^2 \Big)$$

$$\frac{d}{dt}\langle H \rangle = \langle \frac{\partial}{\partial t} H \rangle = \omega\dot{\omega}R^2$$

$$\boxed{\frac{d^3}{dt^3}R^2 + 4\omega^2\frac{d}{dt}R^2 + 4\omega\dot{\omega}R^2 = 0}$$

Why the equation-of-motion closes ? An incident ?

Conformal Symmetry

The Schrodinger Group Symmetry

temporal translation: $H = -i\partial_t$

spatial translation: $P^i = -i\partial_i$

spatial rotation: $M^{ij} = ix_i\partial_j - ix_j\partial_i,$

Galilean boost: $K^i = -it\partial_i - mx_i$

dilation: $D = -2it\partial_t - ix_i\partial_i - i\frac{d}{2}$

special Schrodinger transformation: $\boxed{C = -it^2\partial_t - itx_i\partial_i - \frac{1}{2}m\vec{x}^2 - i\frac{d}{2}t.}$

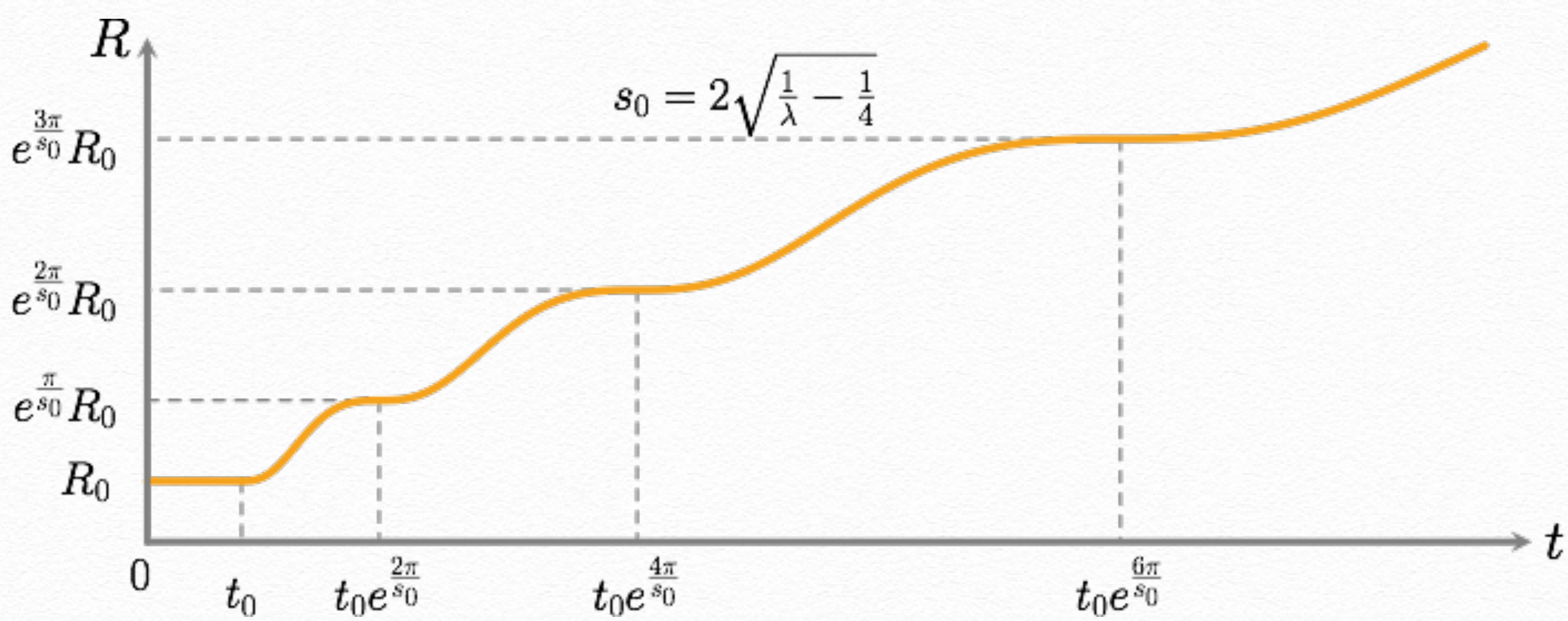
Ref: D. T. Son

Generalization to relativistic case to probe conformal symmetry

Expansion Dynamics

$\lambda < 4$

$$\frac{\langle \hat{R}^2 \rangle (t)}{R_0^2} = \frac{t}{t_0} \frac{1}{\sin^2 \varphi} \left[1 - \cos \varphi \cdot \cos \left(s_0 \ln \frac{t}{t_0} + \varphi \right) \right]$$

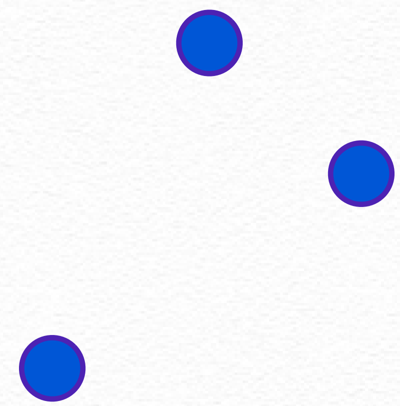


Why plateaus ?

$$\frac{d^n}{dt^n} \langle \hat{R}^2 \rangle |_{t=t_0} = 0$$

Efimov Physics with Three Bosons

Problem: Three bosons interacting through a short-range interaction



1970

$$\left[-\frac{\hbar^2 \mathrm{d}^2}{2m\mathrm{d}\rho^2} - \frac{s_0^2 + 1/4}{m\rho^2}\right]\psi = E\psi$$

$$\boxed{\psi = \sqrt{\rho}\cos[s_0 \log(\rho/\rho_0)]}$$

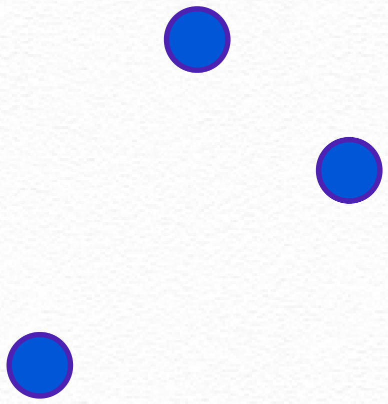
$$\rho \longrightarrow e^{2\pi/s_0}\rho$$

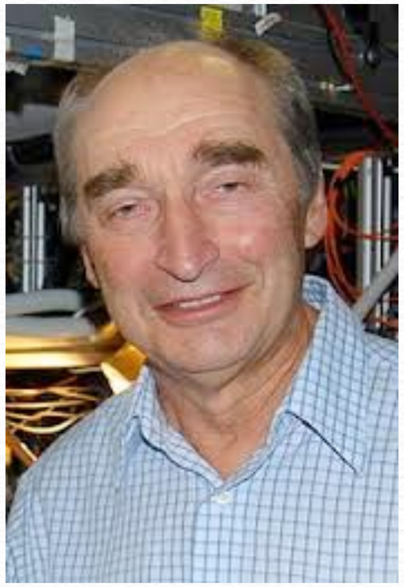
$$E_{\mathrm{T}}^{(n+1)}/E_{\mathrm{T}}^{(n)} \simeq e^{-2\pi/s_0}$$

Discrete Scaling Symmetry

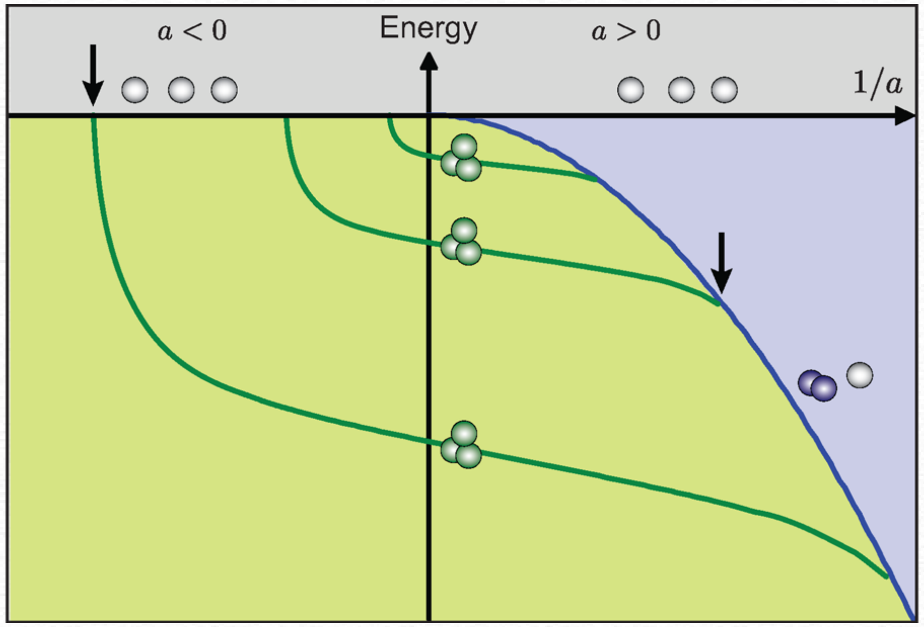
Efimov Physics with Three Bosons

Problem: Three bosons interacting through a short-range interaction





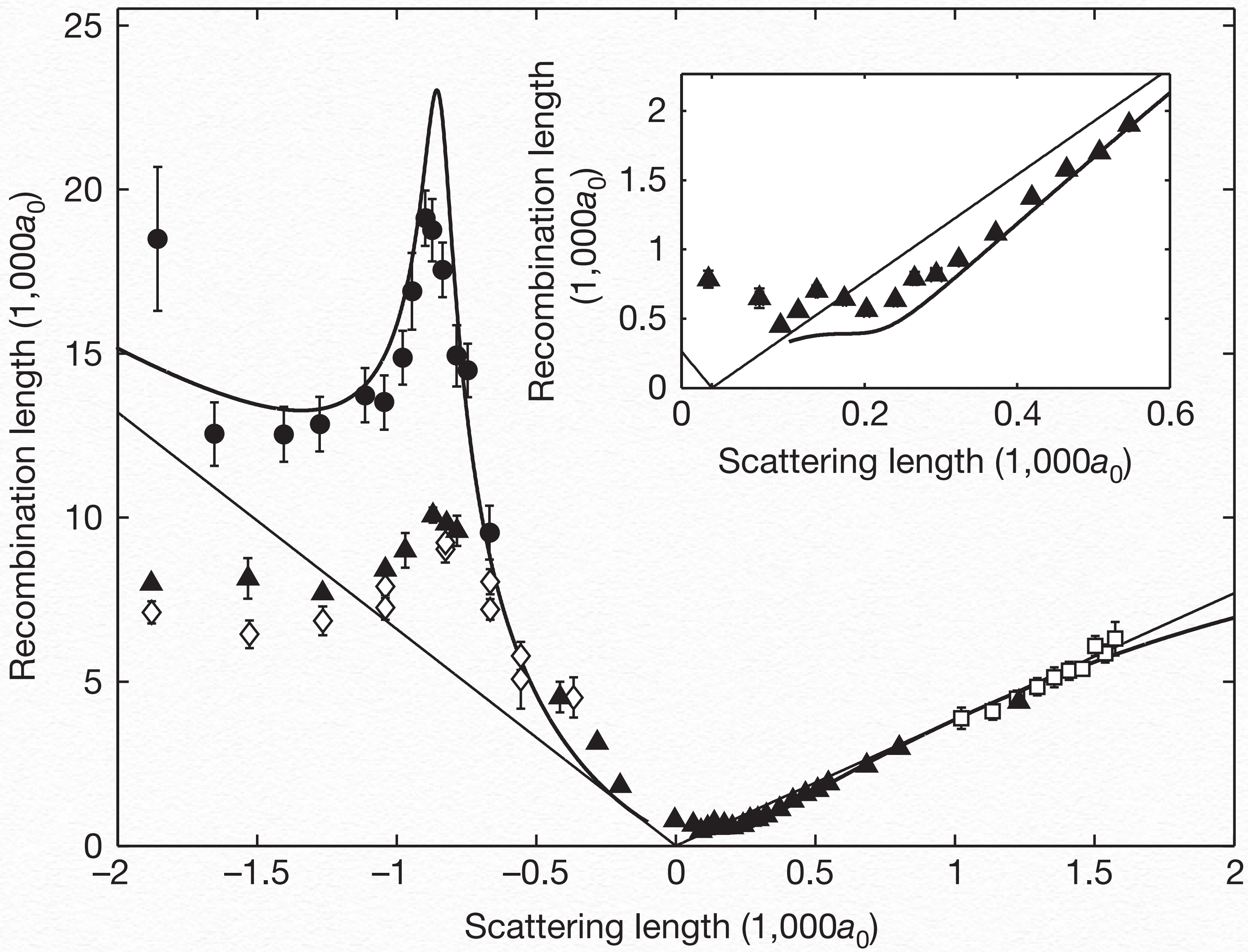
1970



Universal Discrete Scaling Symmetry

Renormalization Group View: Limit Circle Solution

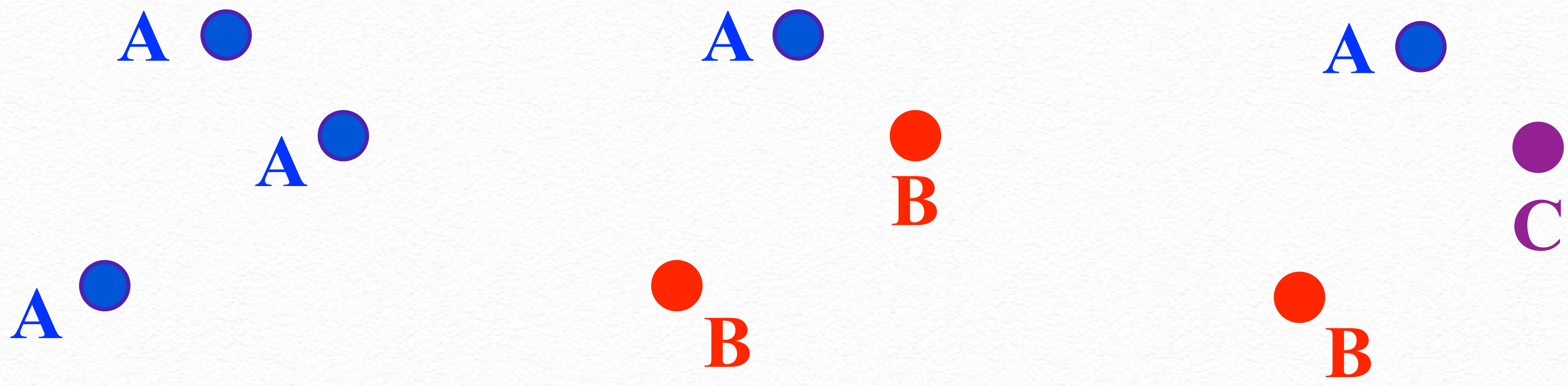
First Experimental Evidence of Efimov Effect



Three bosons of Cs atoms

Grimm's group @ Innsbruck, Nature 2006

Experimental Evidences of Efimov Effect



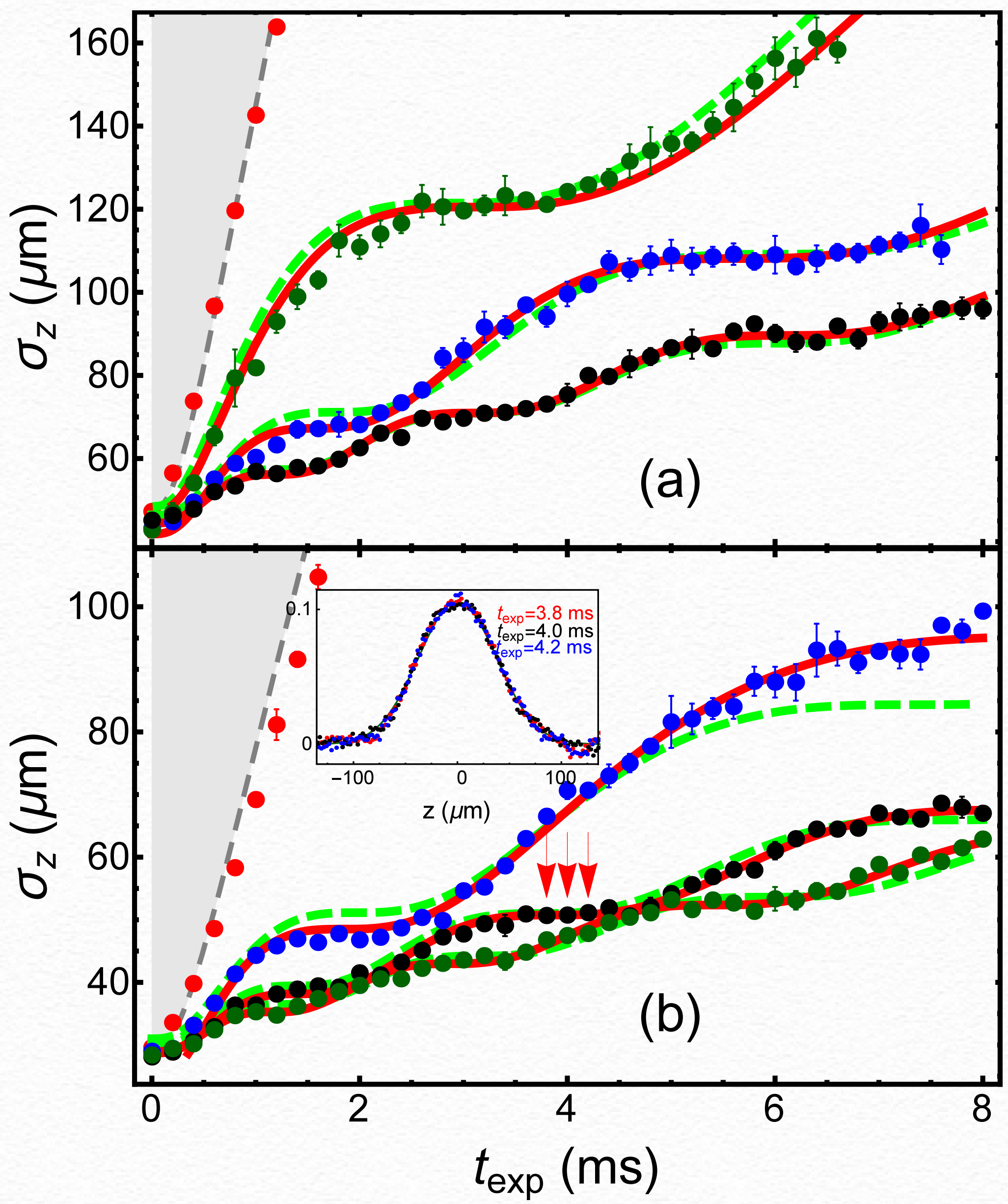
AAA	ABB	ABC
^{133}Cs(Innsbruck, 2006)	^{41}K-^{87}Rb (Florence 2009? Osaka City 2017)	^{6}Li(Heidelberg 2008, Penn State 2009, Tokyo 2010)
^{39}K(Florence, 2009)	^{40}K-^{87}Rb (JILA, 2013)	
^{7}Li(Rice, Bar Ilan, 2009)	^{6}Li-^{133}Cs (Heidelberg and Chicago, 2014)	
^{85}Rb(JILA, 2012)	^{7}Li-^{87}Rb (Tubingen, 2015)	

Efimov States v.s. Efimovian Expansion

The Efimov Effect	The Efimovian Expansion
$-\frac{\hbar^2 d^2}{2md^2\rho}\psi - \frac{\lambda}{\rho^2}\psi = E\psi$	$\frac{d^3}{dt^3}\langle\hat{R}^2\rangle + \frac{4}{\lambda t^2}\frac{d}{dt}\langle\hat{R}^2\rangle - \frac{4}{\lambda t^3}\langle\hat{R}^2\rangle = 0.$
Spatial continuous scaling symmetry	Temporal continuous scaling symmetry
Short-range boundary condition	Initial time
$\psi = \sqrt{\rho}\cos[s_0\log(\rho/\rho_0)]$	$\frac{\langle\hat{R}^2\rangle(t)}{R_0^2} = \frac{t}{t_0}\frac{1}{\sin^2\varphi}\left[1-\cos\varphi\cdot\cos\left(s_0\ln\frac{t}{t_0}+\varphi\right)\right]$
Spatial discrete scaling symmetry $\rho \rightarrow e^{2\pi/s_0}\rho$	Temporal discrete scaling symmetry $t \rightarrow e^{2\pi/s_0}t$

Experimental Observation

by Haibin Wu in East China Normal University



Non-interacting

Unitary Fermions

$$\frac{\langle \hat{R}^2 \rangle (t)}{R_0^2} = \frac{t}{t_0} \frac{1}{\sin^2 \varphi} \left[1 - \cos\varphi \cdot \cos\left(s_0 \ln \frac{t}{t_0} + \varphi \right) \right]$$

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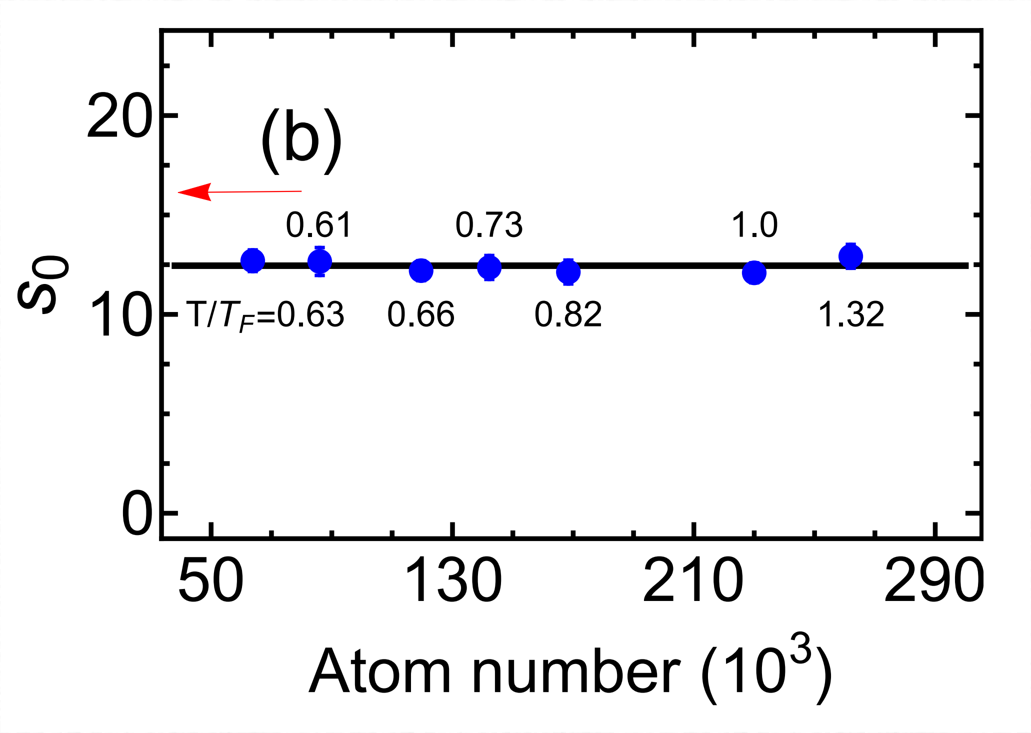

Independent of Temperature

Independent of State of Matter

Experimental Observation

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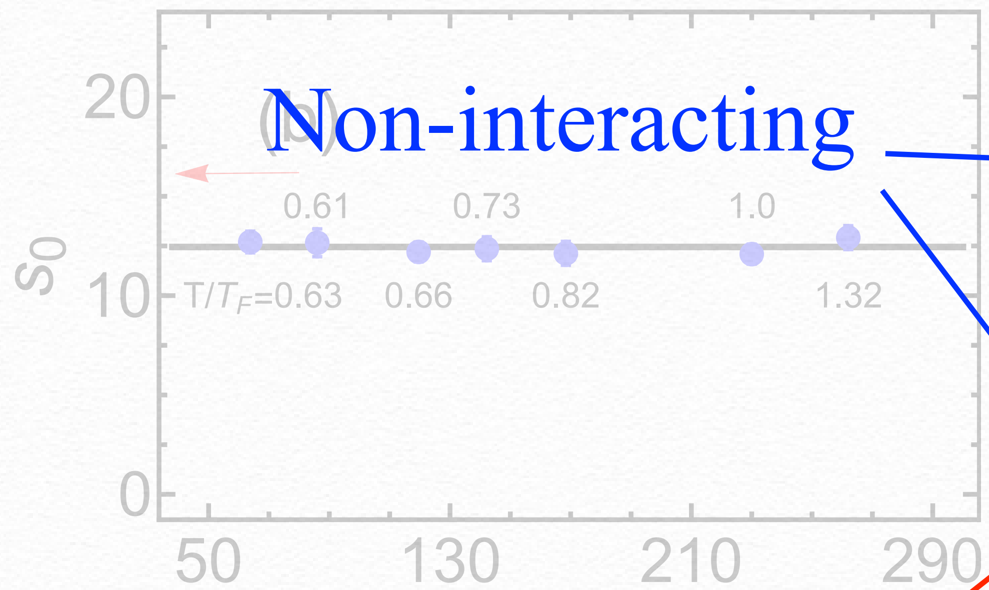
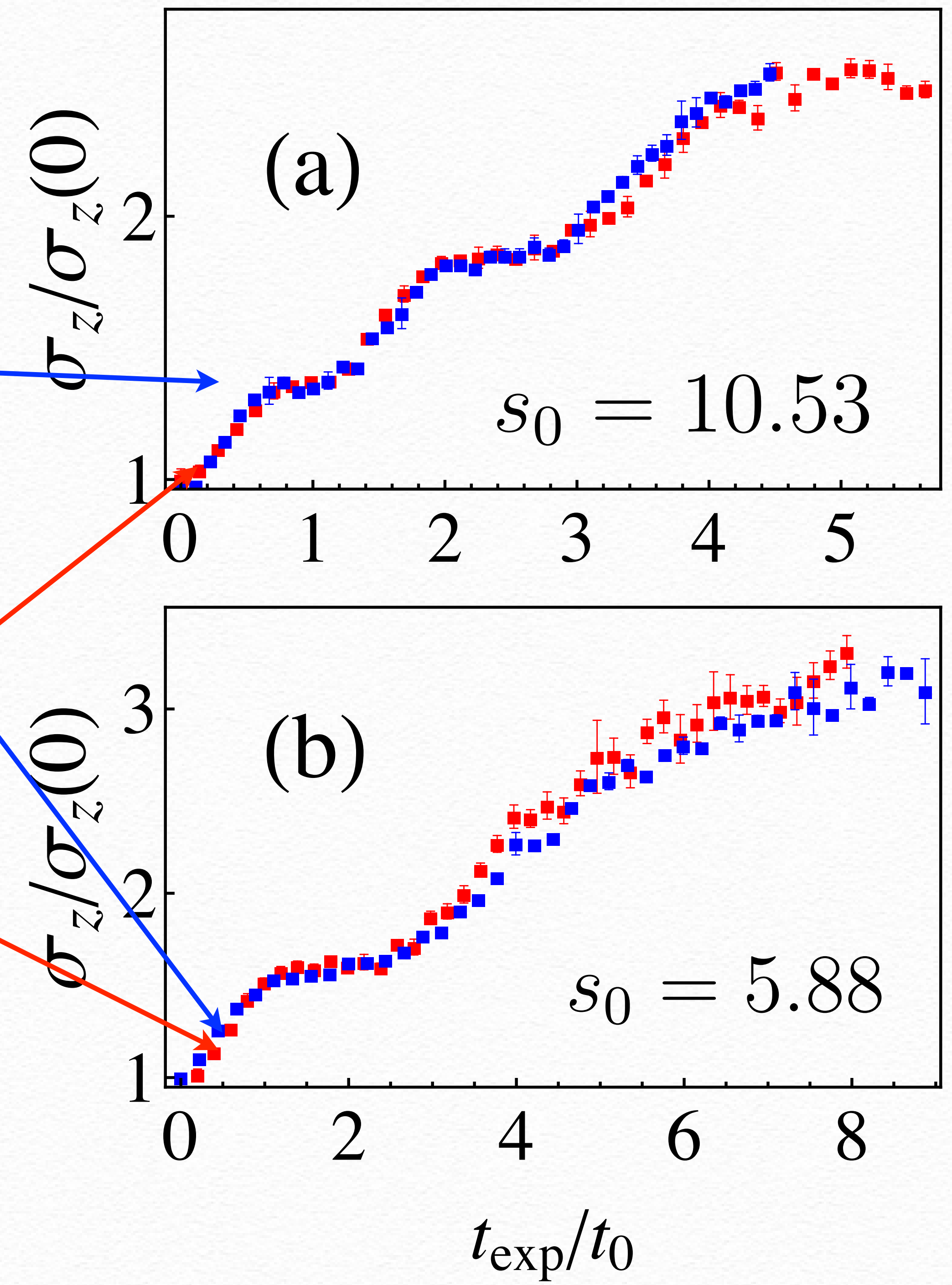


Non-interacting

Unitary fermions



Independent of Temperature

Independent of State of Matter

Symmetry and Dynamics

Dynamics $\longleftarrow$ Symmetry of a Hamiltonian

II. Constraint

Ref. Jinlong Yu, Ning Sun and HZ, PRL, 119, 225302 (2017)

Question to Address

- Starting from an initial state $|\Psi_0\rangle$
- Wave function evolves as $e^{-i(\hat{H}_0+\hat{V})t}|\Psi_0\rangle$

$$\hat{V} = U\sum_i n_i(n_i - 1) \quad \text{for bosons}$$

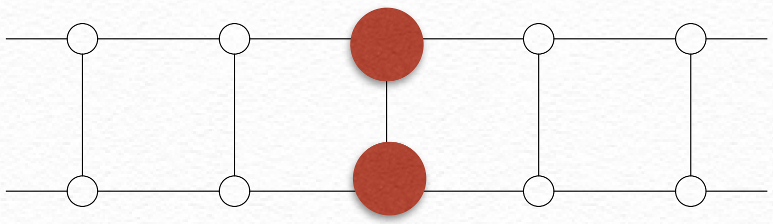
$$\hat{V} = U\sum_i n_{i\uparrow}n_{i\downarrow} \quad \text{for fermions}$$

- Measure an operator with repulsive and attractive models

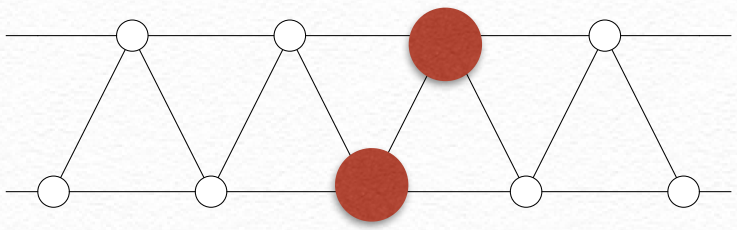
$$\boxed{\langle O(t)\rangle_{+U} = \pm\langle O(t)\rangle_{-U}} \quad ?$$

Example 1: Hubbard Model

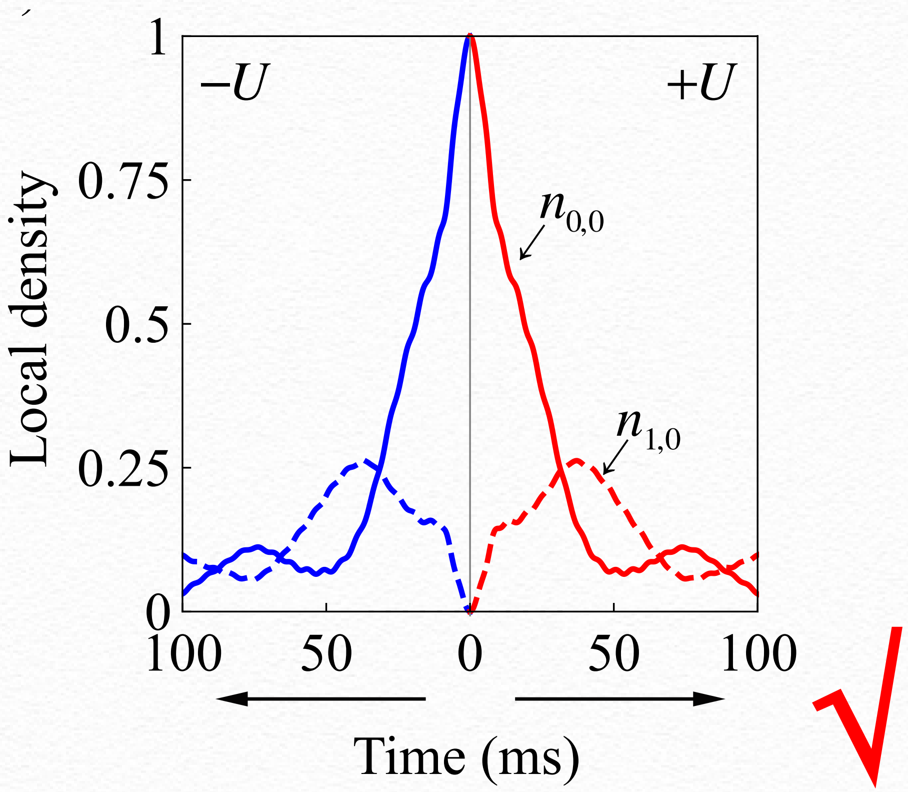
$$H_0 = -J \sum_{\langle ij \rangle,\sigma} c^{\dagger}_{i\sigma} c_{j\sigma}$$

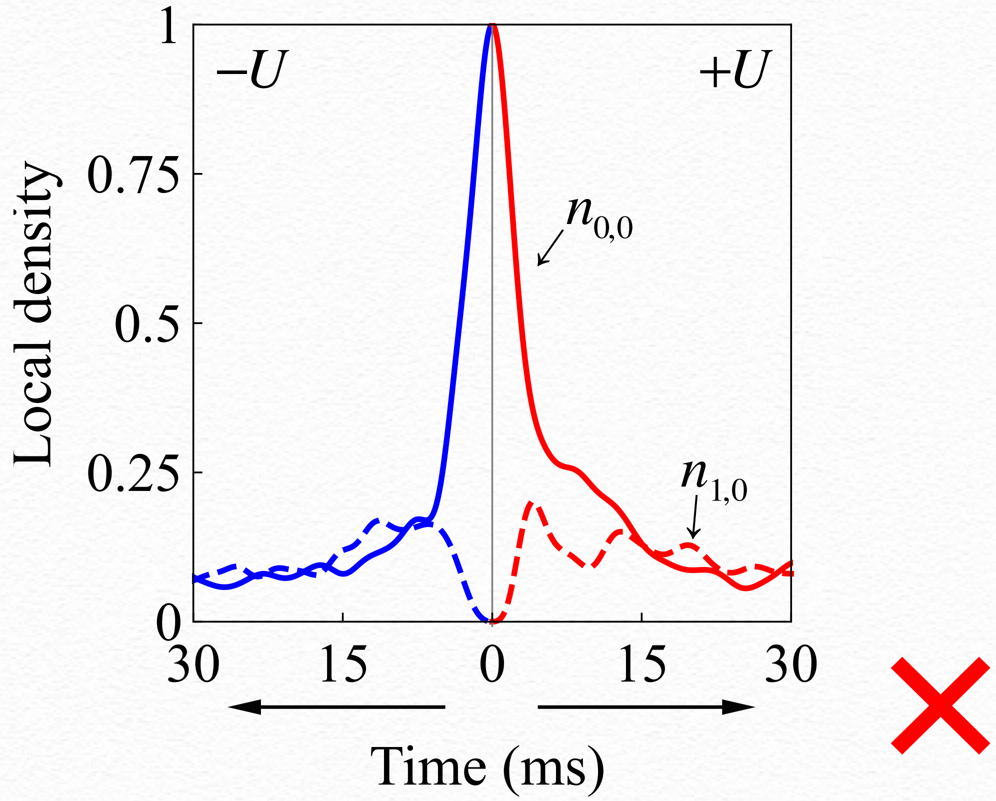


Square Lattice

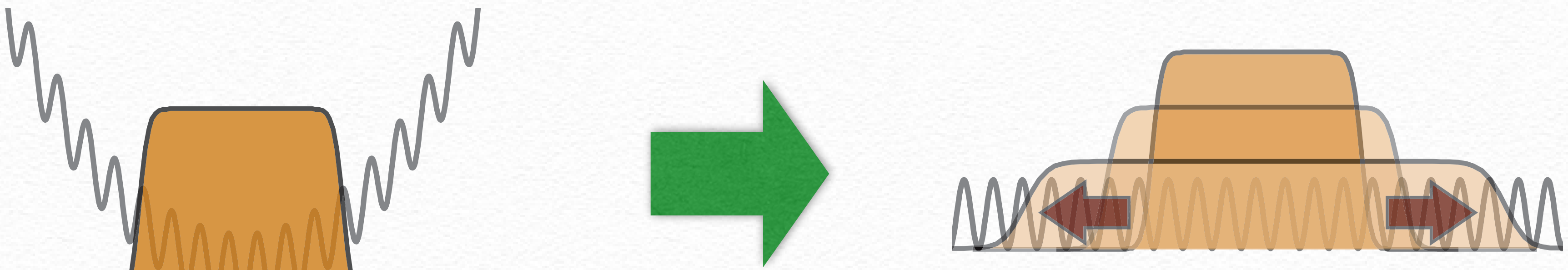


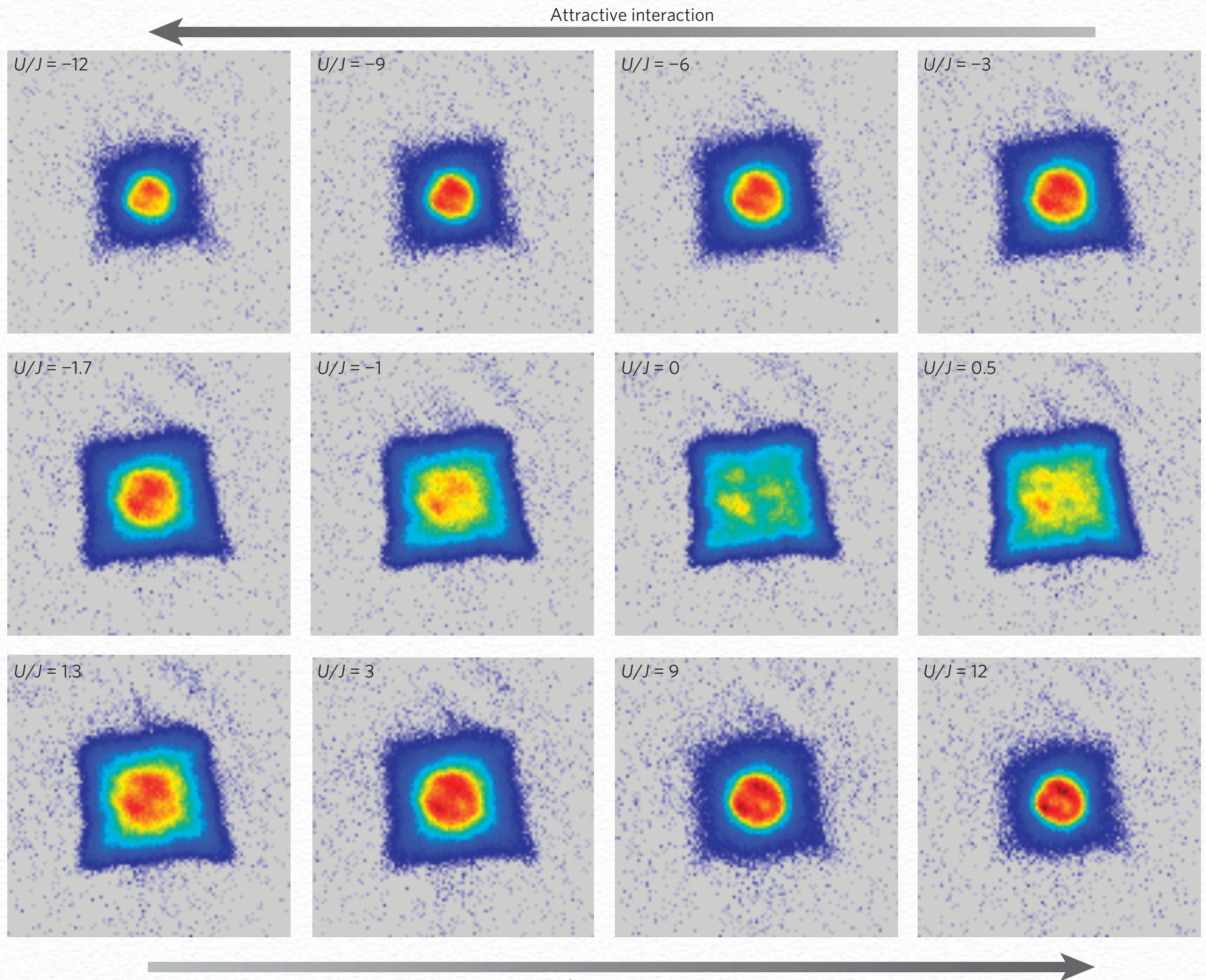
Triangular Lattice





Example 1: Hubbard Model

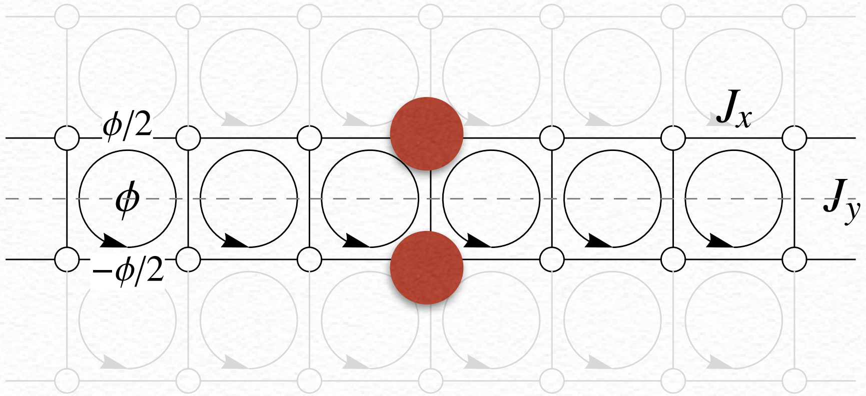




Munich Group, U. Schneider, et.al. Nat. Phys. 2012

Example 2: Hofstader-Hubbard Model

$$\hat{H}_0 = -J_x \sum_{i,\sigma} \left[e^{i(i_y-1/2)\phi} \hat{c}^{\dagger}_{i_x,i_y,\sigma} \hat{c}_{i_x-1,i_y,\sigma} + \text{h.c.} \right] - J_y \sum_{i,\sigma} \left[\hat{c}^{\dagger}_{i_x,i_y,\sigma} \hat{c}_{i_x,i_y-1,\sigma} + \text{h.c.} \right]$$



Chirality of two-particle motion

$$\Delta y_{\text{COM}}(t) = \frac{\langle \hat{O}^R_-(t) \rangle}{\langle \hat{O}^R_+(t) \rangle} - \frac{\langle \hat{O}^L_-(t) \rangle}{\langle \hat{O}^L_+(t) \rangle}$$

$$\hat{O}^R_\pm = \sum_{i_x>0} (\hat{n}_{i_x,i_y=1} \pm \hat{n}_{i_x,i_y=0}),$$

$$\hat{O}^L_\pm = \sum_{i_x<0} (\hat{n}_{i_x,i_y=1} \pm \hat{n}_{i_x,i_y=0}).$$

Harvard Group, M. E. Tai, et.al. Nature 2017

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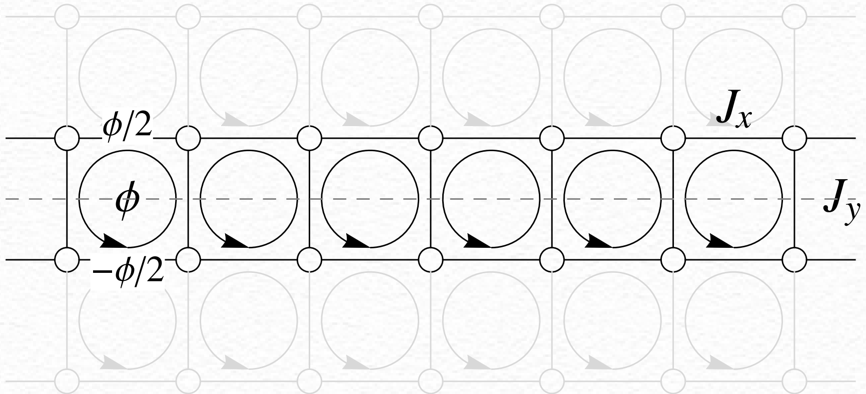

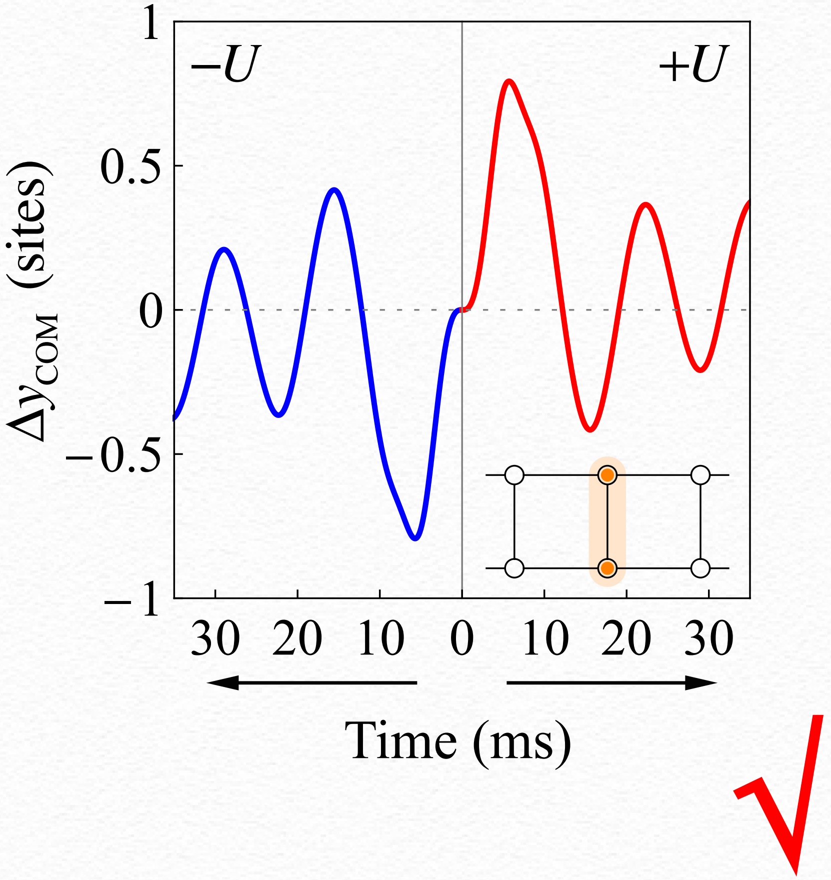

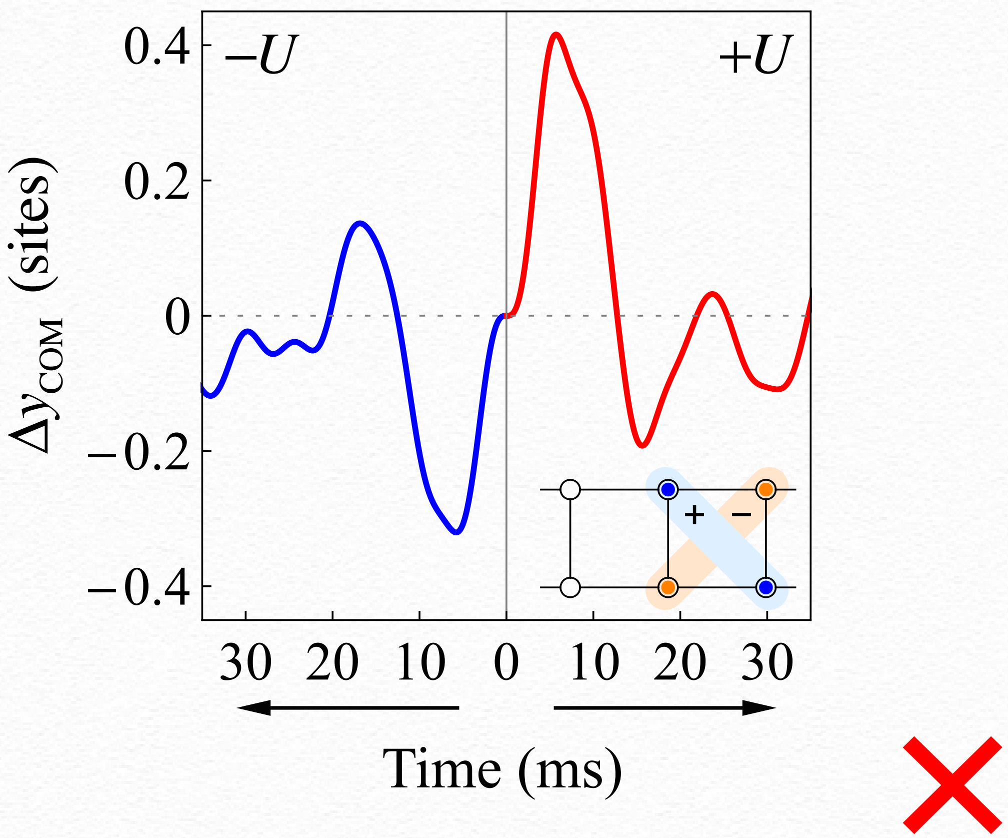

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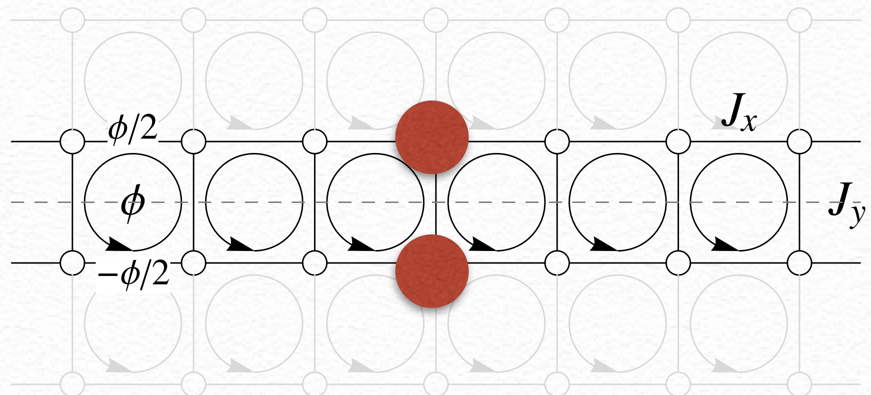
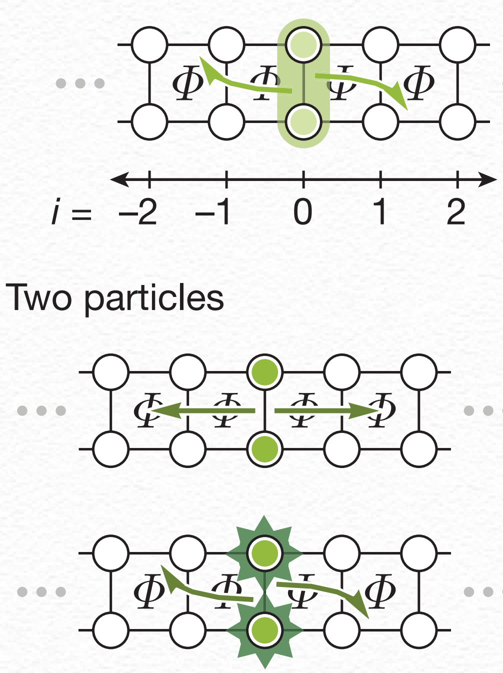


$$\Delta y_{COM}|_{U>0} = -\Delta y_{COM}|_{U<0}$$

$$\boxed{\Delta y_{COM}|_{U=0} = 0}$$

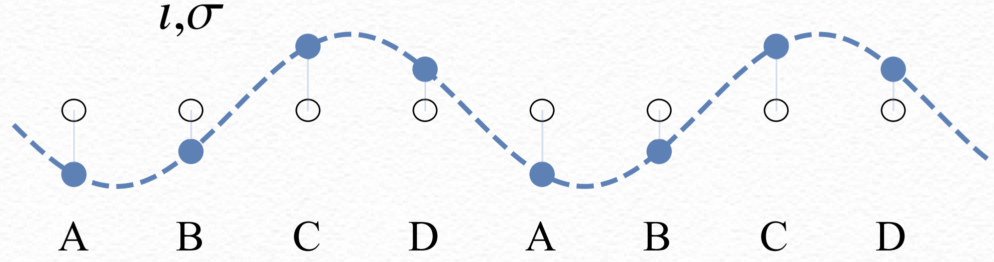
Interaction induced chirality



Harvard Group, M. E. Tai, et.al. Nature 2017

Example 3: Aubry-Andre-Hubbard Model

$$\hat{H}_0 = \sum_{i,\sigma} \left[-J(\hat{c}^\dagger_{i,\sigma}\hat{c}_{i+1,\sigma} + \text{h.c.}) + \Delta\cos(2\pi\beta i + \theta)\hat{n}_{i,\sigma} \right]$$

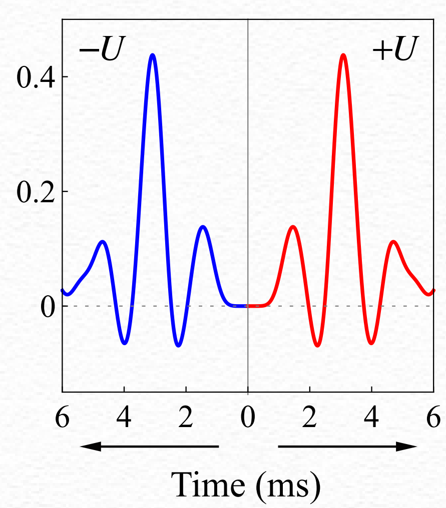


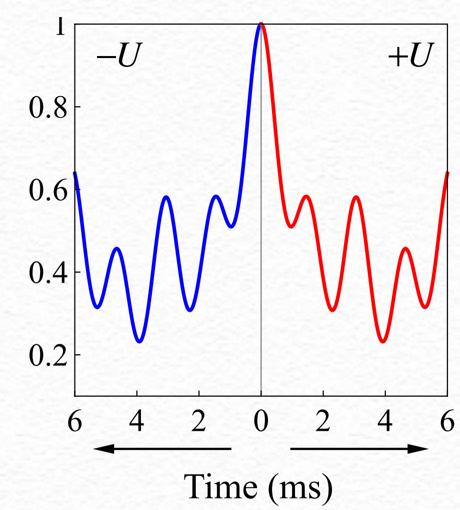
$$\hat{\mathcal{I}} = \frac{\sum_{i\in\text{even}} \hat{n}_{i\sigma} - \sum_{i\in\text{odd}} \hat{n}_{i\sigma}}{\sum_{i\in\text{even}} \hat{n}_{i\sigma} + \sum_{i\in\text{odd}} \hat{n}_{i\sigma}}$$

$$|\Psi_0\rangle_{\text{uniform}} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{c}^\dagger_{i\uparrow}\hat{c}^\dagger_{i\downarrow}|0\rangle$$

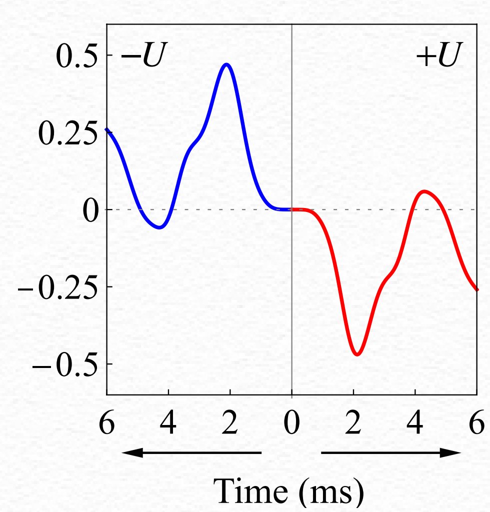
$$|\Psi_0\rangle_{\text{CDW}} = \frac{1}{\sqrt{N/2}} \sum_{i\in\text{even}} \hat{c}^\dagger_{i\uparrow}\hat{c}^\dagger_{i\downarrow}|0\rangle$$

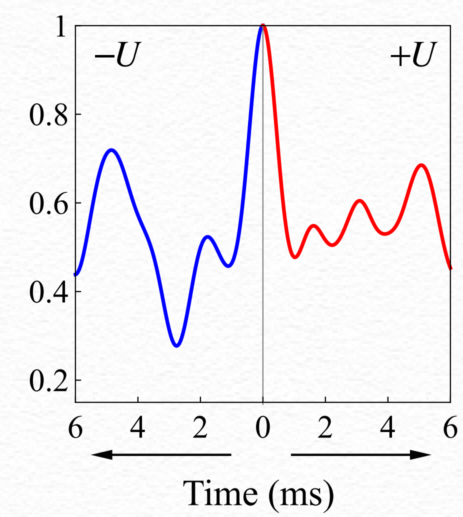
$\beta = 1/4$



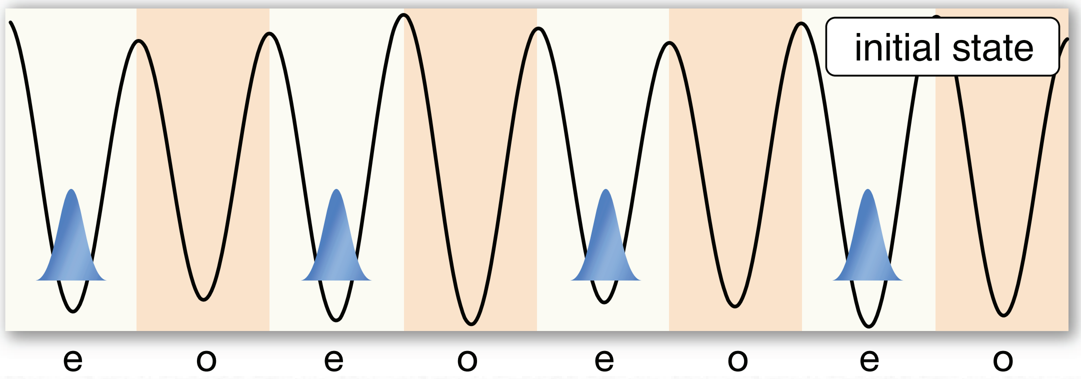


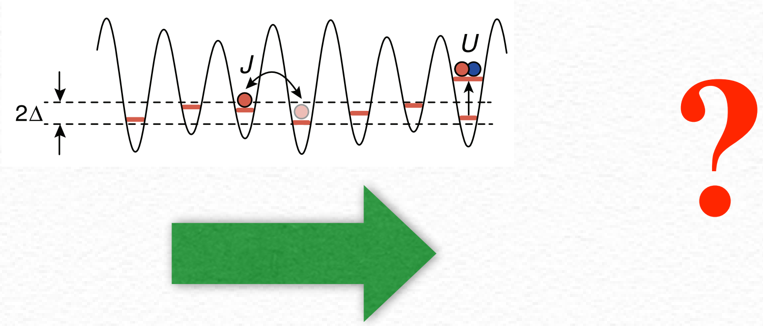
$\beta = 1/6$

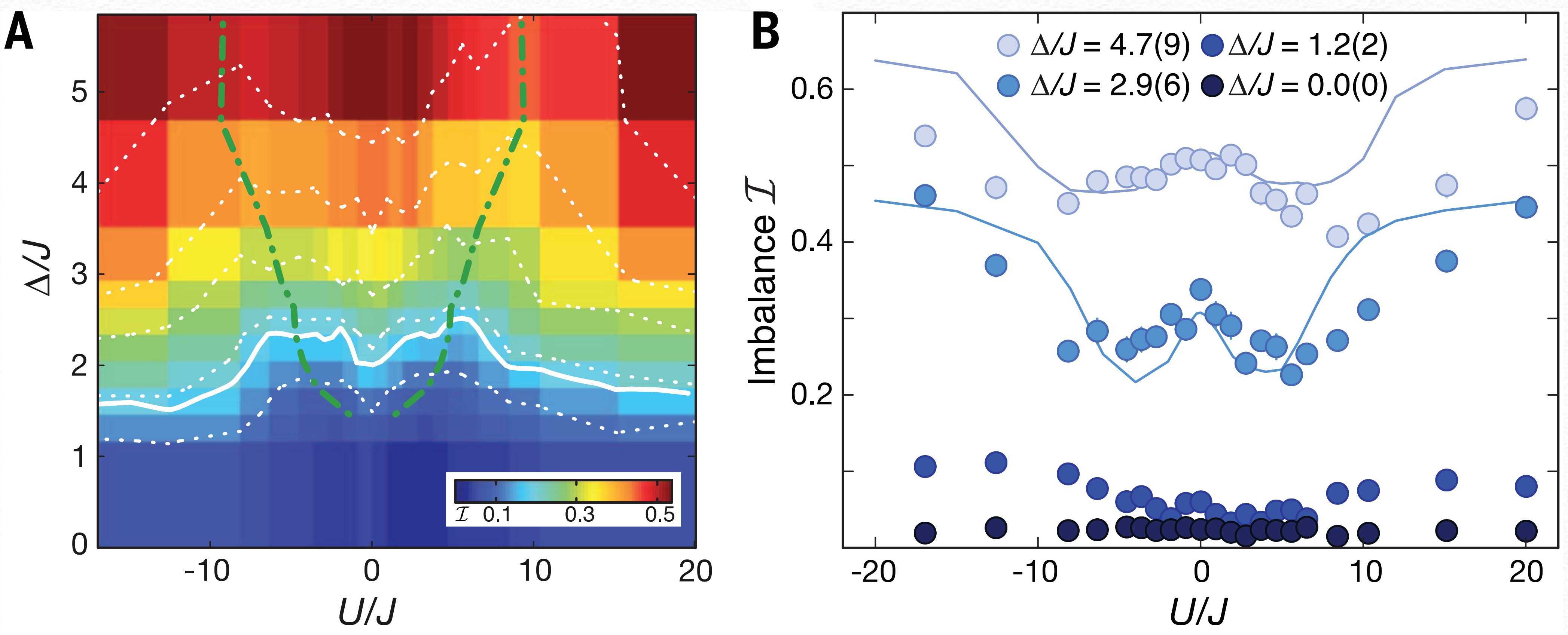




Example 3: Aubry-Andre-Hubbard Model







Munich Group, M. Schreiber, et.al. Science 2015

Summary of Three Examples

$$\langle O(t)\rangle_{+U} = \pm\langle O(t)\rangle_{-U}$$?

Example I (Munich 2012) — **Different Lattice Geometry**

Example II (Harvard 2017) — **Different Initial State**

Example III (Munich 2015) — **Different Onsite Potential**

Single-Particle/ Initial State Properties $\Rightarrow$ **Interaction Effect on Dynamics**

Theorem: Symmetry Protected Dynamical Symmetry

For the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$

if we can find an antiunitary operator $\hat{S} = \hat{R}\hat{W}$

(i) $\hat{S}^{-1}\hat{H}_0\hat{S} = -\hat{H}_0 \qquad [\hat{S}, \hat{V}] = 0$

(ii) $\hat{S}^{-1}|\Psi_0\rangle = e^{i\chi}|\Psi_0\rangle$

(iii) $\hat{S}^{-1}\hat{O}\hat{S} = \pm\hat{O}$

$$\Rightarrow \quad \langle O(t)\rangle_{+U} = \pm\langle O(t)\rangle_{-U}$$

Example 1: Revisit

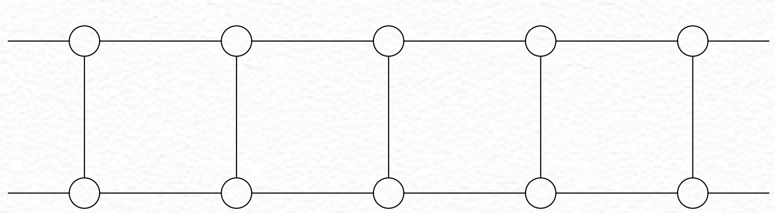
$$H_0 = -J \sum_{\langle ij \rangle, \sigma} c^\dagger_{i\sigma} c_{j\sigma}$$

Bipartite Lattice Symmetry

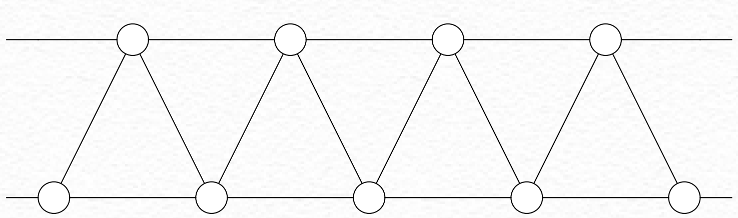
$$\hat{W}^{-1} \hat{c}_{i\sigma} \hat{W} = -\hat{c}_{i\sigma}, \quad \text{if} \;\; i \in A,$$

$$\hat{W}^{-1} \hat{c}_{i\sigma} \hat{W} = \hat{c}_{i\sigma}, \quad \text{if} \;\; i \in B.$$

$$\Rightarrow \quad \hat{S}^{-1} \hat{H}_0 \hat{S} = -\hat{H}_0$$



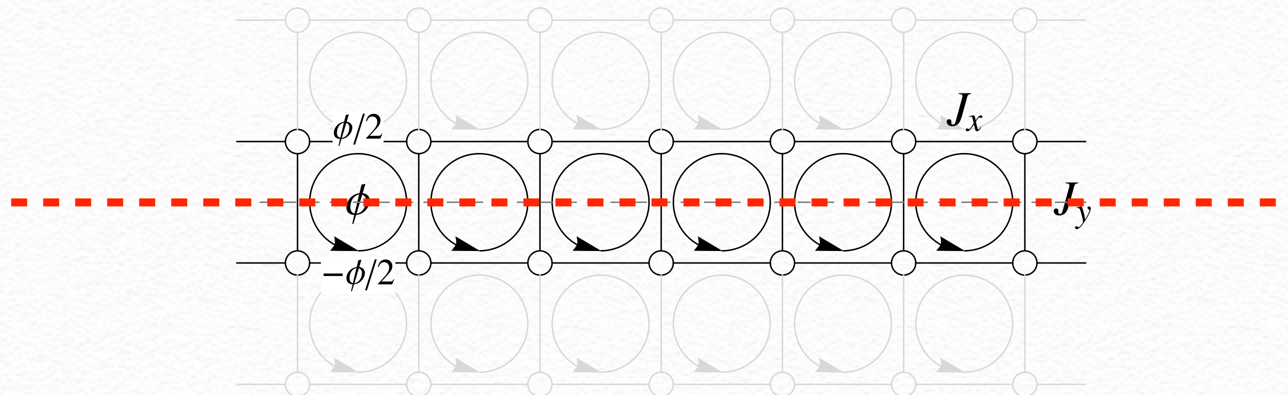
Square Lattice √



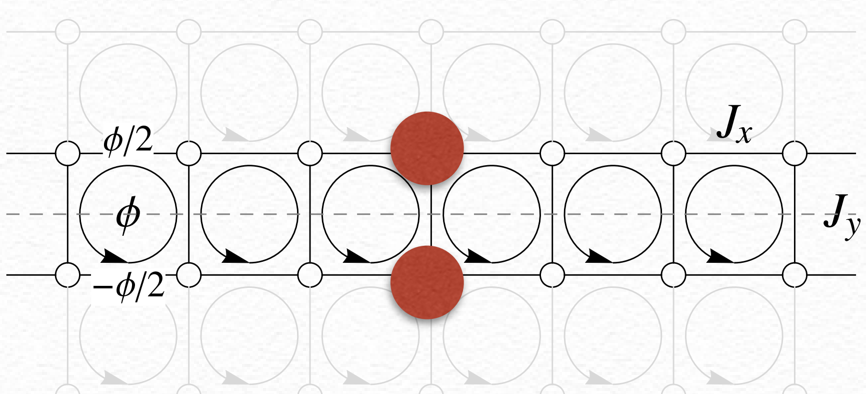
Triangular Lattice ✗

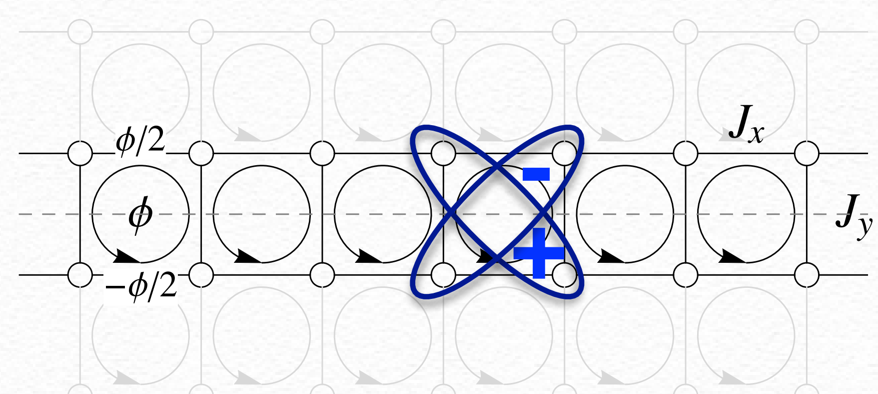
Example 2: Revisit

$$\hat{H}_0 = -J_x \sum_{i,\sigma}\left[e^{i(i_y-1/2)\phi}\hat{c}^{\dagger}_{i_x,i_y,\sigma}\hat{c}_{i_x-1,i_y,\sigma} + \text{h.c.}\right] - J_y \sum_{i,\sigma}\left[\hat{c}^{\dagger}_{i_x,i_y,\sigma}\hat{c}_{i_x,i_y-1,\sigma} + \text{h.c.}\right]$$



$$\hat{W}^{-1}\hat{c}_{i_x,i_y,\sigma}\hat{W} = (-1)^{i_x+i_y}\hat{c}_{i_x,1-i_y,\sigma}$$



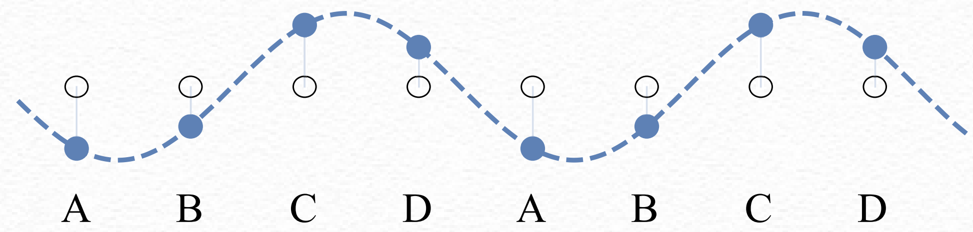


$$\Delta y_{COM}|_{U=0} = 0 \quad \surd$$

$$\Delta y_{COM}|_{U=0} \neq 0 \quad \times$$

Example 3: Revisit

$$\hat{H}_0 = \sum_{i,\sigma} \left[-J(\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i+1,\sigma} + \text{h.c.}) + \Delta \cos(2\pi\beta i + \theta)\hat{n}_{i,\sigma} \right]$$



$$\beta = p/q$$

$$\hat{W}^{-1}\hat{c}_{i,\sigma}\hat{W} = (-1)^i \hat{c}_{i+q/2,\sigma}$$

	Uniform Initial State	CDW Initial State
q is even q/2 is even	√	√
q is even q/2 is odd	√	✕
q is odd	✕	✕

Application of Theorem

Example I
(Munich 2012)

$\hat{W}$

Bipartite Lattice Symmetry

$$\langle O(t) \rangle_{+U} = \pm \langle O(t) \rangle_{-U}$$

$\hat{W}$

Bipartite Lattice +
Reflection
Symmetry

$\hat{W}$

Bipartite Lattice +
Translation
Symmetry

Example II
(Harvard 2017)

Example III
(Munich 2015)

Summary

I. Logarithmical periodic expansion dynamics can detect the scaling symmetry of the bulk system.

Dynamics $\rightleftarrows$ **Symmetry of a Hamiltonian**

II. The symmetry of bulk system can put constraint on dynamics with positive and negative interactions.

Thank You Very Much for Attention !