Fundamental Aspects of Quantum Dynamics:

III: Entropy

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Quantum Few- and Many-Body Physics in Ultracold Atoms
Wuhan
April 2018

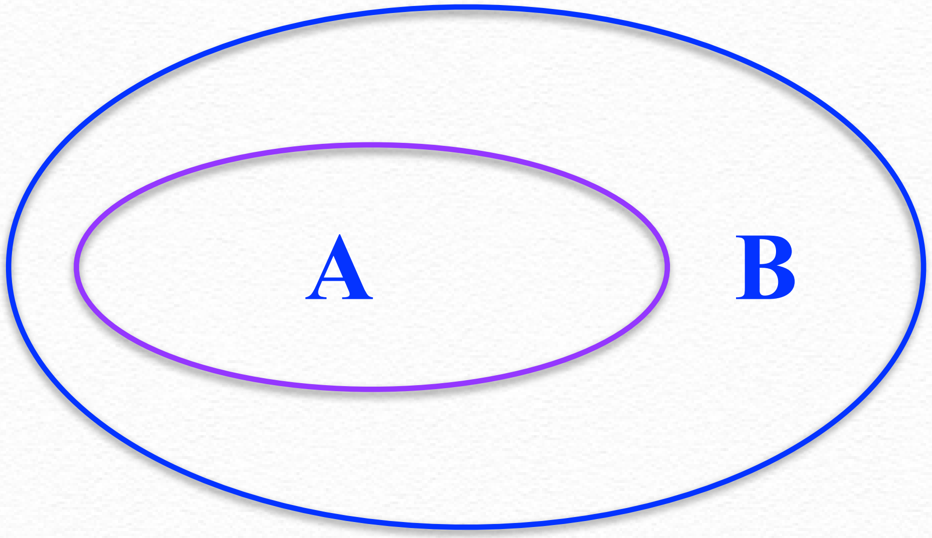
Entropy and Dynamics

Entropy → Detected by Dynamics

Related by Dynamics

Out-of-Time-Order Correlation → Detected by Dynamics

Entanglement Entropy



$$\rho_A \equiv \mathrm{tr}_B(\rho_{AB})$$

$$\rho_B \equiv \mathrm{tr}_A(\rho_{AB})$$

For separable states: $\mathrm{tr}(\rho_A^2) = \mathrm{tr}(\rho_B^2) = \mathrm{tr}(\rho_{AB}^2) = 1.$

For entangled states: $\mathrm{tr}(\rho_A^2) = \mathrm{tr}(\rho_B^2) < \mathrm{tr}(\rho_{AB}^2) = 1$

nth Renyi Entropy

$$S_n(A) \equiv \frac{1}{1-n} \log \mathrm{tr}\,(\rho_A^n)$$

2nd

$$S_2(A) = -\log \mathrm{tr}\,(\rho_A^2)$$

1st

$$\lim_{n\to 1} S_n(A) = -\mathrm{tr}(\rho_A \log \rho_A) \equiv S(A)$$

Measurement of Second Renyi Entropy

Theorem: $\mathrm{tr}(V_2 \rho_1 \otimes \rho_2) = \mathrm{tr}(\rho_1 \rho_2)$

Exchanging 1 an 2

Creating two identical copies: $\rho_1 = \rho_2$

$$\boxed{\langle V_2 \rangle = tr(\rho^2)}$$

For instance: $|\psi\rangle = (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^n (\hat{a}_1^\dagger + \hat{a}_2^\dagger)^m |0\rangle$

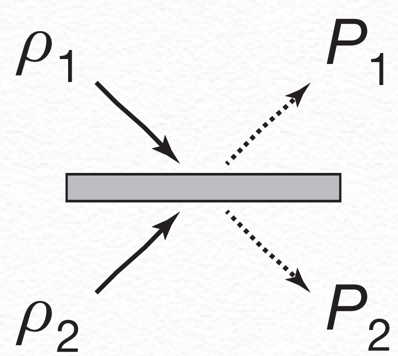
$$\langle V_2 \rangle = (-1)^n$$

Hong-Ou-Mandel Interference

$$\hat{a}_1 \rightarrow \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$$
$$\hat{a}_2 \rightarrow \frac{1}{\sqrt{2}}(-\hat{a}_1 + \hat{a}_2)$$

$$|\psi\rangle = (\hat{a}_1^\dagger - \hat{a}_2^\dagger)^n(\hat{a}_1^\dagger + \hat{a}_2^\dagger)^m|0\rangle \rightarrow \hat{a}_2^{\dagger,n}\hat{a}_1^{\dagger,m}|0\rangle$$

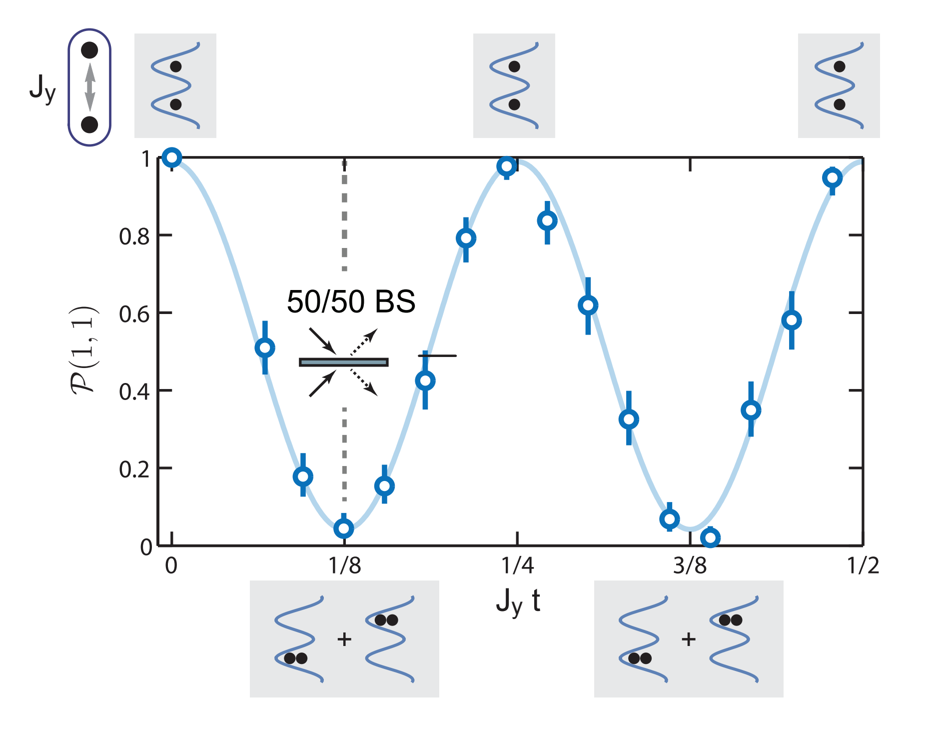
$$\langle V_2\rangle = tr(\rho^2)$$

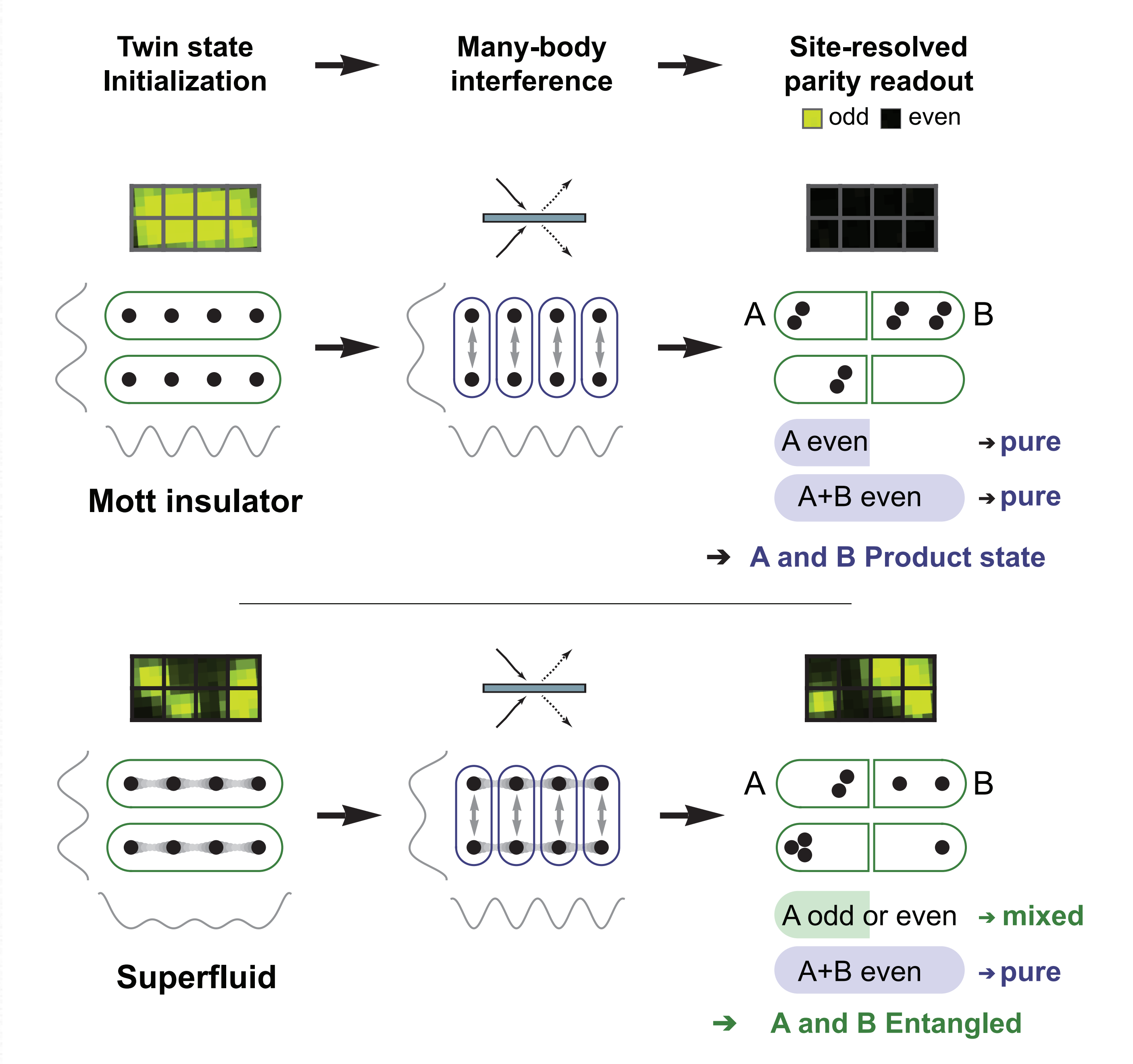


$\langle P_i\rangle$	=	$\mathrm{Tr}\,(\rho_1\,\rho_2)$	$\underline{\underline{\rho_1 = \rho_2}}$	$\mathrm{Tr}\,(\rho^2)$
Average parity		Quantum state overlap		Purity

$$P_i = \prod_k p_i^{(k)}$$

Hong-Ou-Mandel Interference for BHM

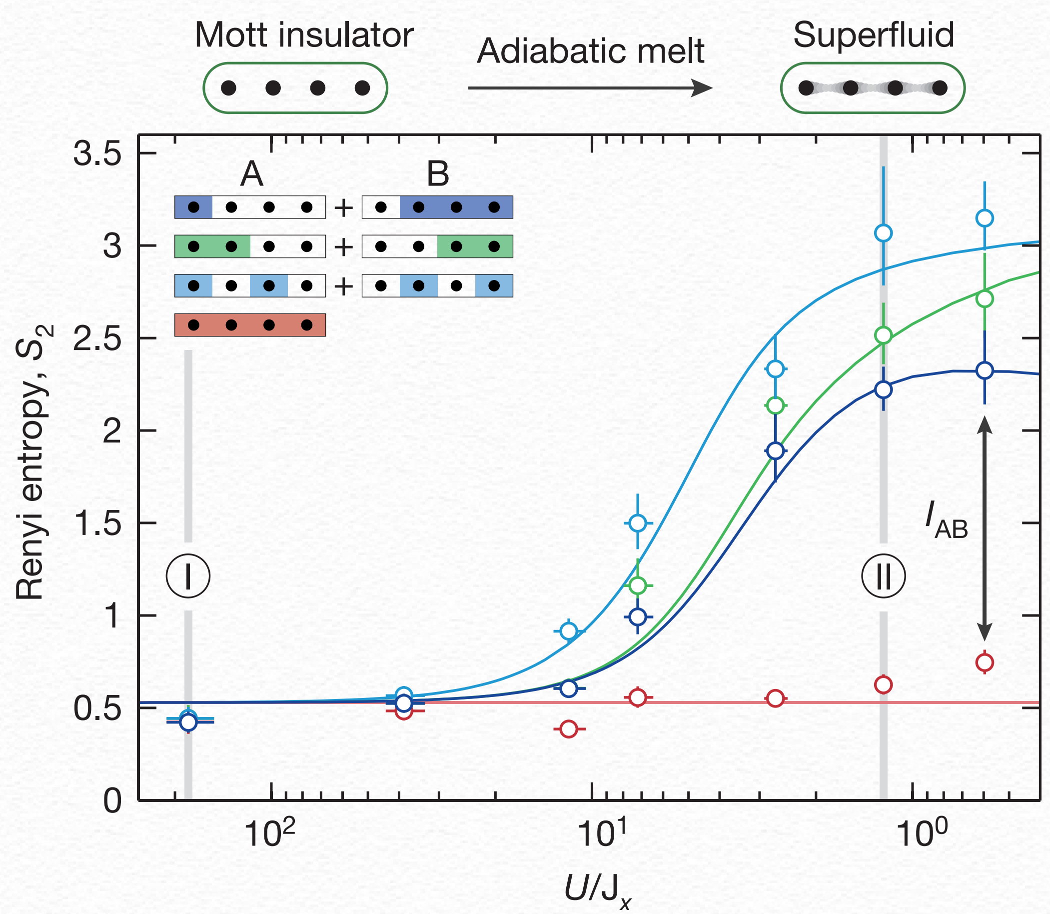


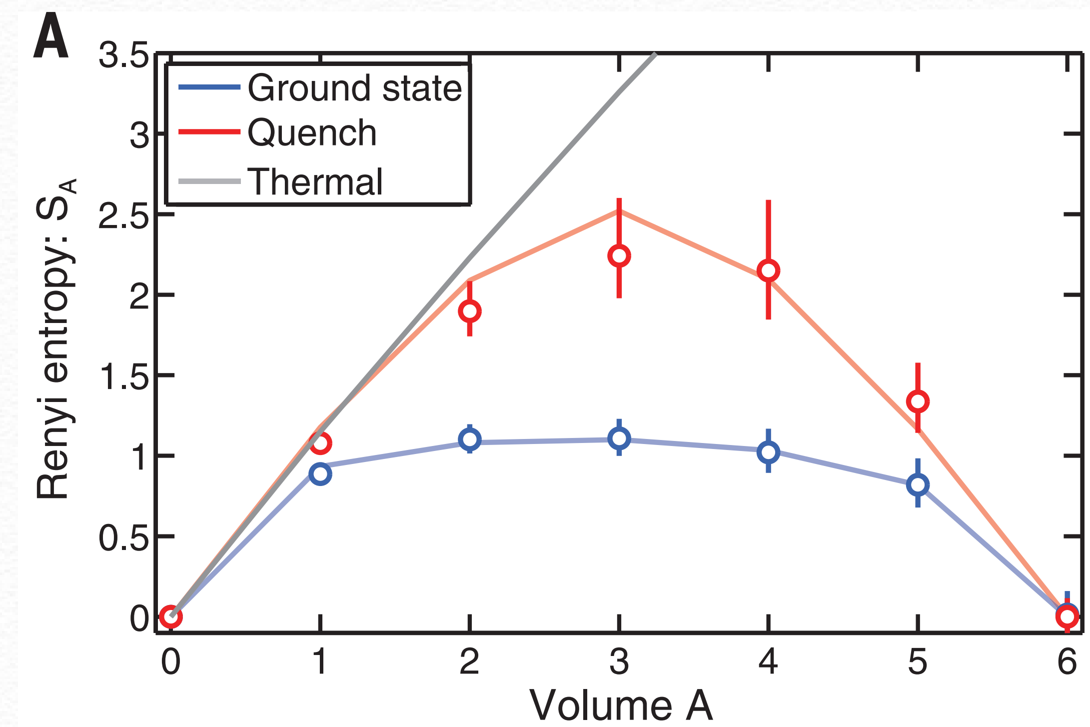


Dynamical realization of Hong-Ou-Mandel

Harvard Group, Nature 2016

Entanglement Entropy for BHM





Equilibrium

After Quench

Harvard Group, Nature 2016

Science 2016

Out-of-Time-Ordered Correlation (OTOC)

Normal correlation you can find in any textbook:

$$\langle \hat{W}^\dagger(t)\hat{W}(t)\hat{V}^\dagger(0)\hat{V}(0)\rangle_\beta$$

OTOC $$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle_\beta$$

Since 2014…

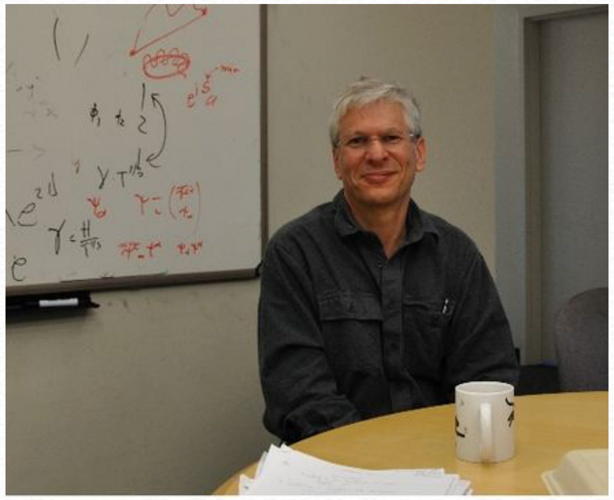
Condensed Matter/ Quantum Information

High Energy/Gravity

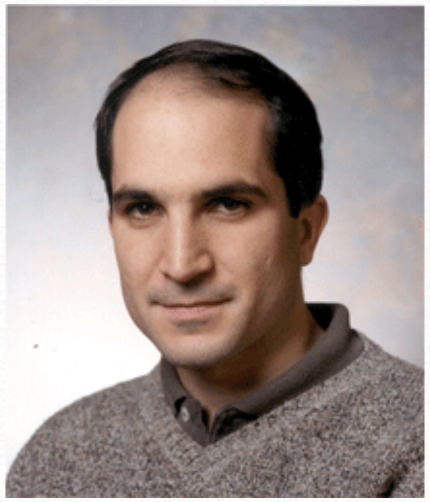
Kitaev

Sachdev

…….



Shenker



Maldacena

Witten

…….

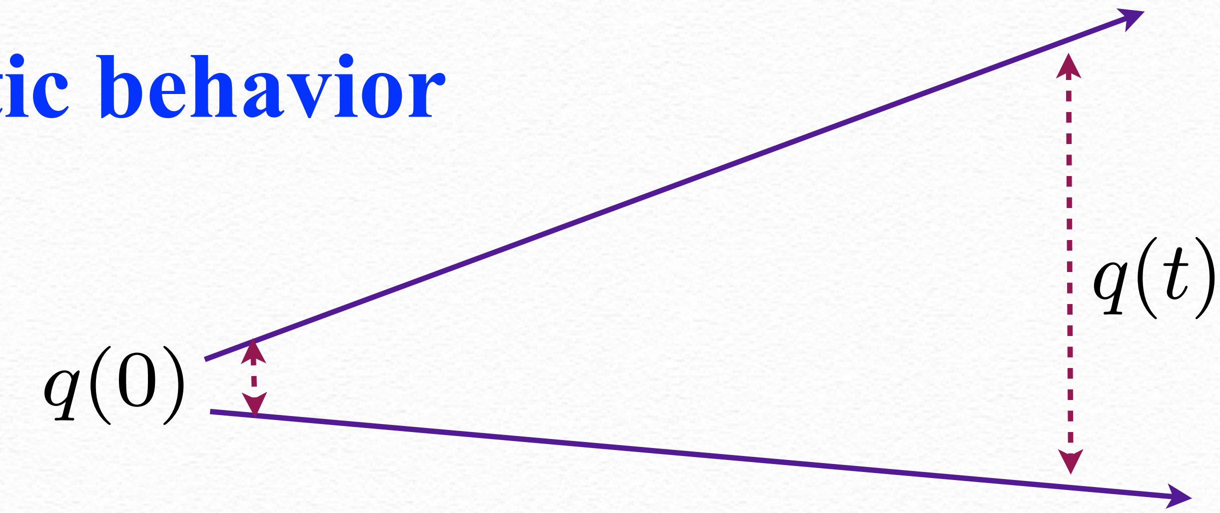
Out-of-Time-Ordered Correlator

$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle_\beta$$

$$\hat{W}(t) = e^{i\hat{H}t}\hat{W}e^{-i\hat{H}t}$$

- **OTOC diagnoses chaotic behavior**

$$\frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t} = \{q(t), p(0)\}$$



λ_L **Lyapunov exponent**

$$\boxed{C(t) = \langle |[W(t), V(0)]|^2\rangle_\beta}$$

$$C(t) = \langle \underbrace{V^\dagger(0)W^\dagger(t)W(t)V(0) + W^\dagger(t)V^\dagger(0)V(0)W(t)}_{\text{“Accessible correlators”}}$$

$$\underbrace{-W^\dagger(t)V^\dagger(0)W(t)V(0) - V^\dagger(0)W^\dagger(t)V(0)W(t)}_{\text{“Out-of-time-ordered correlators”}}\rangle_\beta$$

Larkin, Ovchinnikov 1969

Out-of-Time-Ordered Correlator

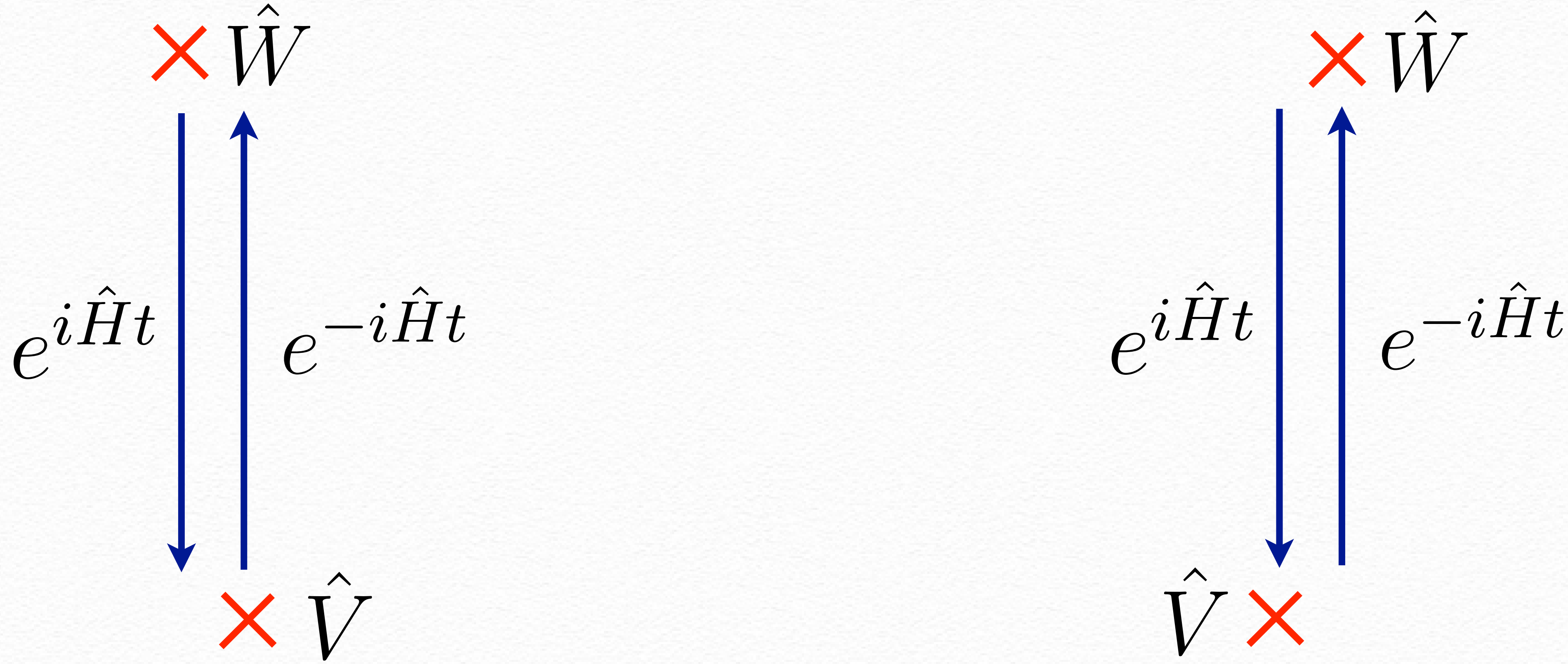
$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle_\beta$$

$$\hat{W}(t) = e^{i\hat{H}t}\hat{W}e^{-i\hat{H}t}$$

- **OTOC diagnoses the butterfly effect**

$$\mathrm{OTOC} = \langle y|x\rangle$$

$$|x\rangle = \hat{W}(t)\hat{V}(0)|\rangle \qquad |y\rangle = \hat{V}(0)\hat{W}(t)|\rangle$$



Out-of-Time-Ordered Correlator

$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle_\beta$$

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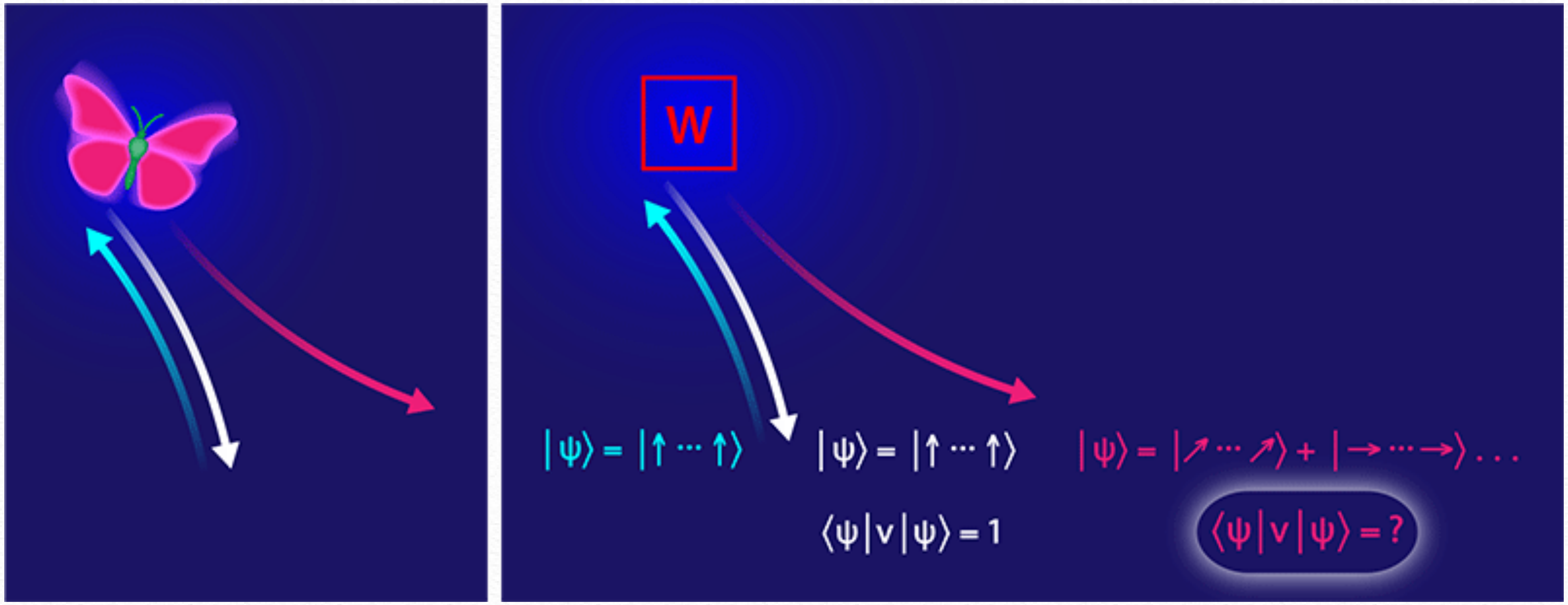


by Brian Swingle* and Norman Y. Yao†

Out-of-Time-Ordered Correlator

$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle_\beta$$

$$\hat{W}(t) = e^{i\hat{H}t}\hat{W}e^{-i\hat{H}t}$$

- **Delocalization of information is closely related to the decay of the OTOC, and the butterfly effect in quantum system implies the information-theoretic definition of scrambling.**

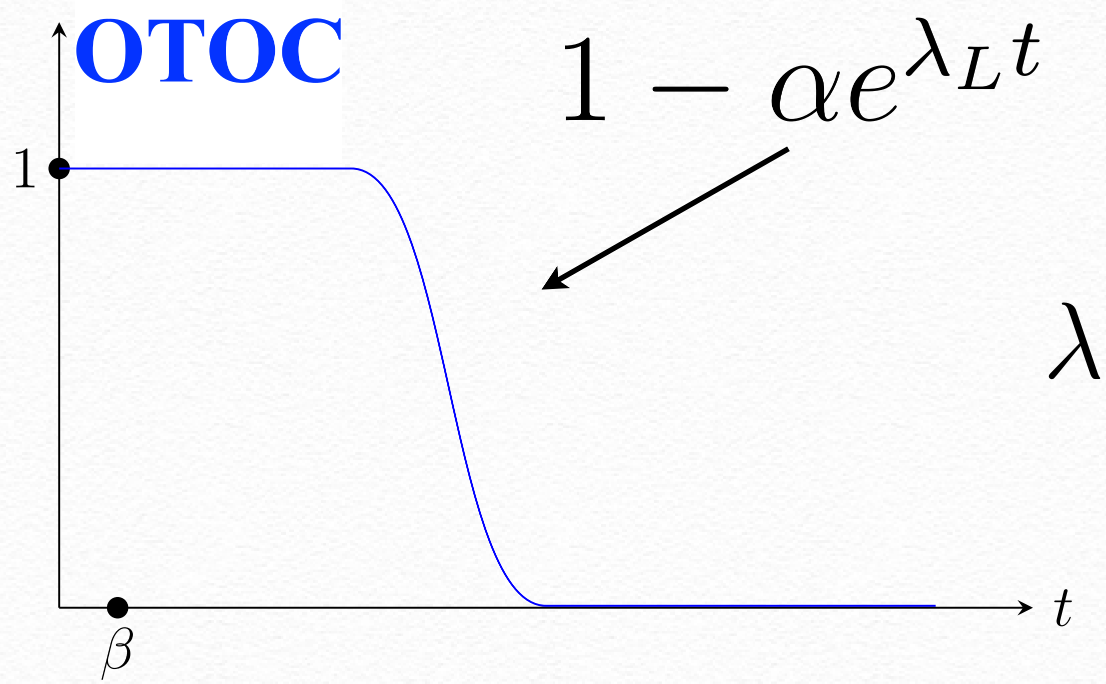
$$C(t) = \langle |[W(t), V(0)]|^2 \rangle_\beta$$

$$\hat{W}(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}[\hat{H}, \ldots, [\hat{H}, \hat{W}], \ldots,]$$

Hosur, Qi, Roberts and Yoshida, 2015

Out-of-Time-Ordered Correlator

- **OTOC diagnoses the butterfly effect**



λ_L **Lyapunov exponent**

- **OTOC has also emerged in studying gravity models. The calculation with a black hole shows that**

$$\lambda_L = \frac{2\pi}{\beta}$$

- **OTOC has a upper bound** $\lambda_L \leqslant \frac{2\pi}{\beta}$

Kitaev, KITP, 2015; Maldacena, Shenker and Stanford, 2015

Holographic Duality

Holographic duality:

A *quantum many body system* (<u>strongly interacting, emergent conformal field symmetry</u>) in D-dimension

can be mapped to

a *gravity theory* (<u>hopefully, classical</u>) in D+1-dimension

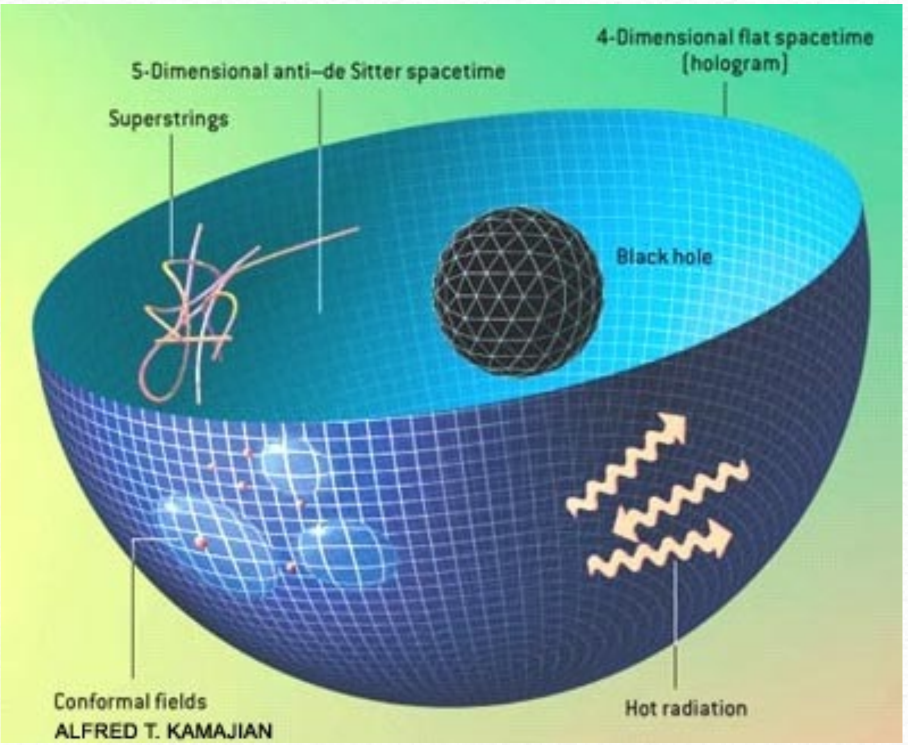
$$\langle \hat{A}_i \hat{A}_j \rangle_{QFT} = \langle \hat{B}_i \hat{B}_j \rangle_G$$

Condensed Matter /Cold Atom Physicists:

A way to solve strongly interacting quantum many-body problem

High-Energy/Gravity Physicists:

A way to quantize gravity



Holographic Duality and Cold Atom Physics

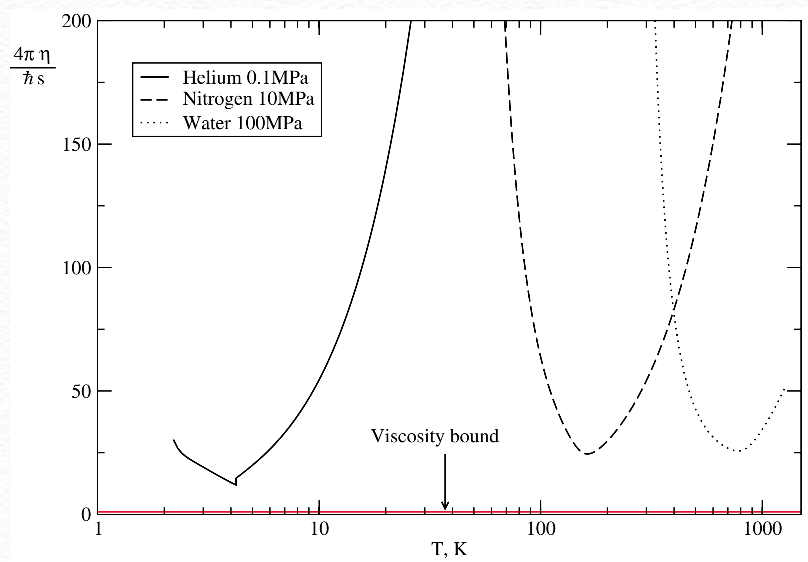
PRL **94,** 111601 (2005) PHYSICAL REVIEW LETTERS week ending 25 MARCH 2005

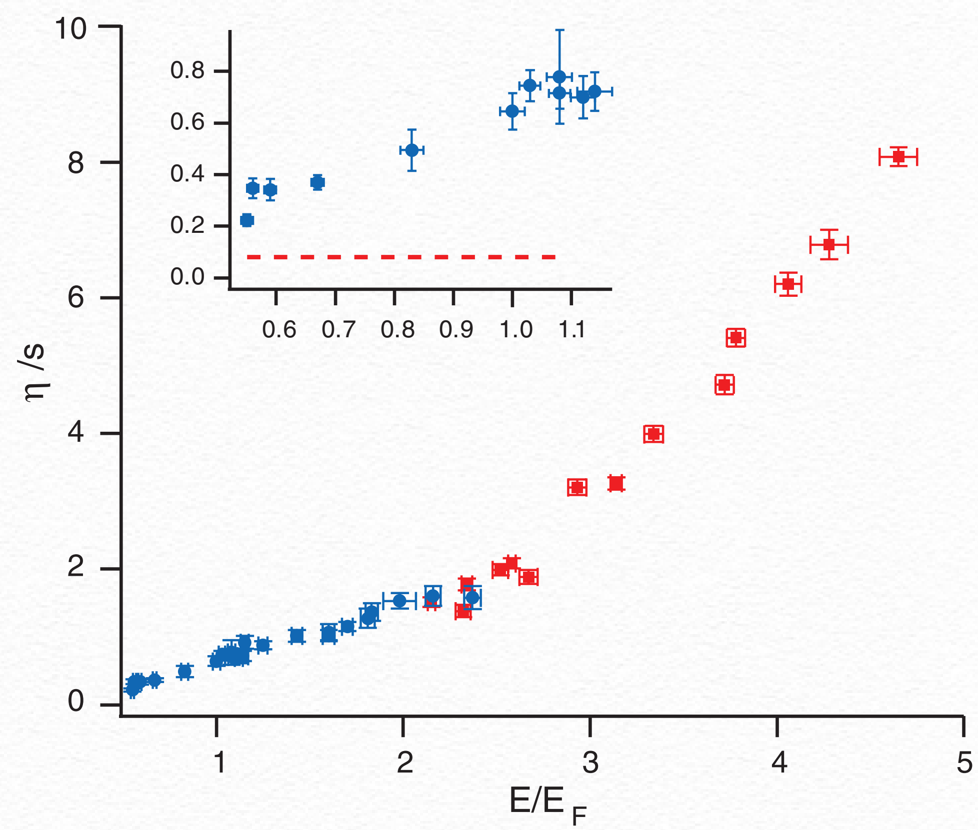
Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics

P. K. Kovtun,[1] D. T. Son,[2] and A. O. Starinets[3]

D. T. Son

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$





Universal Quantum Viscosity in a Unitary Fermi Gas

C. Cao,[1] E. Elliott,[1] J. Joseph,[1] H. Wu,[1] J. Petricka,[2] T. Schäfer,[3] J. E. Thomas[1]*

Science, 2011

Out-of-Time-Ordered Correlator

- OTOC has also emerged in studying gravity models. The calculation with a black hole shows that

$$\lambda_L = \frac{2\pi}{\beta}$$

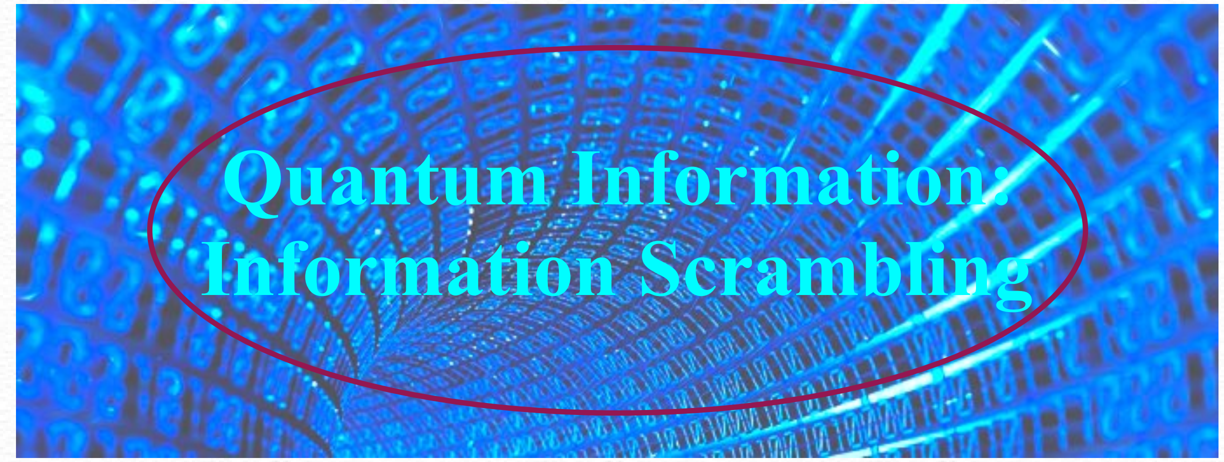
- OTOC has a upper bound $\lambda_L \leqslant \frac{2\pi}{\beta}$

A quantum system is holographically dual to a black hole $\Rightarrow$ The Lyapunov exponent of the system saturates the bound

The Lyapunov exponent of the system saturates the bound $\overset{?}{\Rightarrow}$ A quantum system is holographically dual to a black hole

An example is the SYK model

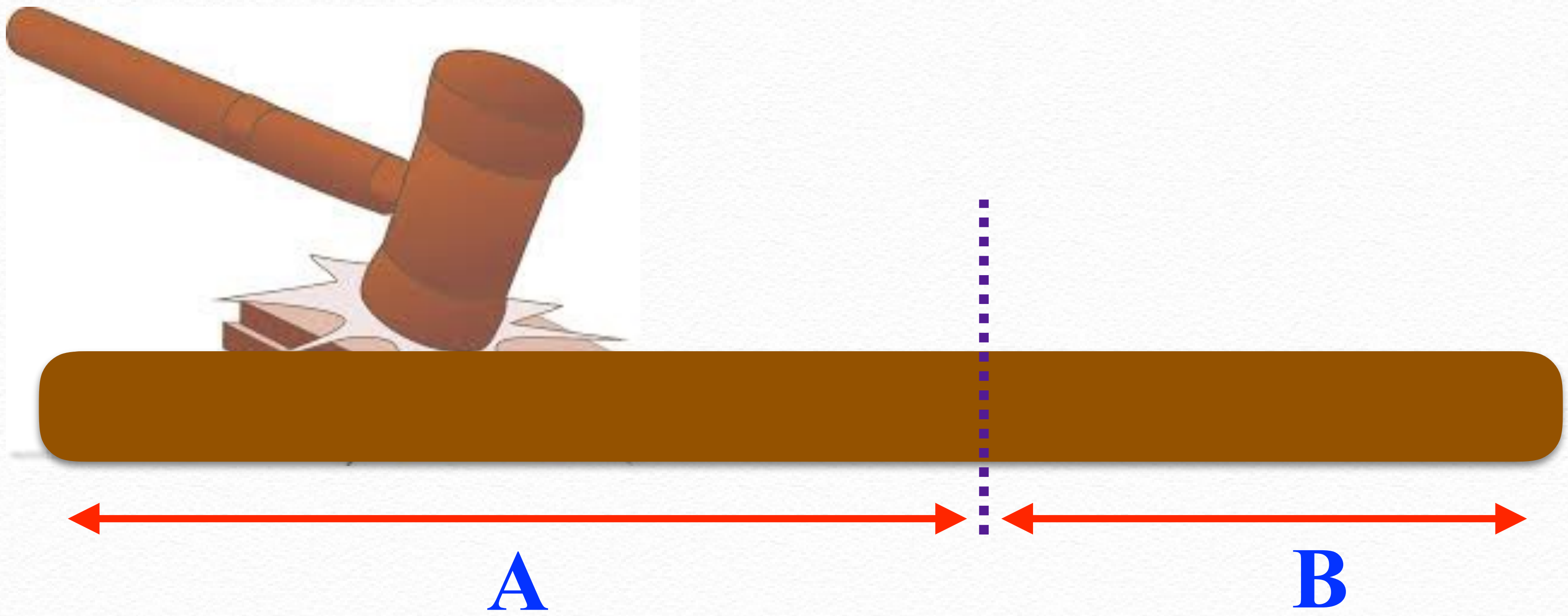
Out-of-Time-Ordered Correlator



Out-of-Time-Ordered Correlator

$$\langle W^\dagger(t) V^\dagger(0) W(t) V(0) \rangle$$

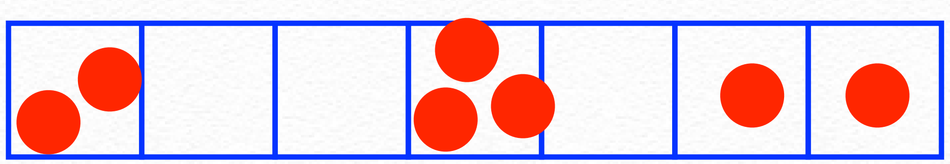
Quench Experiment

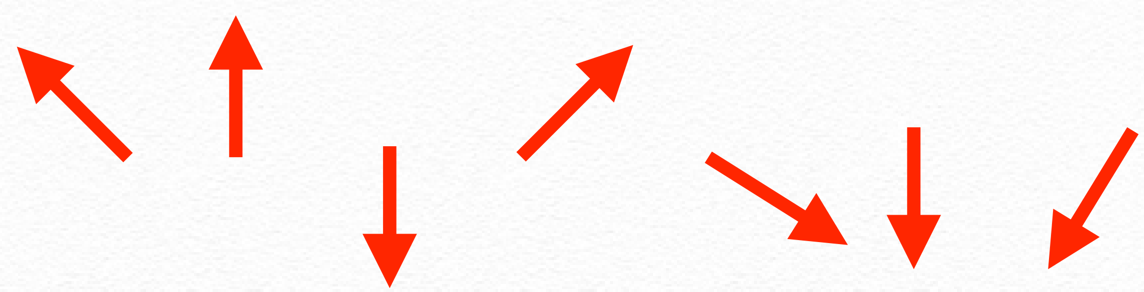


Local quench

$\hat{b}_i^\dagger|\Psi\rangle$

$S_i^-|\Psi\rangle$

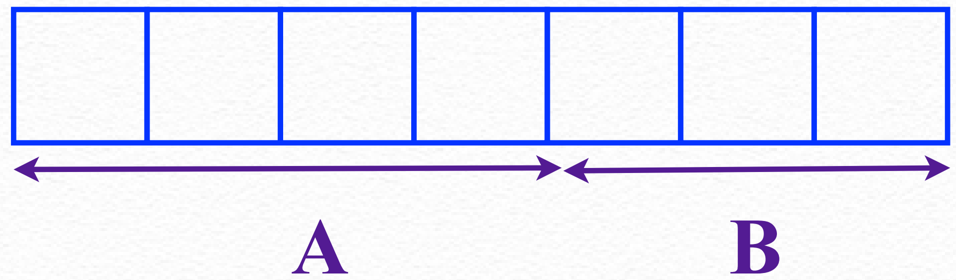




$$\exp(-S_A^{(2)}) = \sum_{M\in B} \mathrm{Tr}[\hat{M}(t)\hat{V}(0)\hat{M}(t)\hat{V}(0)]$$

R. Fan, P. Zhang, H. Shen and H. Zhai, Science Bulletin, 2017

OTOC v.s. Renyi Entropy



$$\exp(-S_A^{(2)}) = \sum_{M \in B} \mathrm{Tr}[\hat{M}(t)\hat{V}(0)\hat{M}(t)\hat{V}(0)]$$

Non-Equilibrium Properties

Quench the system by arbitrary operator $\mathcal{O}$

Entanglement Entropy

=

Equilibrium Properties

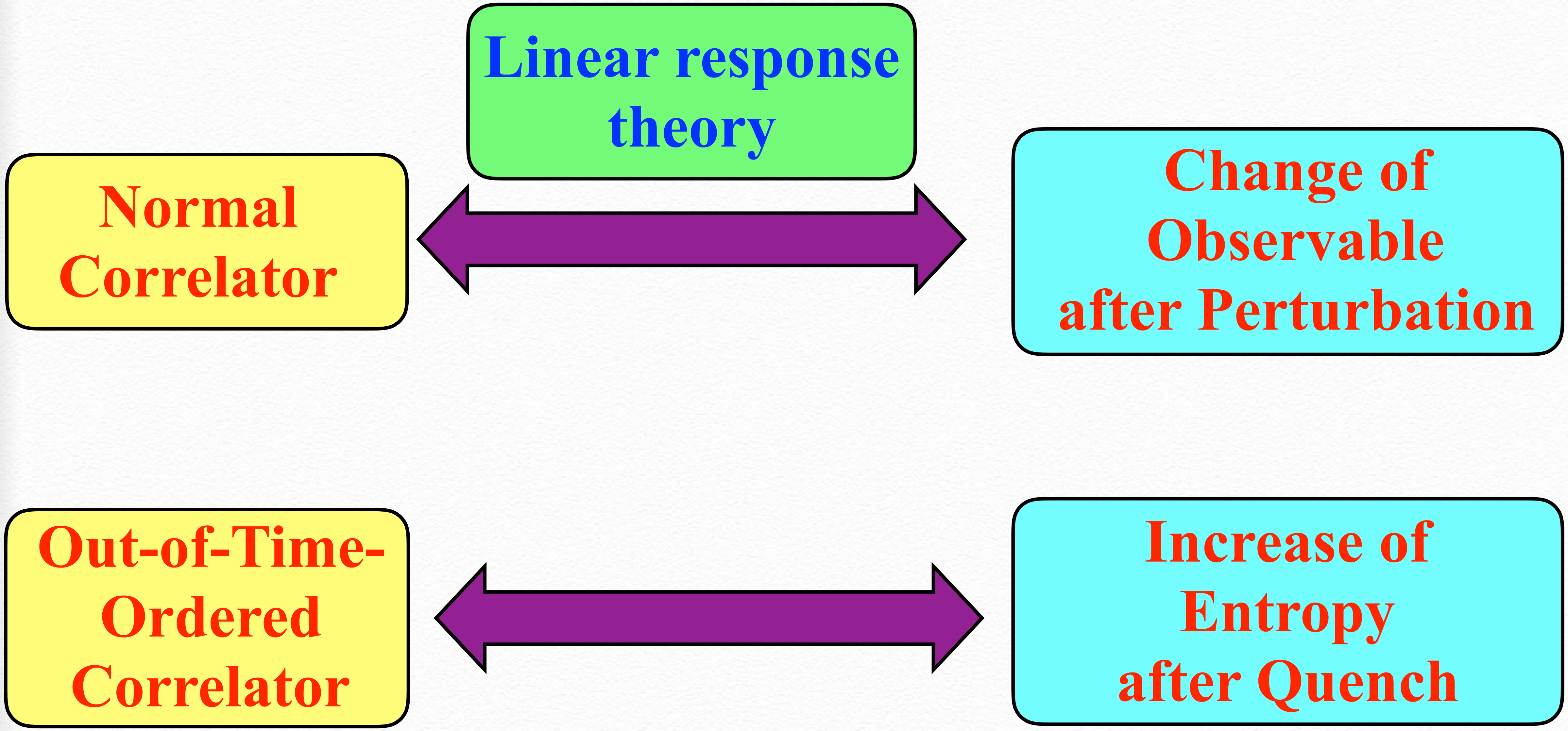
$$\hat{V} = \hat{O}\hat{O}^\dagger$$

$\hat{M}$ **is a complete set of operators in B**

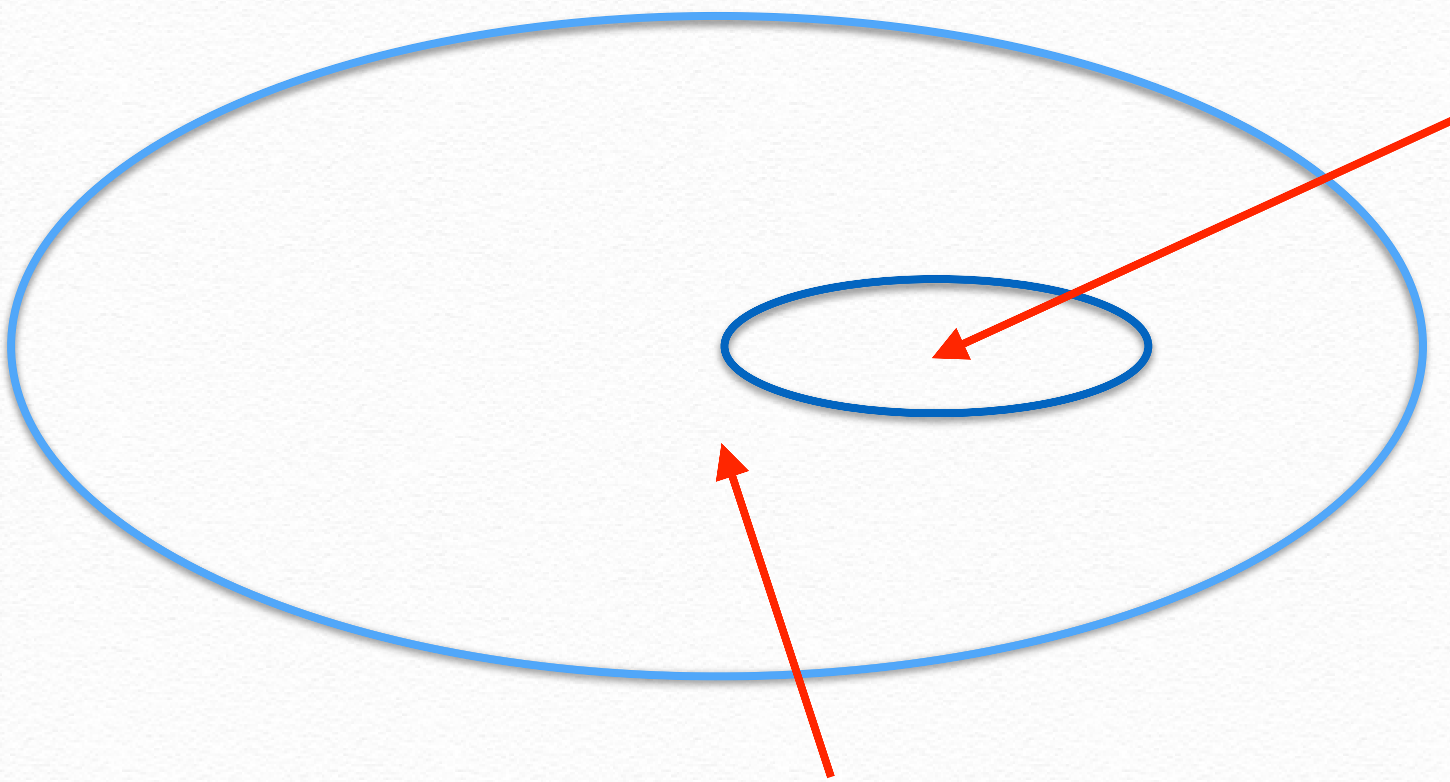
OTOC

R. Fan, P. Zhang, H. Shen and H. Zhai, Science Bulletin, 2017

OTOC v.s. Renyi Entropy



Quantum Thermalization



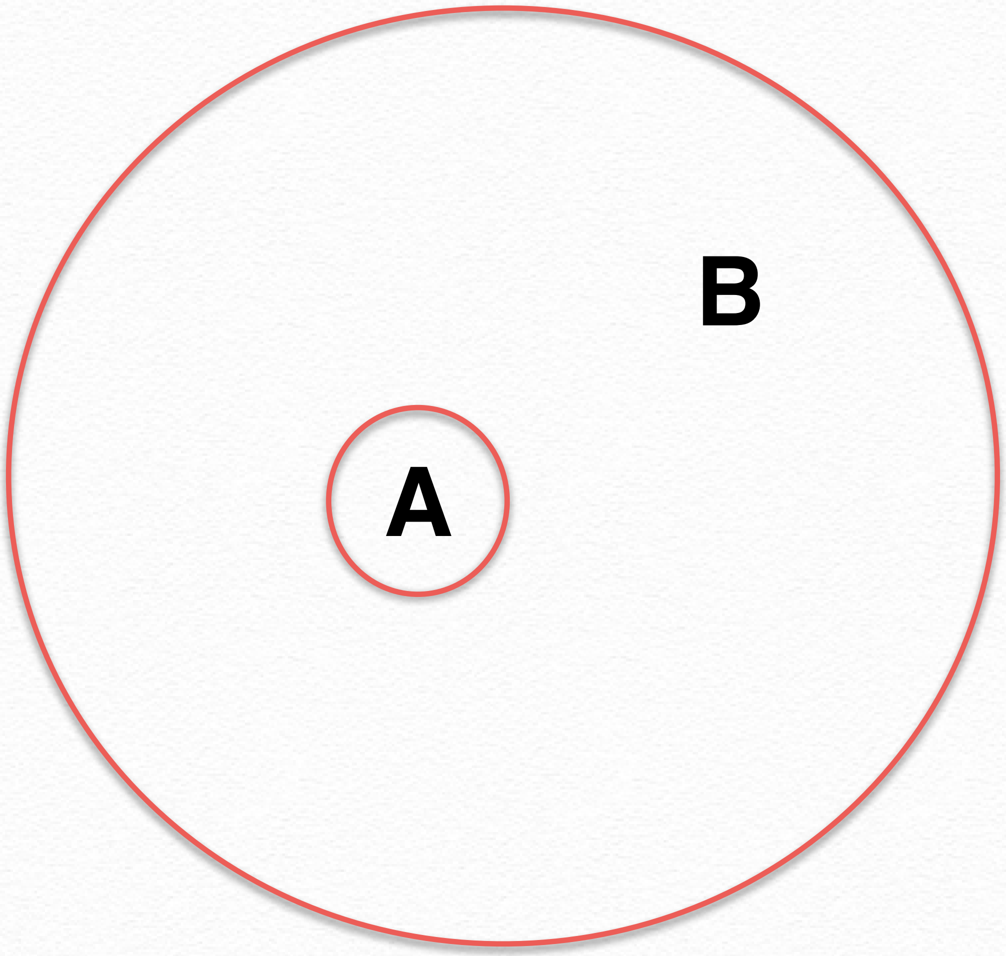
Statistical Mechanics:
Local observation is consistent with a thermal ensemble of statistical mechanics

Quantum Mechanics:
The whole system evolves as a unitary quantum evolution

Eigen-State Thermolization Hypothesis

Partition the system into two parts A and B. Suppose $V_A/V_B \to 0$ in the **thermodynamics limit** $V_A, V_B \to \infty$. If the ETH holds true for **every local** operator in A, then

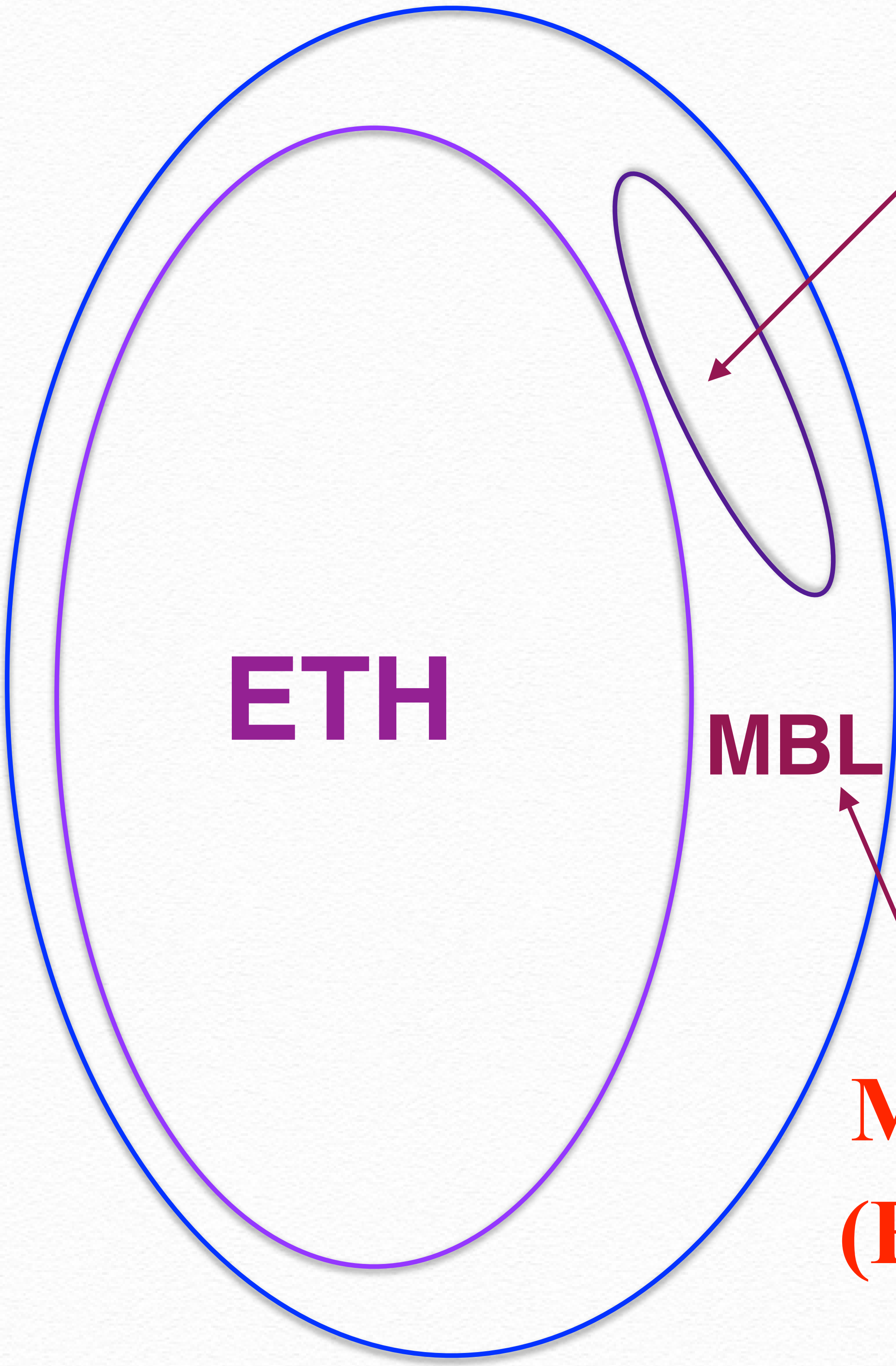
$$\rho_A \equiv \mathrm{Tr}_B(|\alpha\rangle\langle\alpha|) = e^{-\hat{H}_A/k_B T_\alpha}.$$



Implication:

Entanglement entropy obeys volume law

Many-Body Localization



Exactly Integrable:

Non-Interacting System:
Free fermions, bosons
Anderson localization

Bethe-Ansatz Integrable

Many-Body Localization:
(Emergent Integrability)

Many-Body Localization and Cold Atom Physics

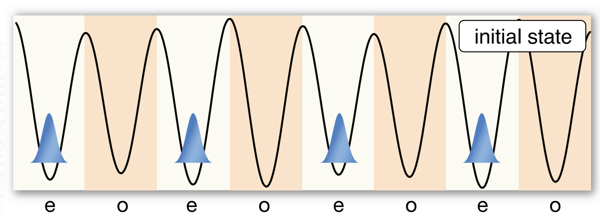
RESEARCH ARTICLE

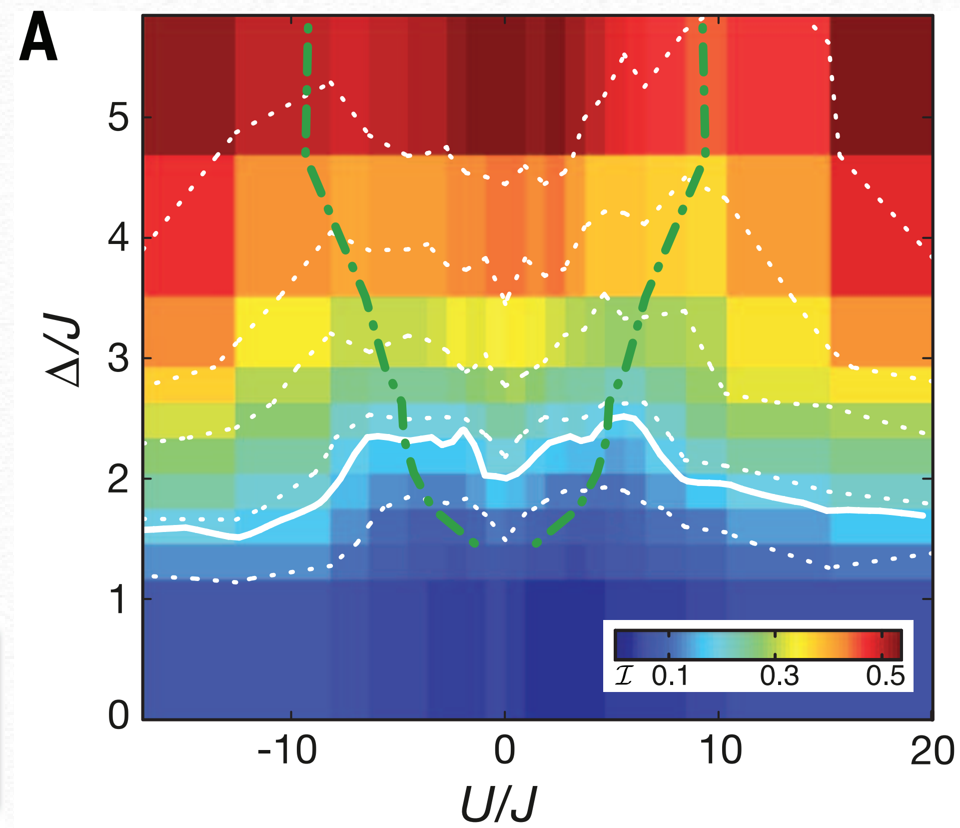
QUANTUM GASES

Observation of many-body localization of interacting fermions in a quasirandom optical lattice

Michael Schreiber,[1,2] Sean S. Hodgman,[1,2] Pranjal Bordia,[1,2] Henrik P. Lüschen,[1,2]
Mark H. Fischer,[3] Ronen Vosk,[3] Ehud Altman,[3]
Ulrich Schneider,[1,2,4] Immanuel Bloch[1,2]*

Science, 2015





QUANTUM SIMULATION

Exploring the many-body localization transition in two dimensions

Jae-yoon Choi,[1*†] Sebastian Hild,[1*] Johannes Zeiher,[1] Peter Schauß,[1‡]
Antonio Rubio-Abadal,[1] Tarik Yefsah,[1§] Vedika Khemani,[2] David A. Huse,[2,3]
Immanuel Bloch,[1,4] Christian Gross[1]

Science, 2016

OTOC v.s. Renyi Entropy

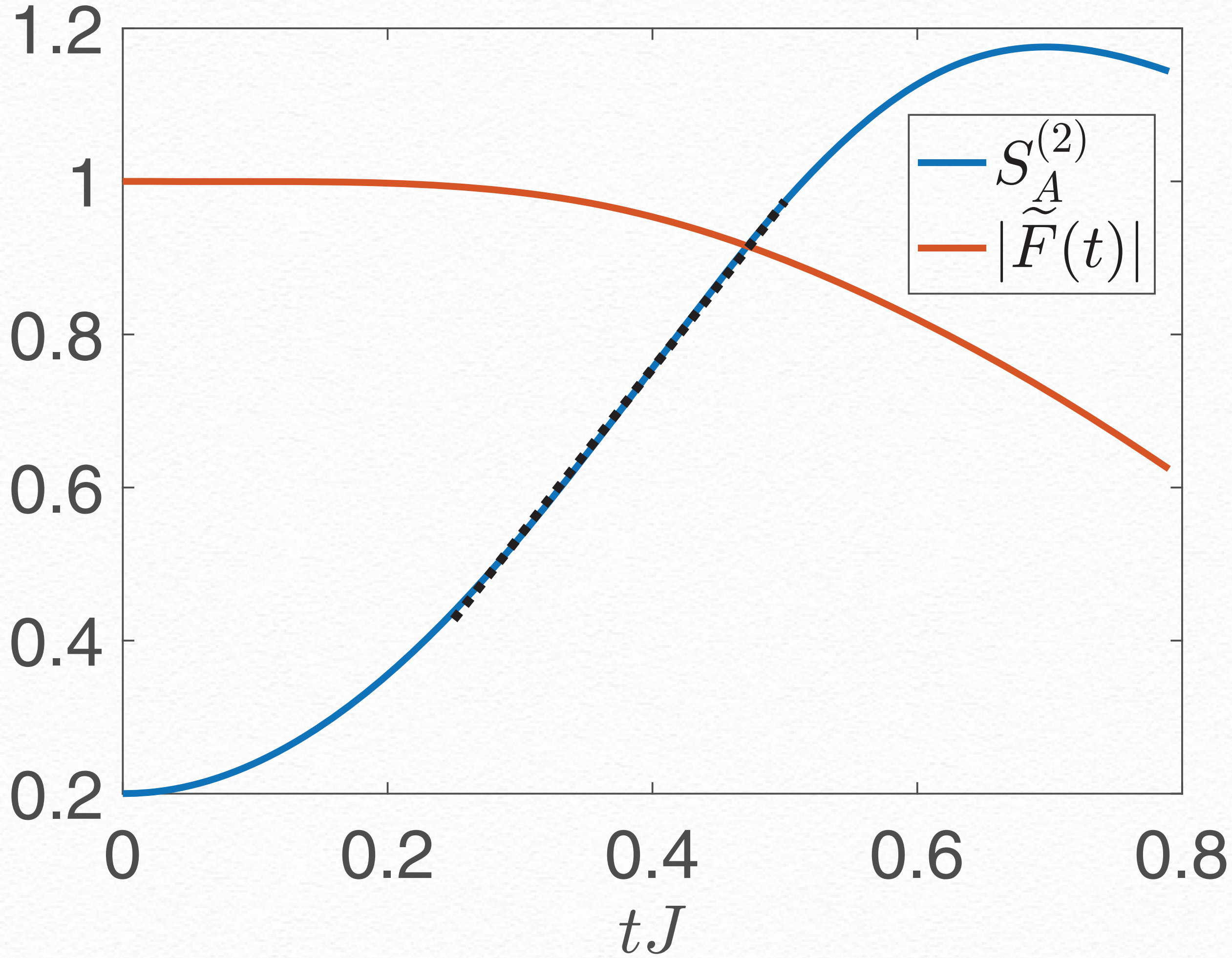
$$\exp(-S_A^{(2)}) = \sum_{M \in B} \mathrm{Tr}[\hat{M}(t)\hat{V}(0)\hat{M}(t)\hat{V}(0)]$$

Thermal Phase (ETH)	Single-Particle Localized	Many-Body Localized
Power-law spreading of entanglement	No spreading of entanglement	Logarithmic spreading of entanglement
OTOC exponential decay	OTOC remains constant	OTOC power-law decay

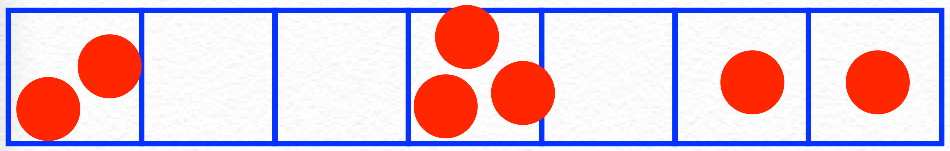
Our Results

OTOC in the Bose-Hubbard Model

Thermal Phase (ETH)



Bose-Hubbard Model

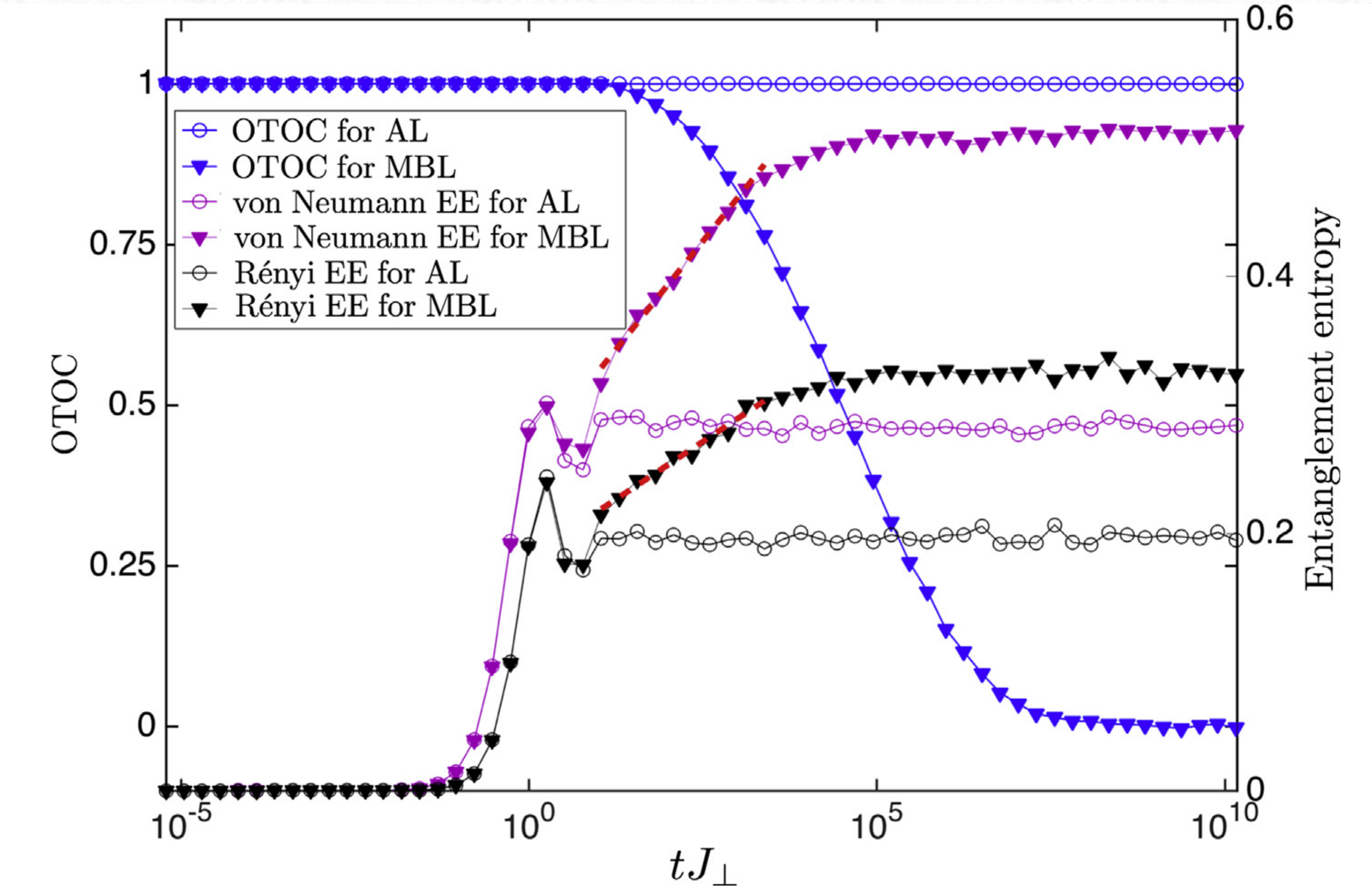


$$\hat{H} = -J\sum_{\langle ij\rangle}(\hat{b}_i^\dagger \hat{b}_j + \mathrm{H.c.}) + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1),$$

H. Shen, P. Zhang, R. Fan and H. Zhai, PRB, 2017

OTOC in the Localization Phase

Single-Particle Localized and MBL



$$\hat{H} = \sum_i J_\perp(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y) + J_z \hat{s}_i^z \hat{s}_{i+1}^z + h_i \hat{s}_i^z.$$

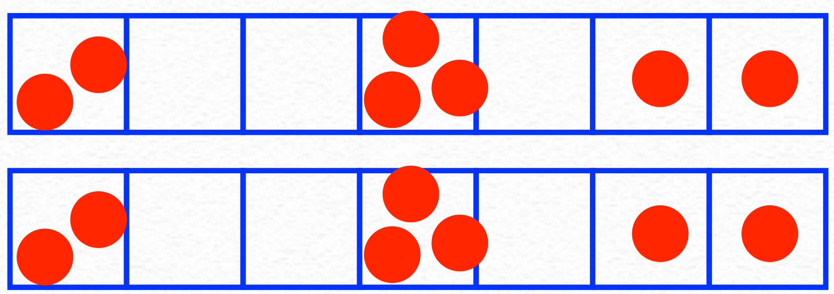
XXZ Model

R. Fan, P. Zhang, H. Shen and H. Zhai, Science Bulletin, 2017

Measurements of OTOC in Cold Atom System

$$F_M(t) = \mathrm{tr}[\hat{W}^\dagger(t)\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger\hat{W}(t)\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger]$$
$$= \mathrm{tr}[\hat{\rho}_1\hat{\rho}_2] = \mathrm{tr}[\hat{S}_{12}\hat{\rho}_1 \otimes \hat{\rho}_2]$$

$$\hat{\rho}_1 = \hat{W}^\dagger e^{-i\hat{H}t}\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger e^{i\hat{H}t}\hat{W}$$
$$\hat{\rho}_2 = e^{-i\hat{H}t}\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger e^{i\hat{H}t}.$$



I. Prepare two copies

H. Shen, P. Zhang, R. Fan and H. Zhai, PRB, 2017

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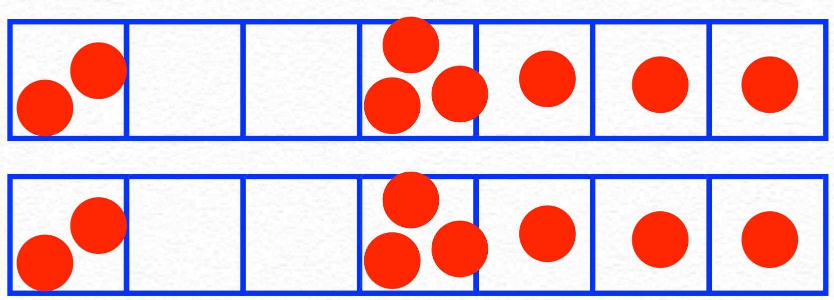
$$\hat{\rho}_2 = e^{-i\hat{H}t}\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger e^{i\hat{H}t}.$$

$$\hat{O} = \hat{b}_i^\dagger$$



II. Prepare two copies

H. Shen, P. Zhang, R. Fan and H. Zhai, PRB, 2017

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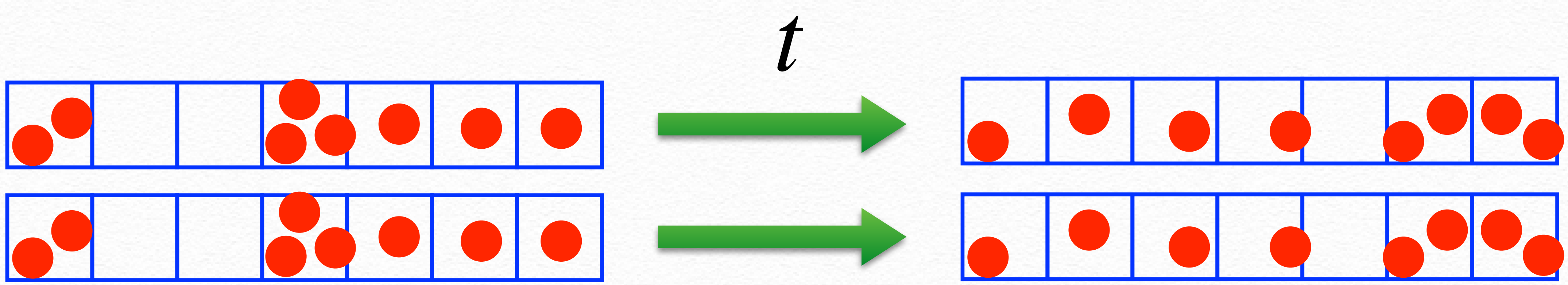
Time evolution



III. Evolute time t

H. Shen, P. Zhang, R. Fan and H. Zhai, PRB, 2017

Measurements of OTOC in Cold Atom System

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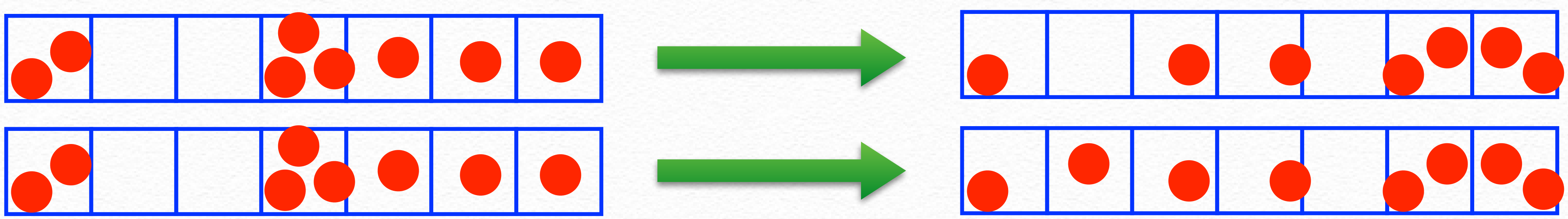
$$\hat{\rho}_2 = e^{-i\hat{H}t}\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger e^{i\hat{H}t}.$$

$$\hat{W} = \hat{b}_j^\dagger$$



IV. Apply a quench to one copy

H. Shen, P. Zhang, R. Fan and H. Zhai, PRB, 2017

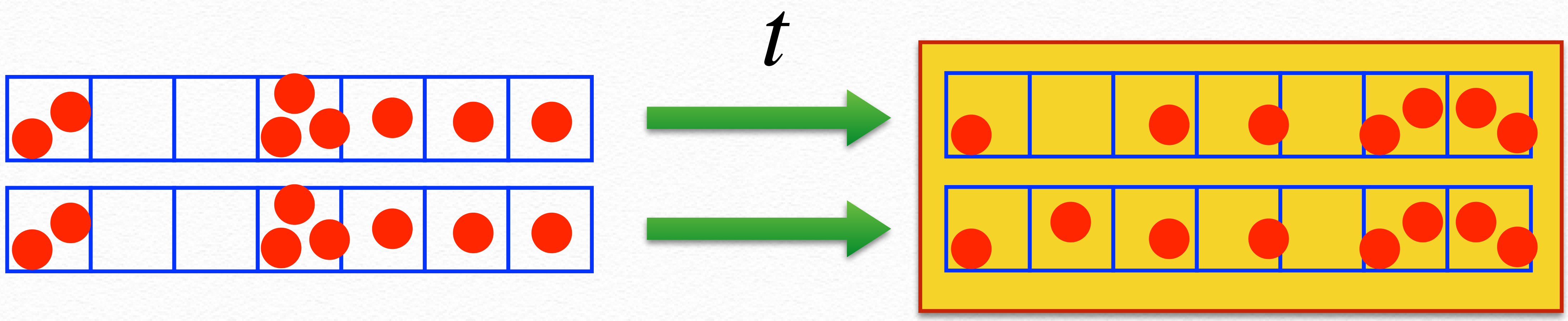
Measurements of OTOC in Cold Atom System

$$F_M(t) = \mathrm{tr}[\hat{W}^\dagger(t)\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger\hat{W}(t)\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger]$$

$$= \mathrm{tr}[\hat{\rho}_1\hat{\rho}_2] = \mathrm{tr}[\hat{S}_{12}\hat{\rho}_1 \otimes \hat{\rho}_2]$$

$$\hat{\rho}_1 = \hat{W}^\dagger e^{-i\hat{H}t}\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger e^{i\hat{H}t}\hat{W}$$

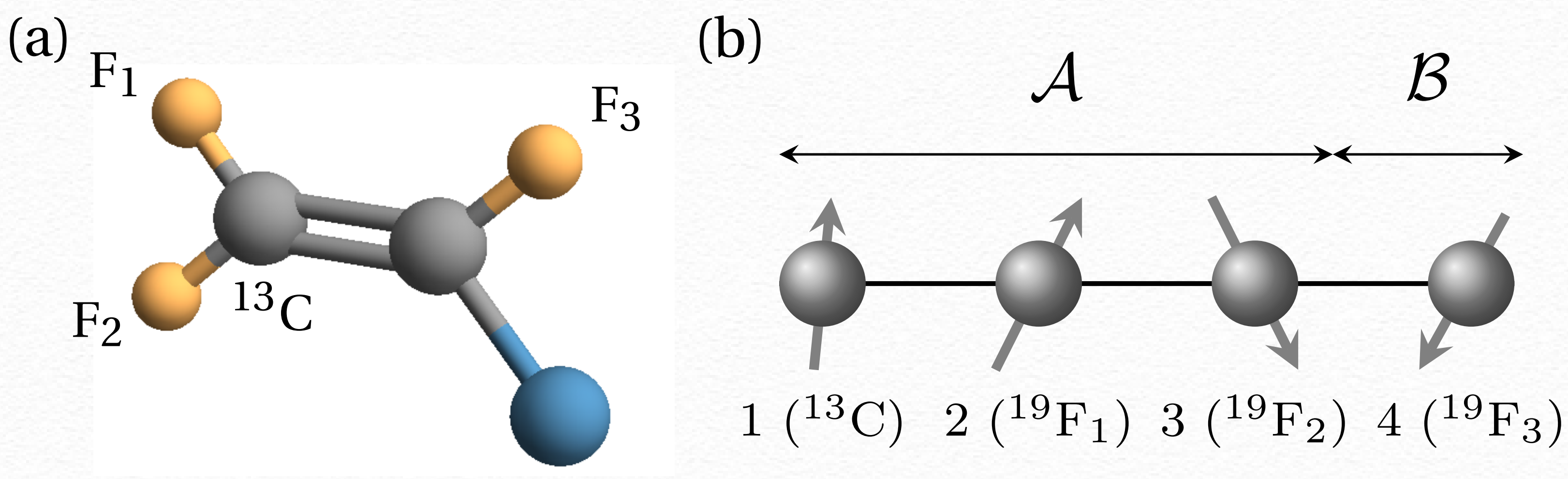
$$\hat{\rho}_2 = e^{-i\hat{H}t}\hat{O}e^{-\beta\hat{H}/2}\hat{O}^\dagger e^{i\hat{H}t}.$$



V. Use Hong-Ou-Mandel Interference

H. Shen, P. Zhang, R. Fan and H. Zhai, PRB, 2017

NMR Quantum Simulation Measuring OTOC



$$F(t) = \langle \hat{B}^\dagger(t)\hat{A}^\dagger(0)\hat{B}(t)\hat{A}(0)\rangle_\beta$$

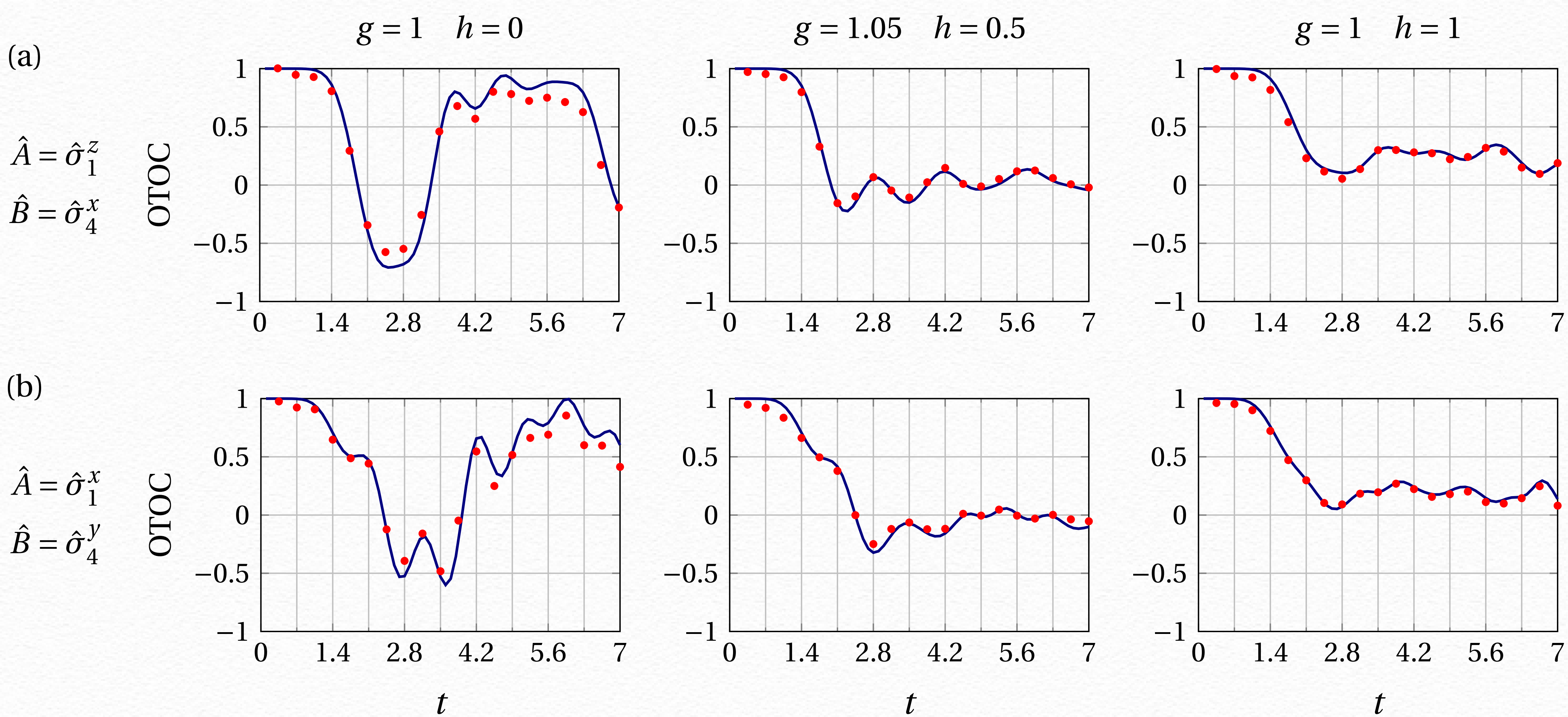
$$\hat{H} = \sum_i \left(-\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + g\hat{\sigma}_i^x + h\hat{\sigma}_i^z\right)$$

- **Zero h: Integrable case**
- **Non-Zero h: Non-Integrable case**

Measurements of OTOC for Ising Chain

$$\hat{H} = \sum_i \left(-\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + g\hat{\sigma}_i^x + h\hat{\sigma}_i^z\right)$$

Integrable Case **Non-Integrable Cases**

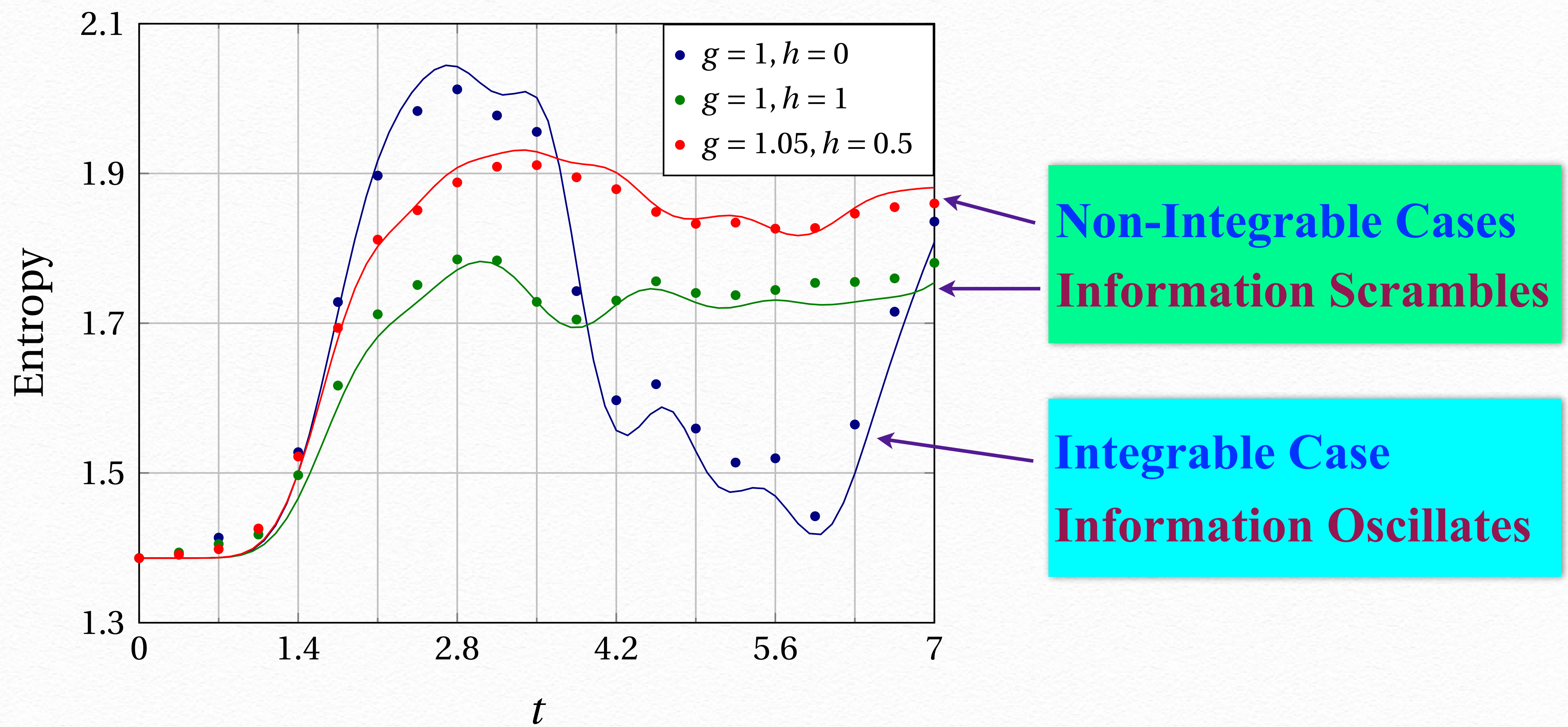


Hosur, Qi, Roberts and Yoshida, 2015

J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, J. Du, PRX (2017)

Measurements of OTOC for Ising Chain

$$\exp(-S_A^{(2)}) = \sum_{M\in B} \mathrm{Tr}[\hat{M}(t)\hat{V}(0)\hat{M}(t)\hat{V}(0)]$$



J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, J. Du, PRX (2017)

Multiple Quantum Coherence

$$F(\phi,t) = Tr(\hat{V} e^{i\hat{H}t} \hat{R}(\phi) e^{-i\hat{H}t} \hat{V} e^{i\hat{H}t} \hat{R}^{\dagger}(\phi) e^{-i\hat{H}t})$$

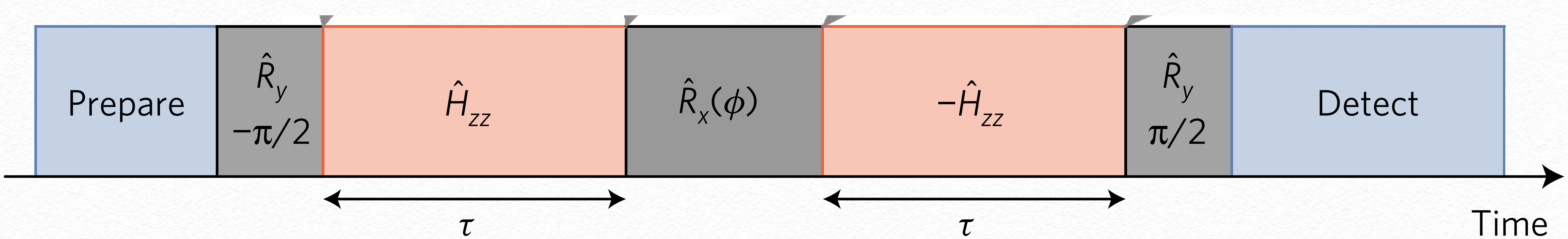
I. Prepare an initial density matrix $\rho_0 = \hat{V}$

II. Evolute the system with Hamiltonian $\hat{H}$

III. Rotate the system by $\hat{R}(\phi)$

IV. Evolute the system with Hamiltonian $-\hat{H}$

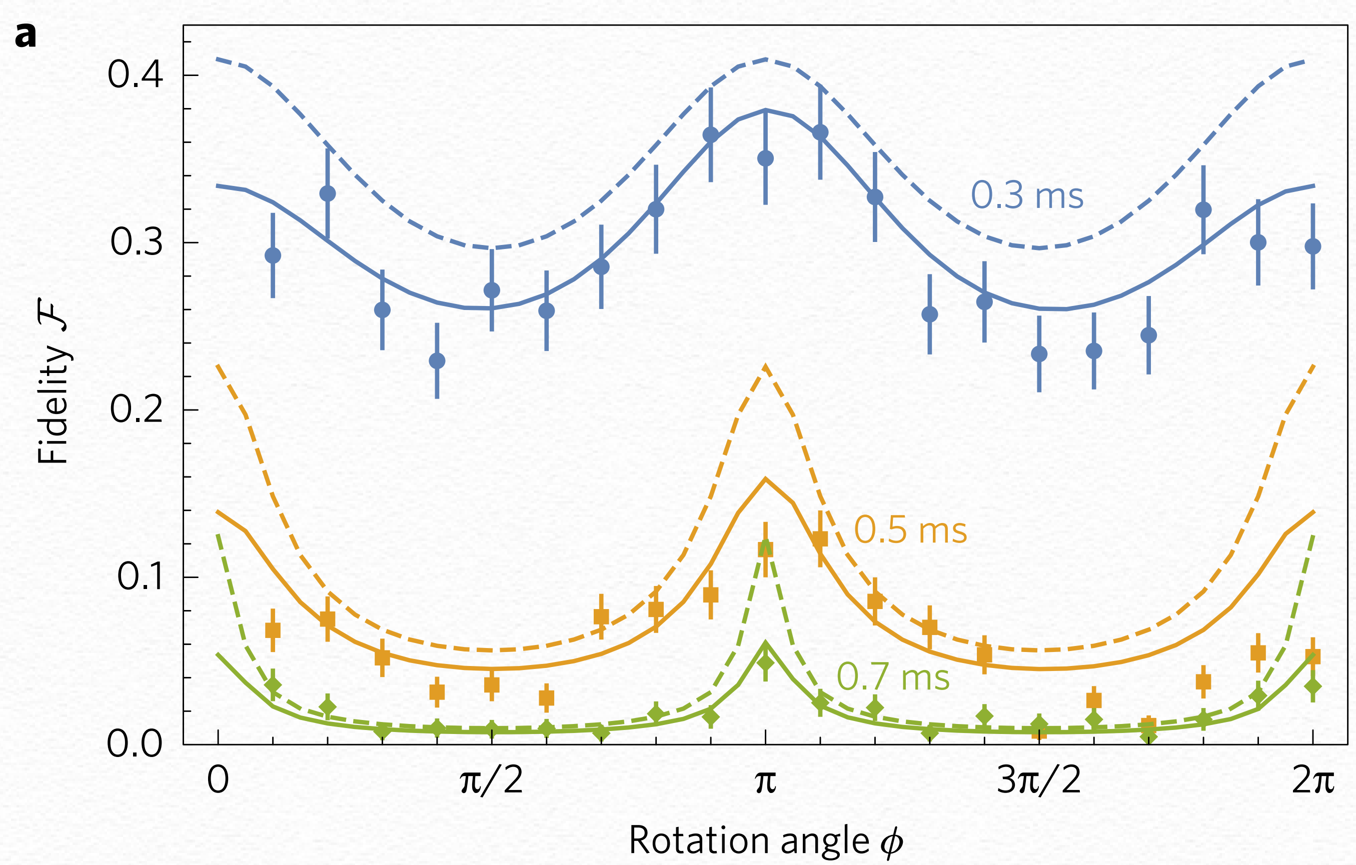
V. Measure observable $\hat{V}$



Multiple Quantum Coherence

$$F(\phi, t) = Tr(\hat{V} e^{i\hat{H}t} \hat{R}(\phi) e^{-i\hat{H}t} \hat{V} e^{i\hat{H}t} \hat{R}^\dagger(\phi) e^{-i\hat{H}t})$$

$$= Tr(\underbrace{e^{-i\hat{H}t} \hat{V} e^{i\hat{H}t}}_{\hat{V}(t)} \hat{R}(\phi) \underbrace{e^{-i\hat{H}t} \hat{V} e^{i\hat{H}t}}_{\hat{V}(t)} \hat{R}^\dagger(\phi))$$



Phonon-mediated, reversible spin-spin interaction

NIST, Boulder group, Nat. Phys. 2017

Summary

Entropy

Detected by
Hong-Ou-Mandel
Dynamics between
Two Identical Systems

Related by
Quench Dynamics

Out-of-Time-Order Correlation

Detected by
Time-Reversible
Many-Body Dynamics

Thank You Very Much for Attention !