# Diagrammatic perturbation theories of a strongly interacting atomic Fermi gas - II

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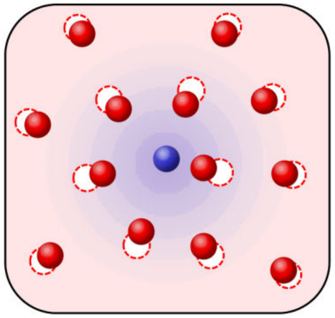
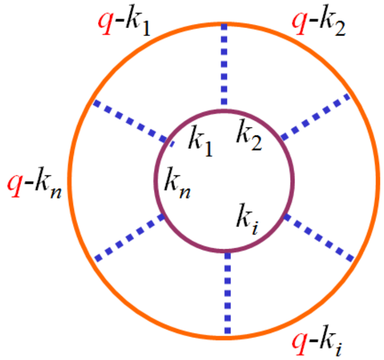
**WIPM, CAS**

# Outline

- The standard procedure to establish **Feynman rules/diagrams**
- Application 1: Polaron problem – the simplest many-body systems



- Application 2: Nozières & Schmitt-Rink theory (pairing instability)
- Application 3: The BCS theory and GPF theory
- Application 4: Beyond-GPF (ε-expansion theory)




- **Any unsolved problems/challenges (FFLO)?**

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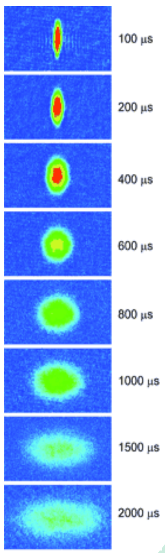
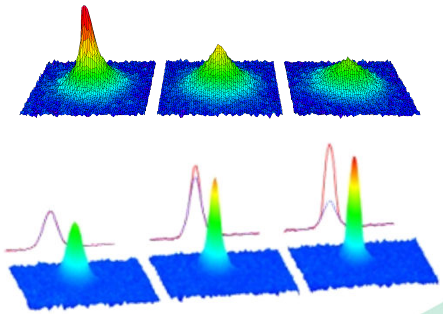
# A brief review of the recent experimental progress

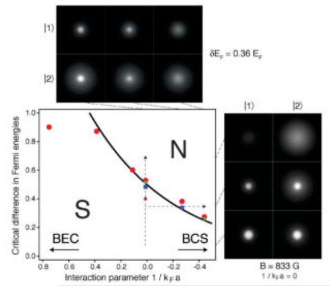
**Duke Science 2002**



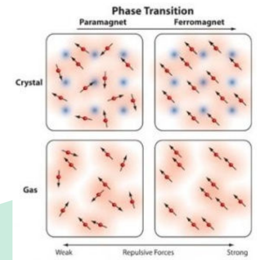

**MIT, JILA PRL 2004**



**MIT, Rice Science 2006**




**MIT, Science 2009**




**Strongly-interacting Fermi gases** 2002

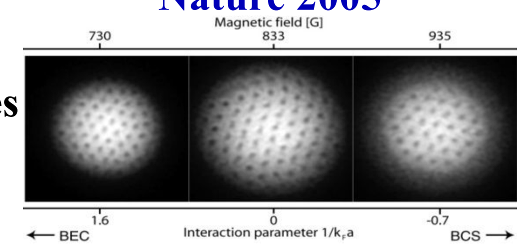
**Fermi superfluidity** 2004-2005

**Imbalanced superfluidity** 2006 - present

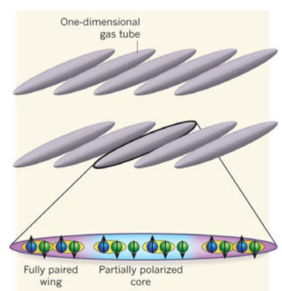
**Stoner ferromagnetism** 2009 – present

**Universal equation of state** 2010-2012

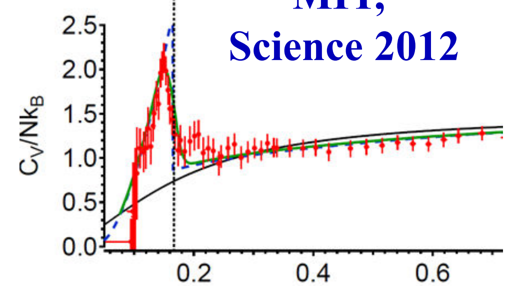
**MIT, Nature 2005**




**Rice, Nature 2010**




**MIT, Science 2012**



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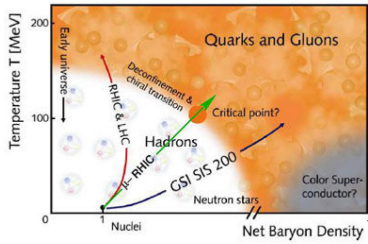
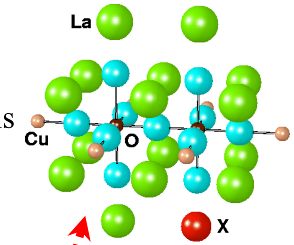
# Phase diagram of BEC-BCS crossover

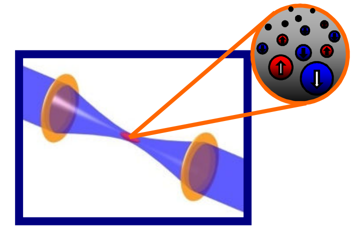
neutron stars & nuclear matter

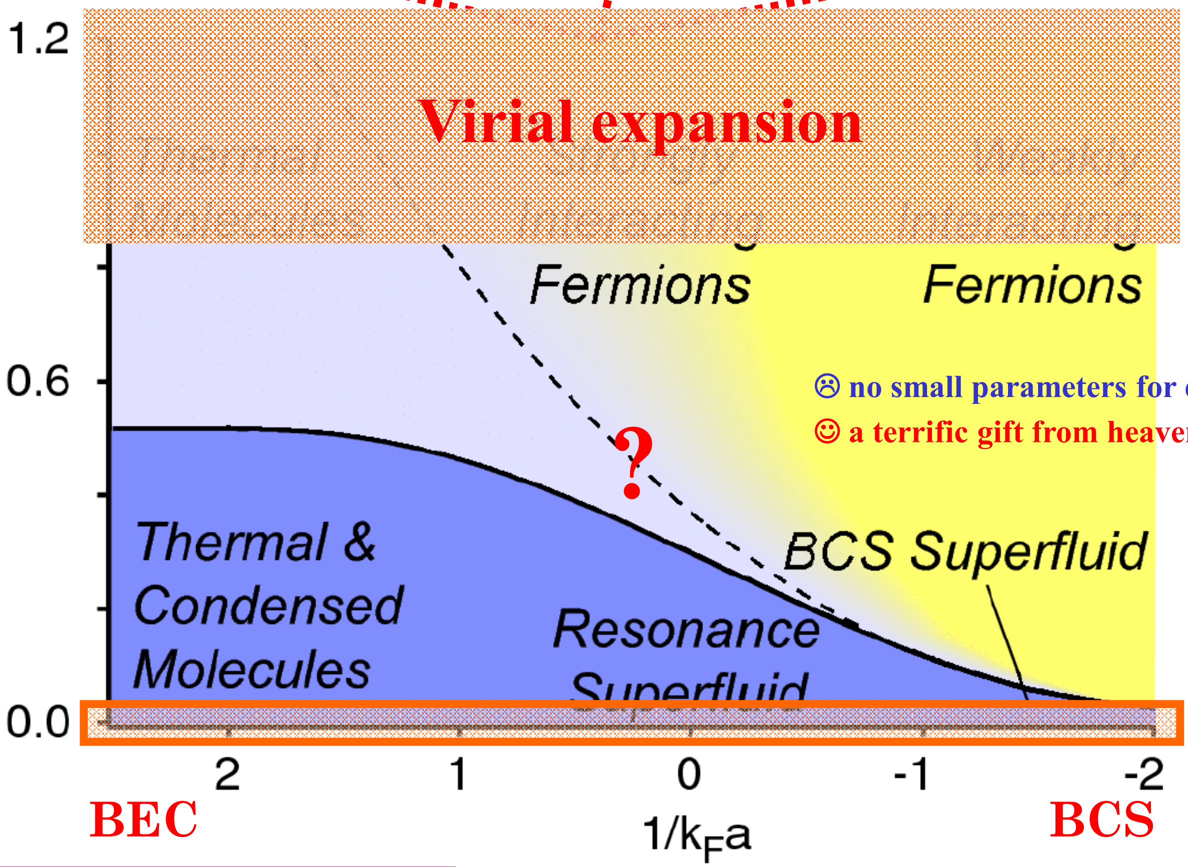



high-$T_c$ materials




ultracold atoms

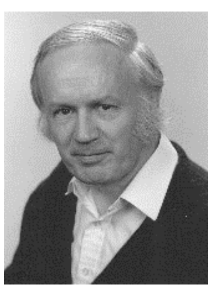





☹ no small parameters for diagrammatic theory…

☺ a terrific gift from heaven for theorists!

1980



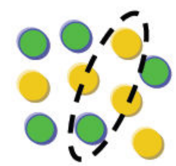
**BEC**

$$\Psi_0 = \exp\left(N_B^{1/2}\sum_{\mathbf{k}}\phi_{\mathbf{k}}c^{+}_{\mathbf{k}}c^{+}_{-\mathbf{k}}\right)|0\rangle$$

smooth crossover!!!

$$v_k = \frac{N_B^{1/2}\phi_k}{\left(1+N_B\phi_k^2\right)^{1/2}}$$

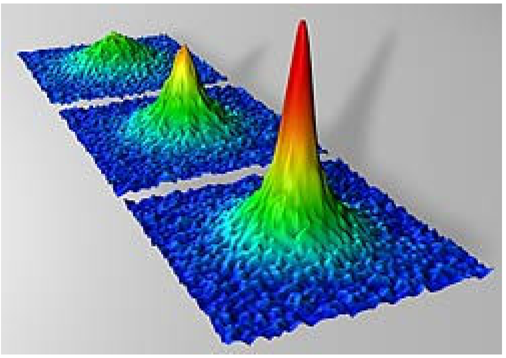
**BCS**



$$\Psi_0 = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}}c^{\dagger}_{\mathbf{k}}c^{\dagger}_{-\mathbf{k}})|0\rangle$$

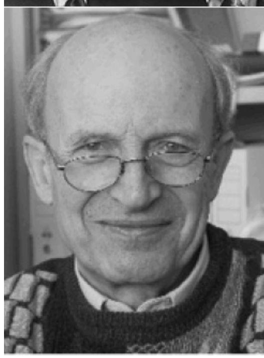
# Application 2:
# BEC-BCS crossover (NSR theory)

We now consider **particle-particle** excitations!



**We may understand the BCS instability, even in the strongly interacting regime!**

(**BCS** 1957) my favorite!          (1980)



**Nozières & Schmitt-Rink (NSR),**
*Journal of Low Temperature Physics* **59**, 195 (1985).

**While I was on sabbatical at Grenoble in 1981, I remember when *Nozières* rushed in one day to tell us that, for a strong attractive interaction between the fermions, the BCS theory transition temperature of 1957 reduced to the BEC formula first derived by Einstein in 1925. Philippe was very excited, as he should have been!**

**Allan Griffin,** *J. Phys.: Condens. Matter* **21** 164220 (2009).

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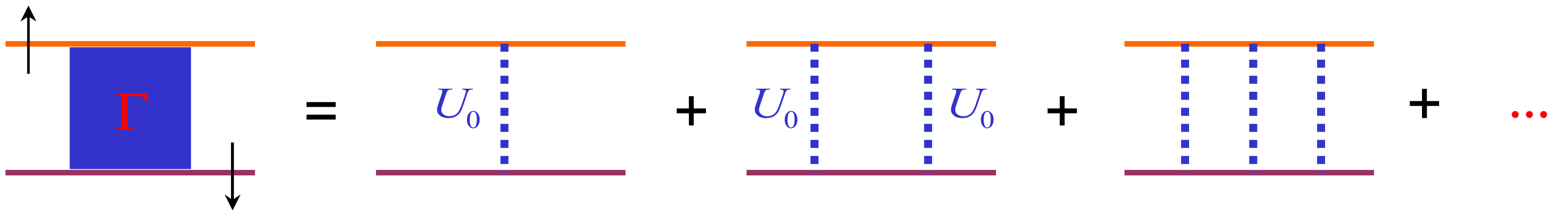
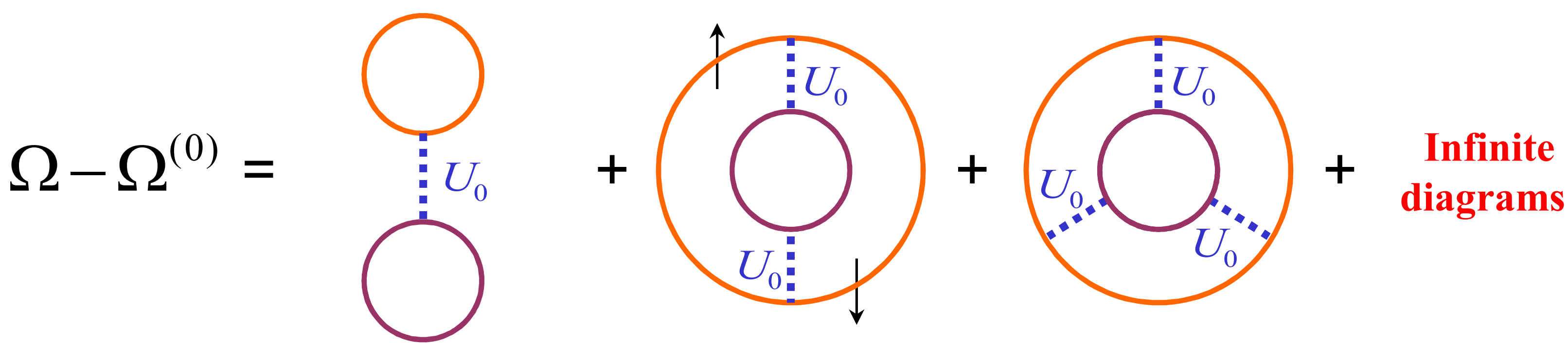
# NSR theory: physical picture




$$\frac{1}{\Gamma(q)} = \frac{m}{4\pi\hbar^2 a} + \sum_{\mathbf{k}} \left[ \frac{f(\xi_{\mathbf{k}}) + f(\xi_{\mathbf{q-k}}) - 1}{i\nu_n - \xi_{\mathbf{k}} - \xi_{\mathbf{q-k}}} - \frac{m}{\hbar^2 \mathbf{k}^2} \right]$$

Consider a spin-1/2 Fermi gas with equal population in each spin state. In the above ladder diagrams, the spin up and down fermions scatter successively. They seem to form a pair due to the attractive interaction. In this sense, the vertex function describes the motion of pair. Indeed, as we shall see, it can be regarded as the *Green function* of the (bosonic) pairs!

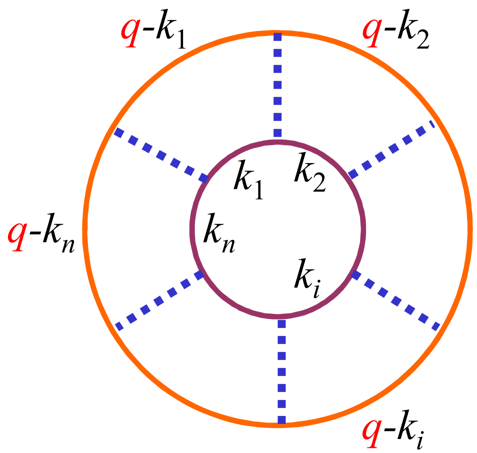
According to NSR, let us check this idea by using the thermodynamic potential, which, as shown before, is given by the following ***ring*** diagrams (obtained from the ladders),




Let us calculate it!

## NSR theory: derivation

Consider, for example, the following $n$-th order ring diagram,




$$= V\sum_q \frac{1}{n}(-1)^{n+1}U_0^n \sum_{k_1,k_2,\ldots,k_n}[G^{(0)}(k_1)G^{(0)}(q-k_1)]\cdots[G^{(0)}(k_n)G^{(0)}(q-k_n)]$$

Then, sum all the ring diagrams (using $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$):

$$\Omega-\Omega^{(0)} = V\sum_q\sum_n\frac{(-1)^{n+1}}{n}[U_0\sum_k G^{(0)}(k)G^{(0)}(q-k)]^n = V\sum_q \ln[1+U_0\sum_k G^{(0)}(k)G^{(0)}(q-k)]$$

Or, using the vertex function,

$$\Omega-\Omega^{(0)} = V\sum_q \ln[1+U_0\chi(q)] = V\sum_q \ln[\frac{1+U_0\chi(q)}{-U_0}] = V\sum_q \ln[-\Gamma^{-1}(q)]$$

To be contrasted with $\Omega_B^{(0)} = V\sum_q \ln\{-[G_B^{(0)}(q)]^{-1}\}$ for free bosons!

## NSR theory: why phase transition?

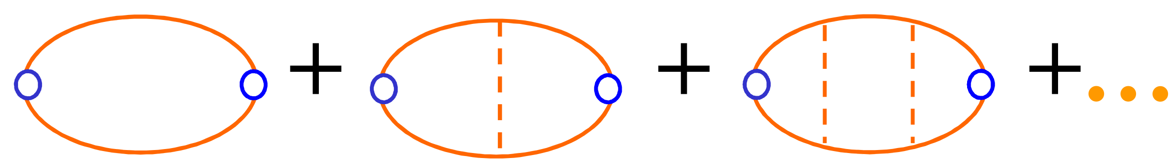
Apparently, we will have trouble if $\Gamma^{-1}(\mathbf{q}, i\nu_n=0) = 0$. What will happen then? **BCS superfluid phase transition** with symmetry breaking in number conservation! This is simply so-called **Thouless criterion**:

$$\max_q [\Gamma^{-1}(\mathbf{q}, i\nu_n = 0)]_{T=T_c} = 0.$$

As the temperature decreases to the transition temperature, the inverse of vertex function increases and touches zero from below! (note that, the phase transition does not necessarily occur at $q = 0$, *i.e.*, in general the Cooper pair may acquire a finite momentum!). **But, why this happens? Why symmetry breaking?**

There must be a divergent response function!

**Indeed, the copperon response function diverges at $T_c$!**

$\chi_{\text{copperon}} =$ 

$$= \frac{\chi(q)}{U_0^{-1} + \chi(q)} = \chi(q)\Gamma(q)$$

# NSR theory: Thouless criterion in the BCS limit

Recall, the vertex function,

$$\frac{1}{\Gamma(q)} = \frac{m}{4\pi\hbar^2 a} + \sum_{\mathbf{k}} \left[ \frac{f(\xi_{\mathbf{k}}) + f(\xi_{\mathbf{q}-\mathbf{k}}) - 1}{i\nu_n - \xi_{\mathbf{k}} - \xi_{\mathbf{q}-\mathbf{k}}} - \frac{m}{\hbar^2 \mathbf{k}^2} \right].$$

In **the BCS limit**, we set the chemical potential to the Fermi energy, $\mu=\varepsilon_F$, then, we may check that the maximum in $\Gamma^{-1}(\mathbf{q}, i\nu_n=0)$ occurs at $q=0$ (which means the Cooper pair has zero momentum!). Therefore, we find the BCS transition temperature is determined by,

$$\frac{m}{4\pi\hbar^2 a} + \sum_{\mathbf{k}} \left[ \frac{1-2f(\xi_{\mathbf{k}})}{2\xi_{\mathbf{k}}} - \frac{m}{\hbar^2 \mathbf{k}^2} \right] = 0.$$

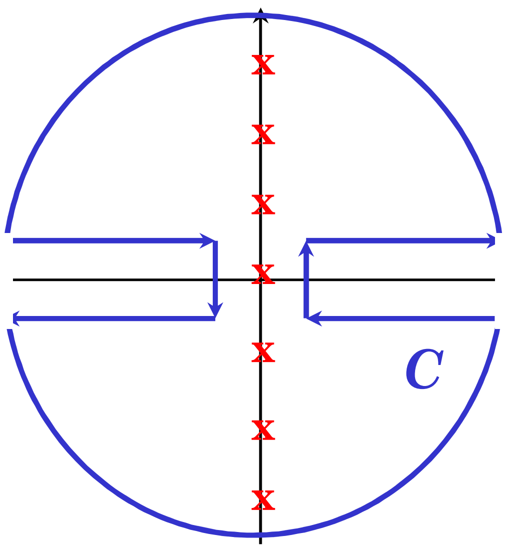
This gives the following BCS superfluid transition temperature:

$$k_B T_{c,BCS} = \frac{8}{e^2} \frac{\gamma}{\pi} \varepsilon_F e^{\frac{\pi}{2k_F a}}, \quad \left( \frac{1}{k_F a} \to -\infty \right).$$

**What happens away from the BCS limit?** The chemical potential is no longer fixed to the Fermi energy, i.e., $\mu < \varepsilon_F$. We need to determine the chemical potential self-consistently, using the NSR thermodynamic potential $\Omega = \Omega^{(0)} + V\sum_q \ln[-\Gamma^{-1}(q)]$!

# NSR theory: derivation

To proceed, we need a mathematic trick. We wish to prove for any function $h(x)$,




$$k_B T \sum_{i\nu_n} h(i\nu_n) e^{+i\nu_n 0^+} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega}-1} \mathrm{Im}\, h(i\nu_n \to \omega + i0^+). \qquad (1)$$

This is because the left-hand-side of equation can be written as a contour integral over $C$ (see the left graph):

$$k_B T \sum_{i\nu_n} h(i\nu_n) e^{+i\nu_n 0^+} = \frac{1}{2\pi i} \oint_C \frac{e^{\omega 0^+}}{e^{\beta\omega}-1} h(\omega)$$

Due to the convergence factor, the integral at two half circles vanishes. The contribution near the real axis gives the right hand side of equation (1).

Recall that, $\Omega - \Omega^{(1)} = V \int \frac{d\mathbf{q}}{(2\pi)^3} k_B T \sum_{i\nu_n} \ln[-\Gamma^{-1}(\mathbf{q}, i\nu_n)] e^{+i\nu_n 0^+}$, we therefore reach at,

phase shift

$$\Omega - \Omega^{(1)} = V \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega}-1} \mathrm{Im} \ln[-\Gamma^{-1}(\mathbf{q}, i\nu_n \to \omega + i0^+)] \equiv -V \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega}-1} \delta(\mathbf{q}, \omega)$$

The only parameter, the chemical potential, is to be determined by the number equation:

$$n - n^{(1)} \equiv \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega}-1} \frac{\partial \delta(q, \omega)}{\partial \mu}$$

# NSR theory: the calculation of *EoS*

How can we use the NSR theory to calculate equation of state? Easy!

(1) For a given $T$ and $a$, select $\mu$, solve the phase shift: $\delta(\mathbf{q},\omega)=-\mathrm{Im}\ln[-\Gamma^{-1}(\mathbf{q},\omega+i0^{+})]$

(2) Using the number equation calculate the number density $n$, $T_F$ and $k_F$;

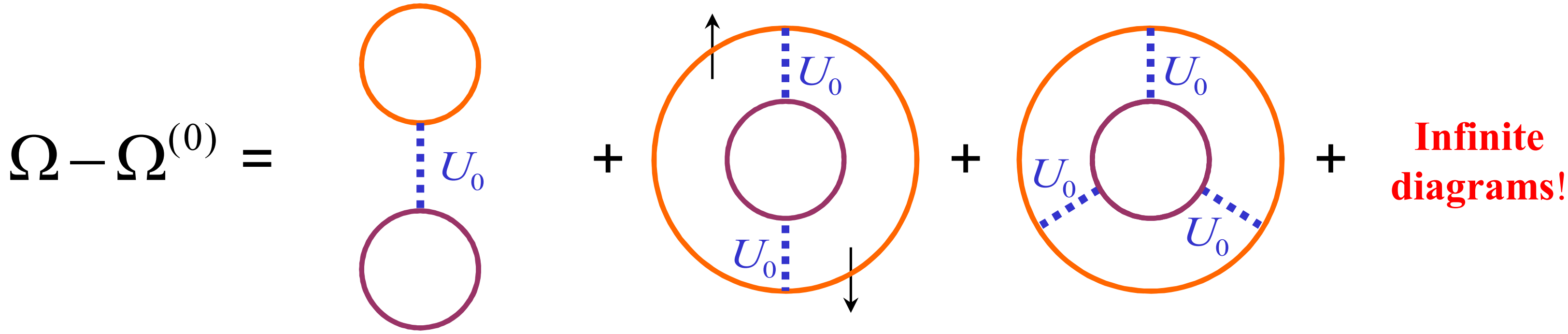
(3) Then, using $S=-\partial\Omega/\partial T$ calculate the entropy;

(4) Finally, calculate the energy $E=\Omega+TS+N\mu$;

(5) Present the equation of state as functions of $T/T_F$ and $1/k_Fa$.

Note that, always check the **Thouless criterion**, if instability occurs, increase $T$ or reduce $\mu$!

# NSR theory: derivation




With all the **two-particle** scattering processes being taken into account, we have the NSR thermodynamic potential:

$$\Omega - \Omega^{(1)} = -V \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega} - 1} \delta(\mathbf{q}, \omega).$$

Here the phase shift $\delta(\mathbf{q},\omega) = -\mathrm{Im}\ln[-\Gamma^{-1}(\mathbf{q},\omega + i0^+)]$, where,

$$\Gamma^{-1}(\mathbf{q}, \omega + i0^+) = \frac{m}{4\pi\hbar^2 a} + \int \frac{d\mathbf{k}}{(2\pi)^3} \left[ \frac{f(\xi_{\mathbf{q}/2+\mathbf{k}}) + f(\xi_{\mathbf{q}/2-\mathbf{k}}) - 1}{\omega + i0^+ - \xi_{\mathbf{q}/2+\mathbf{k}} - \xi_{\mathbf{q}/2-\mathbf{k}}} - \frac{m}{\hbar^2 \mathbf{k}^2} \right].$$

Note that, **$\delta(q, \omega=0) = 0$**. Otherwise we will encounter a singularity (*i.e.*, Bose condensation) in the integral over frequency.

# Kramers-Kronig relation

Let $\chi(\omega) = \chi_1(\omega) + i\chi_2(\omega)$ be a complex function of the complex variable $\omega$, where $\chi_1(\omega)$ and $\chi_2(\omega)$ are real. Suppose this function is analytic in the closed upper half-plane of $\omega$ and vanishes like $1/|\omega|$ or faster as $|\omega| \to \infty$. Slightly weaker conditions are also possible. The Kramers–Kronig relations are given by

$$\chi_1(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega' - \omega}\, d\omega'$$

and

$$\chi_2(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_1(\omega')}{\omega' - \omega}\, d\omega',$$

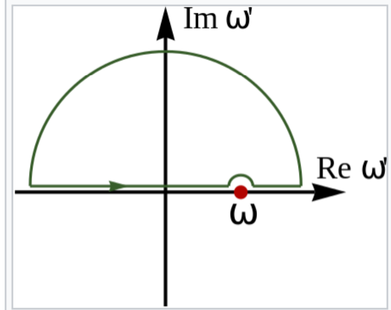
where $\mathcal{P}$ denotes the Cauchy principal value. So the real and imaginary parts of such a function are not independent, and the full function can be reconstructed given just one of its parts.

# Kramers-Kronig relation

The proof begins with an application of Cauchy's residue theorem for complex integration. Given any analytic function $\chi$ in the closed upper half plane, the function $\omega' \to \chi(\omega')/(\omega' - \omega)$ where $\omega$ is real will also be analytic in the upper half of the plane. The residue theorem consequently states that

$$\oint \frac{\chi(\omega')}{\omega' - \omega} \, d\omega' = 0$$

for any closed contour within this region. We choose the contour to trace the real axis, a hump over the pole at $\omega' = \omega$, and a large semicircle in the upper half plane. We then decompose the integral into its contributions along each of these three contour segments and pass them to limits. The length of the semicircular segment increases proportionally to $|\omega'|$, but the integral over it vanishes in the limit because $\chi(\omega')$ vanishes at least as fast as $1/|\omega'|$. We are left with the segments along the real axis and the half-circle around the pole. We pass the size of the half-circle to zero and obtain



Integral contour for deriving Kramers–Kronig relations.

$$0 = \oint \frac{\chi(\omega')}{\omega' - \omega} \, d\omega' = \mathcal{P}\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} \, d\omega' - i\pi\chi(\omega).$$

The second term in the last expression is obtained using the theory of residues,[4] more specifically the Sokhotski–Plemelj theorem. Rearranging, we arrive at the compact form of the Kramers–Kronig relations,

$$\chi(\omega) = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} \, d\omega'.$$

The single $i$ in the denominator will effectuate the connection between the real and imaginary components. Finally, split $\chi(\omega)$ and the equation into their real and imaginary parts to obtain the forms quoted above.

# NSR theory: calculation of phase shift

Let us now analyse the phase shift, $\delta(\mathbf{q},\omega) = -\mathrm{Im}\ln[-\Gamma^{-1}(\mathbf{q},\omega+i0^+)]$, where ($\xi_k=\varepsilon_k-\mu$),

$$\Gamma^{-1}(\mathbf{q},\omega+i0^+) = \frac{m}{4\pi\hbar^2 a} + \int \frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{f(\xi_{\mathbf{q}/2+\mathbf{k}})+f(\xi_{\mathbf{q}/2-\mathbf{k}})-1}{\omega+i0^+-\xi_{\mathbf{q}/2+\mathbf{k}}-\xi_{\mathbf{q}/2-\mathbf{k}}} - \frac{m}{\hbar^2\mathbf{k}^2}\right] = \Gamma_0^{-1}(\mathbf{q},\omega+i0^+)+\Gamma_{mb}^{-1}(\mathbf{q},\omega+i0^+),$$

and,

$$\Gamma_{mb}^{-1}(\mathbf{q},\omega+i0^+) = \int \frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{f(\xi_{\mathbf{q}/2+\mathbf{k}})+f(\xi_{\mathbf{q}/2-\mathbf{k}})}{\omega+i0^+-\xi_{\mathbf{q}/2+\mathbf{k}}-\xi_{\mathbf{q}/2-\mathbf{k}}}\right].$$

(Important!) Numerical trick: we calculate first $\mathrm{Im}\Gamma^{-1}$, and then use Kramers-Kronig relation to obtain the real part $\mathrm{Re}\Gamma^{-1}$, *i.e.*, [Recall that $1/(x+i0^+) = \mathbf{P}(1/x) - i\pi\delta(x)$]

$$\mathrm{Im}\Gamma_{mb}^{-1}(\mathbf{q},\omega+i0^+) = -\pi\int \frac{d\mathbf{k}}{(2\pi)^3}\left[f(\xi_{\frac{\mathbf{q}}{2}+\mathbf{k}})+f(\xi_{\frac{\mathbf{q}}{2}-\mathbf{k}})\right]\delta(\omega-\frac{\varepsilon_q}{2}-2\varepsilon_k+2\mu)$$

$$\mathrm{Re}\Gamma_{mb}^{-1}(\mathbf{q},\omega+i0^+) = \mathbf{P}\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega' \frac{\mathrm{Im}\Gamma_{mb}^{-1}(\mathbf{q},\omega'+i0^+)}{\omega'-\omega}.$$

(General feature) (i) We can obtain analytically,

$$\Gamma_0^{-1}(\mathbf{q},\omega+i0^+) = \frac{m}{4\pi\hbar^2 a} + \frac{im^{3/2}}{4\pi\hbar^3}\sqrt{\omega+i0^+-\frac{\varepsilon_q}{2}+2\mu};$$

(ii) The imaginary part is nonzero only if $\omega-\varepsilon_q/2+2\mu>0$; (iii) It thus easy to see that, as ω goes to infinity, the phase shift is: **$\delta(\mathbf{q}, -\infty)=0$** and **$\delta(\mathbf{q}, +\infty)= \pi/2$**; (iv) Of course, **$\delta(\mathbf{q}, 0)= 0$.**

# NSR theory: BEC regime

Consider now the BEC side with a positive scattering length $a>0$. Physically, we expect that the chemical potential is sightly larger than the half of the binding energy, *i.e.*, $2\mu \approx -\hbar^2/(ma^2) = \varepsilon_B$

Using,

$$\Gamma^{-1} \approx \Gamma_0^{-1}(\mathbf{q}, \omega + i0^+) = \frac{m}{4\pi\hbar^2 a} + \frac{im^{3/2}}{4\pi\hbar^3}\sqrt{\omega + i0^+ - \frac{\varepsilon_q}{2} + 2\mu}$$

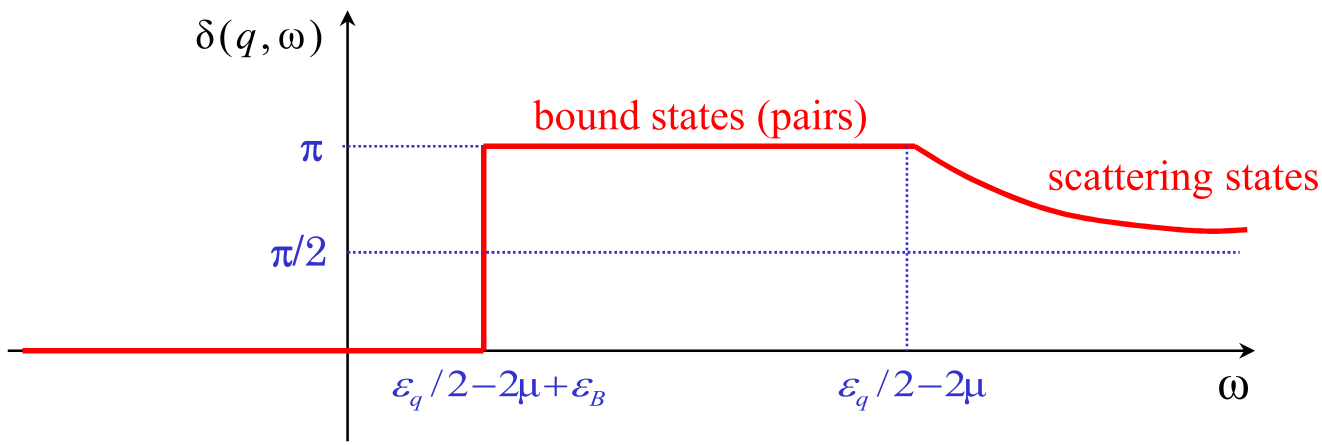
we can find that,

(1) $\Gamma^{-1}$ is real and negative, if $-\infty < \omega < \varepsilon_q/2 - 2\mu + \varepsilon_B$

(2) $\Gamma^{-1}$ is real and positive, if $\varepsilon_q/2 - 2\mu + \varepsilon_B < \omega < \varepsilon_q/2 - 2\mu$

(3) $\mathrm{Re}\Gamma^{-1}$ is a positive const, if $\varepsilon_q/2 - 2\mu < \omega < +\infty$ ; however, $\mathrm{Im}\Gamma^{-1}$ develops!

Therefore, we may conclude (recall $\delta(q,\omega) = -\mathrm{Im}\ln[-\Gamma^{-1}(q, \omega + i0^+)]$):

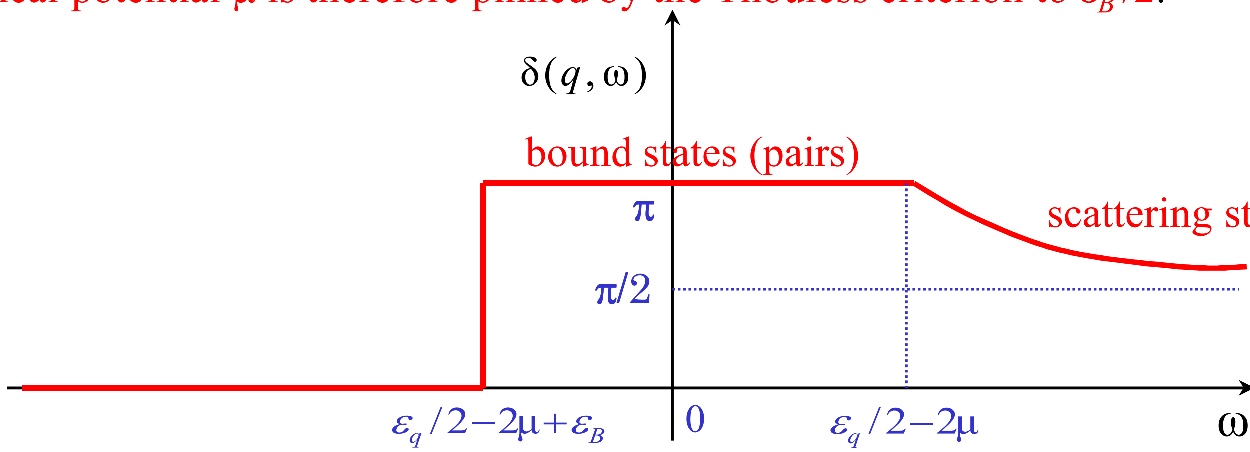



Recall, in case of free bosons, $\delta_B = \pi\Theta(\omega - \hbar^2 q^2/2M + \mu_B)$. Note that $\mu_B = 2\mu - \varepsilon_B$!

In the BEC limit, where $n_F$ is exponentially small, the system is an *ideal* Bose gas of bound pairs!

# NSR theory: BEC regime

You now may criticise the assumption that $2\mu \approx \varepsilon_B << 0$. What happens if we take a positive chemical potential? The phase shift $\delta(q,\omega)$ will simply be shifted to the left $\omega$-axis. As a result, we may have $\delta(q,\omega=0) = \pi$, which implies the instability for condensation. The chemical potential $\mu$ is therefore pinned by the Thouless criterion to $\varepsilon_B/2$.




On the other hand, because $\mu$ is strongly negative, we may approximate the vertex function,

$$\Gamma_0(\mathbf{q},i\nu_n) = \left(\frac{m}{4\pi\hbar^2 a} + \frac{i m^{3/2}}{4\pi\hbar^3}\sqrt{i\nu_n - \frac{\varepsilon_q}{2} + 2\mu}\right)^{-1} \approx \left(\frac{8\pi\hbar^4}{m^2 a}\right)\frac{1}{i\nu_n - \varepsilon_q/2 + \mu_B}.$$

This indicates again that in the BEC limit the system is an *ideal* Bose gas of bound pairs, with mass $M_B=2m$ and a vanishingly small chemical potential $\mu_B=2\mu-\varepsilon_B$ (*i.e.*, no interactions between pairs). Therefore, the transition temperature is,

$$T_c = \left(\frac{n_B}{\zeta(3/2)}\right)^{2/3}\frac{2\pi\hbar^2}{M_B k_B} = \frac{2\pi}{[6\pi^2\zeta(3/2)]^{2/3}} T_F \approx 0.218 T_F.$$

**For a strong interaction, the BCS theory transition temperature of 1957 reduced to the BEC formula derived by Einstein in 1925.**

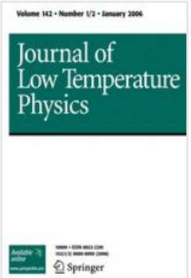
## NSR theory:

$$\Omega-\Omega^{(1)}=V\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{e^{\beta\omega}-1}\mathrm{Im}\ln[-\Gamma^{-1}(\mathbf{q},i\nu_n\to\omega+i0^+)]\equiv-V\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{e^{\beta\omega}-1}\delta(\mathbf{q},\omega)$$

**What happens on the BCS side and in the unitary limit? How do the two Fermi distribution functions contribute to the phase shift?**

You may try: (i) For given $T$ and $a$, calculate the $\Gamma^{-1}$ and solve the number equation for $\mu$; (ii) For a given $1/k_F a$, solve the Thouless criterion for $T_c$; (iii) At $T_c$, check the behaviour of the phase shift, and calculate the number of fermions and number of Cooper pairs.

# NSR theory: the original JLTP paper




Journal of Low Temperature Physics
May 1985, Volume 59, Issue 3, pp 195–211

## Bose condensation in an attractive fermion gas: From weak to strong coupling superconductivity

Authors | Authors and affiliations

P. Nozières, S. Schmitt-Rink

Article

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## Abstract

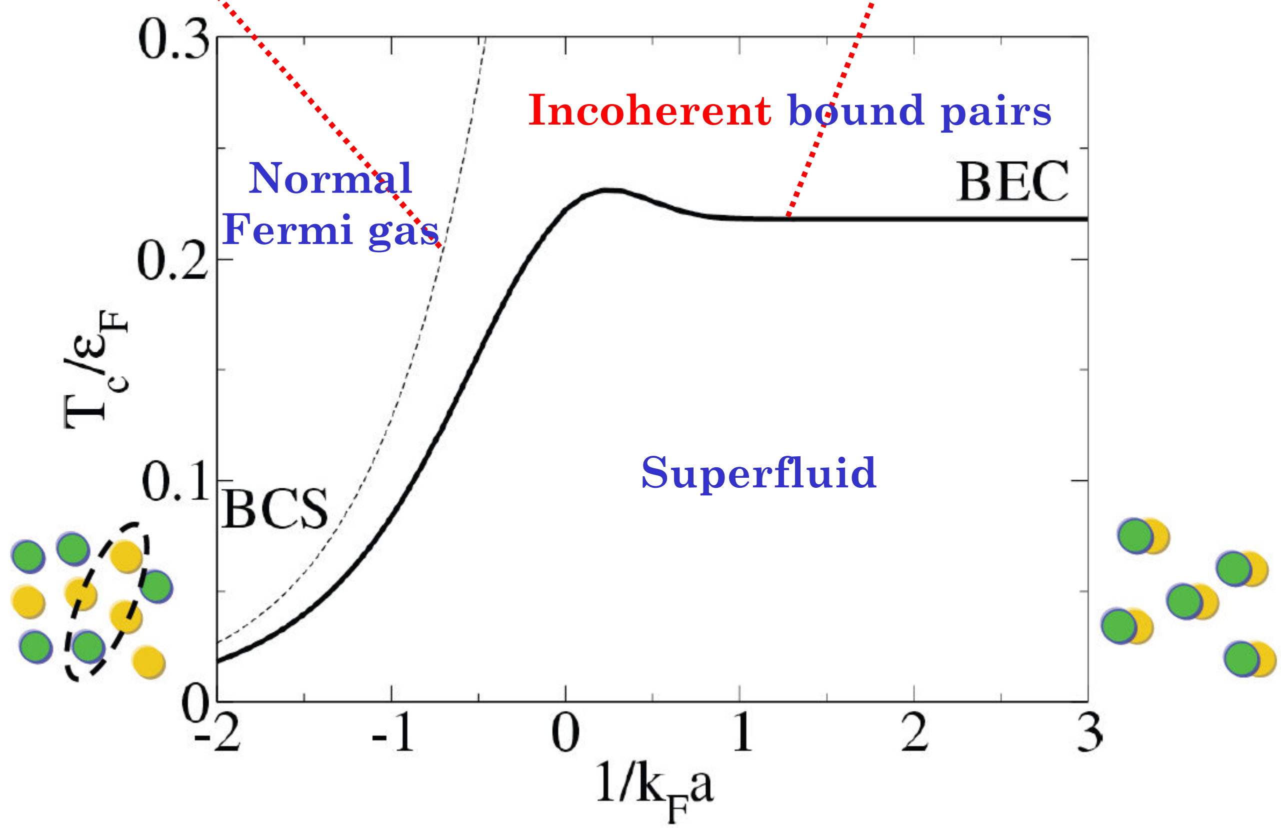
We consider a gas of fermions interacting via an attractive potential. We study the ground state of that system and calculate the critical temperature for the onset of superconductivity as a function of the coupling strength. We compare the behavior of continuum and lattice models and show that the evolution from weak to strong coupling superconductivity is smooth.

# NSR theory: phase diagram!

$$k_B T_C(\mathbf{BCS}) = \frac{8}{e^2}\frac{\gamma}{\pi}\varepsilon_F e^{\frac{\pi}{2k_F a}}$$

$T_{diss}$ is calculated by solving the Thouless criterion, with $\mu=\varepsilon_F$.

$T_c$ is obtained by solving the number equation together with the Thouless criterion.




$$\Omega = \Omega^{(0)} + V\sum_q \ln[-\Gamma^{-1}(q)] = \text{fermions} + \text{pairs}$$

# NSR theory: Functional path-integral reformulation

VOLUME 71, NUMBER 19 PHYSICAL REVIEW LETTERS 8 NOVEMBER 1993

## Crossover from BCS to Bose Superconductivity: Transition Temperature and Time-Dependent Ginzburg-Landau Theory

C. A. R. Sá de Melo,[1] Mohit Randeria,[2] and Jan R. Engelbrecht[3]

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(Received 2 June 1993)

We use functional integral formulation to study the finite temperature crossover from cooperative Cooper pairing to independent bound state formation and condensation, We show the inadequacy of mean field results for normal state properties obtained at the saddle point level as the coupling increases. The importance of quantum (temporal) fluctuations is pointed out and an interpolation scheme for $T_c$ is derived from this point of view. The time-dependent Ginzburg-Landau (TDGL) equation near $T_c$ is shown to describe a damped mode in the BCS limit, and a propagating one in the Bose limit. A singular point is identified at intermediate coupling where a simple TDGL description fails.

PACS numbers: 74.20.−z, 67.40.−w, 74.40.+k, 74.72.−h

**I will introduce very briefly this functional path-integral approach later on.**
**You should read the above classical PRL paper!**

# NSR theory: Functional path-integral reformulation

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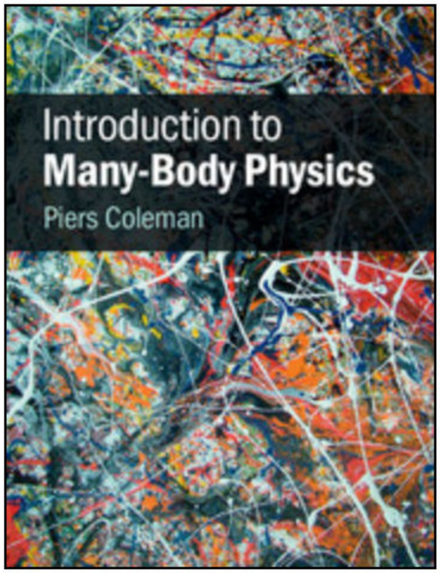


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## Introduction to Many-Body Physics

Piers Coleman
*Rutgers University, New Jersey*

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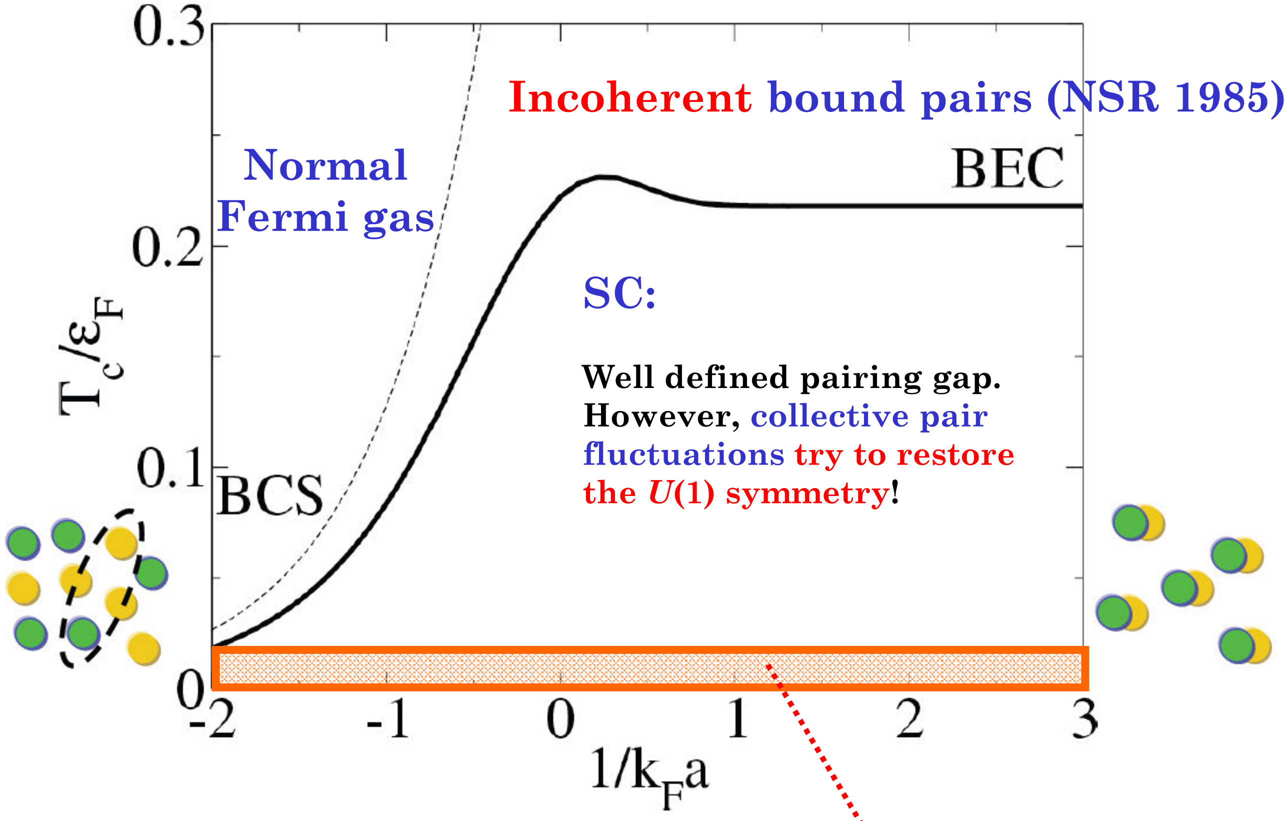
A modern, graduate-level introduction to many-body physics in condensed matter, this textbook
understanding of the correlated behavior of quantum fluids. Starting with an operator-based intr
textbook presents the Feynman diagram approach, Green's functions and finite-temperature mar
interacting systems. Special chapters are devoted to the concepts of Fermi liquid theory, broken
the physics of local-moment metals. A strong emphasis on concepts and numerous exercises ma
condensed matter physics. It will also interest students in nuclear, atomic and particle physics.

• Provides practical knowledge to help graduate students move into experimental research • Disc
spectroscopy to help students better understand how experiments probe response and correlatio
concepts of Fermi liquid theory, broken symmetry, conduction in disordered systems, supercondu

**The basic knowledge of the functional path-integral approach can be learned from some textbooks, for example, "Introduction to Many-Body Physics" by Piers Coleman, Chapter 12.**

# NSR theory: superfluid phase?




At $T$=0, pairs are qualitatively described by the BCS MF gap (**Leggett 1980**).

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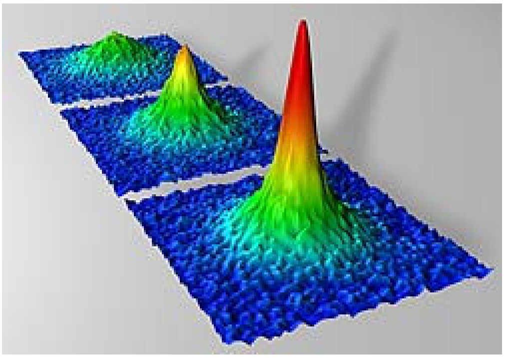
# Application 3:
# BEC-BCS crossover (BCS+GPF theories)

We now consider the superfluid state by using the BCS theory and the Gaussian pair fluctuation (GPF) theory!



**The GPF theory on top of BCS is very useful to describe the BEC-BCS crossover!**

# BCS theory

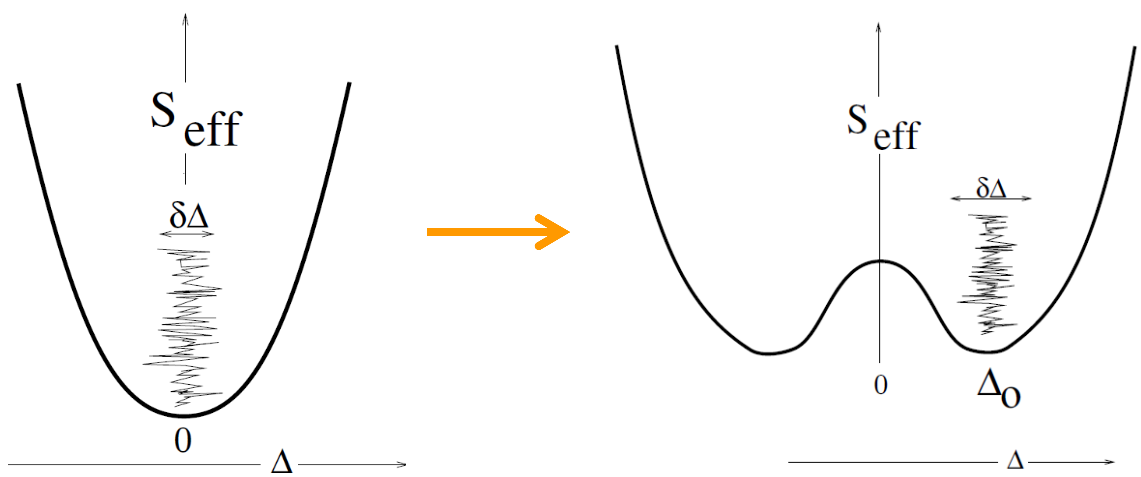
Apparently, we will have trouble if $\Gamma^{-1}(\mathbf{q}, i\nu_n) = 0$. What will then happen? **BCS superfluid phase transition** with symmetry breaking in number conservation! This is simply so-called **Thouless criterion**:

$$\max_q [\Gamma^{-1}(\mathbf{q}, i\nu_n = 0)]_{T=T_c} = 0.$$

How to theoretically describe this **symmetry breaking** in number conservation of fermions? We may introduce an **order parameter** for *condensed* Cooper pairs, i.e.,

$$\Delta(\mathbf{x}) \propto \left\langle \psi_{\downarrow}(\mathbf{x}) \psi_{\uparrow}(\mathbf{x}) \right\rangle \neq 0$$

It sounds ridiculous, right? Yes, it is indeed ridiculous about **60** years ago, when the concept of spontaneous symmetry breaking was just realised!



## BCS theory : standard formulation

Let us consider the Hamiltonian with a contact attractive interaction (as before),

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{+}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U_0 \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} c^{+}_{\mathbf{k}\uparrow} c^{+}_{\mathbf{q}-\mathbf{k}\downarrow} c_{\mathbf{q}-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

For the interaction part, actually, we may focus on a single term with $\mathbf{q} = \mathbf{0}$, i.e.,

$$H_{\text{pair}} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{+}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U_0 \sum_{\mathbf{k}\mathbf{k}'} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

That is, we neglect the pair fluctuations due to nonzero $\mathbf{q}$. **The above *pairing* Hamiltonian is exactly solvable**. In the thermodynamic limit, the solution can be obtained by assuming **a pairing order parameter (real)**:

$$\Delta \equiv U_0 \sum_{\mathbf{k}} \left\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right\rangle = \Delta(\mathbf{q} = 0)$$

and decoupling the interaction Hamiltonian as,

$$U_0 \sum_{\mathbf{k}\mathbf{k}'} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \approx \Delta \sum_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} + \Delta \sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \frac{\Delta^2}{U_0}$$

Note that, the decoupling is exact in the thermodynamic limit.

# BCS theory : standard formulation

The Hamiltonian then becomes,

$$H_{BCS} = \sum_{\mathbf{k}} (c^{+}_{\mathbf{k}\uparrow}, c_{-\mathbf{k}\downarrow}) \begin{bmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{+}_{-\mathbf{k}\downarrow} \end{pmatrix} + \sum_{\mathbf{k}} \xi_{\mathbf{k}} - \frac{\Delta^2}{U_0}$$

Note that, at this point, we are introducing the **Nambu spinor representation**,

$$\Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{+}_{-\mathbf{k}\downarrow} \end{pmatrix}$$

**Homework problem**: Please show the above mean-field Hamiltonian can be diagonalised by making use of the unitary transformation:

$$\begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^{+}_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{bmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{+}_{-\mathbf{k}\downarrow} \end{pmatrix}$$

The resulting eigenvalue (i.e., quasi-particle energy) is given by $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$, and the mean-field Hamiltonian takes the form,

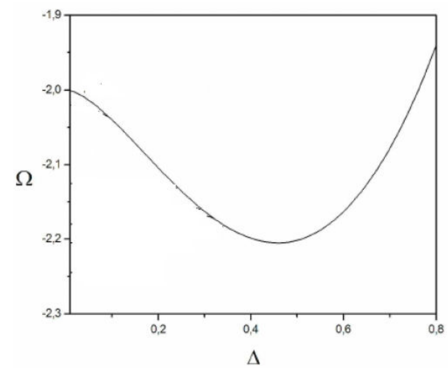
$$H_{BCS} = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^{+}_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}}) - \frac{\Delta^2}{U_0}$$

# BCS theory : standard formulation

Let us consider the case of ***zero*** temperature. What is the thermodynamic potential?

$$\Omega_{BCS} = -\frac{\Delta^2}{U_0} + \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}}\right) = -\frac{m\Delta^2}{4\pi\hbar^2 a} + \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}} + \frac{m\Delta^2}{\hbar^2 \mathbf{k}^2}\right)$$

It is clear that the kinetic energy (second term) increases, but this increase can be compensated by the condensation energy (first term). As a result, the naively picture of the thermodynamic potential is (see right figure),




How to determine the pairing order parameter? It should be the minimum of the thermodynamic potential, so we must have (i.e., the gap equation),

$$0 = -\frac{\partial \Omega_{BCS}}{\partial \Delta^2} = \frac{m}{4\pi\hbar^2 a} + \sum_{\mathbf{k}} \left(\frac{1}{2E_{\mathbf{k}}} - \frac{m}{\hbar^2 \mathbf{k}^2}\right)$$

Note that, we also need to determine the chemical potential by using the number equation:

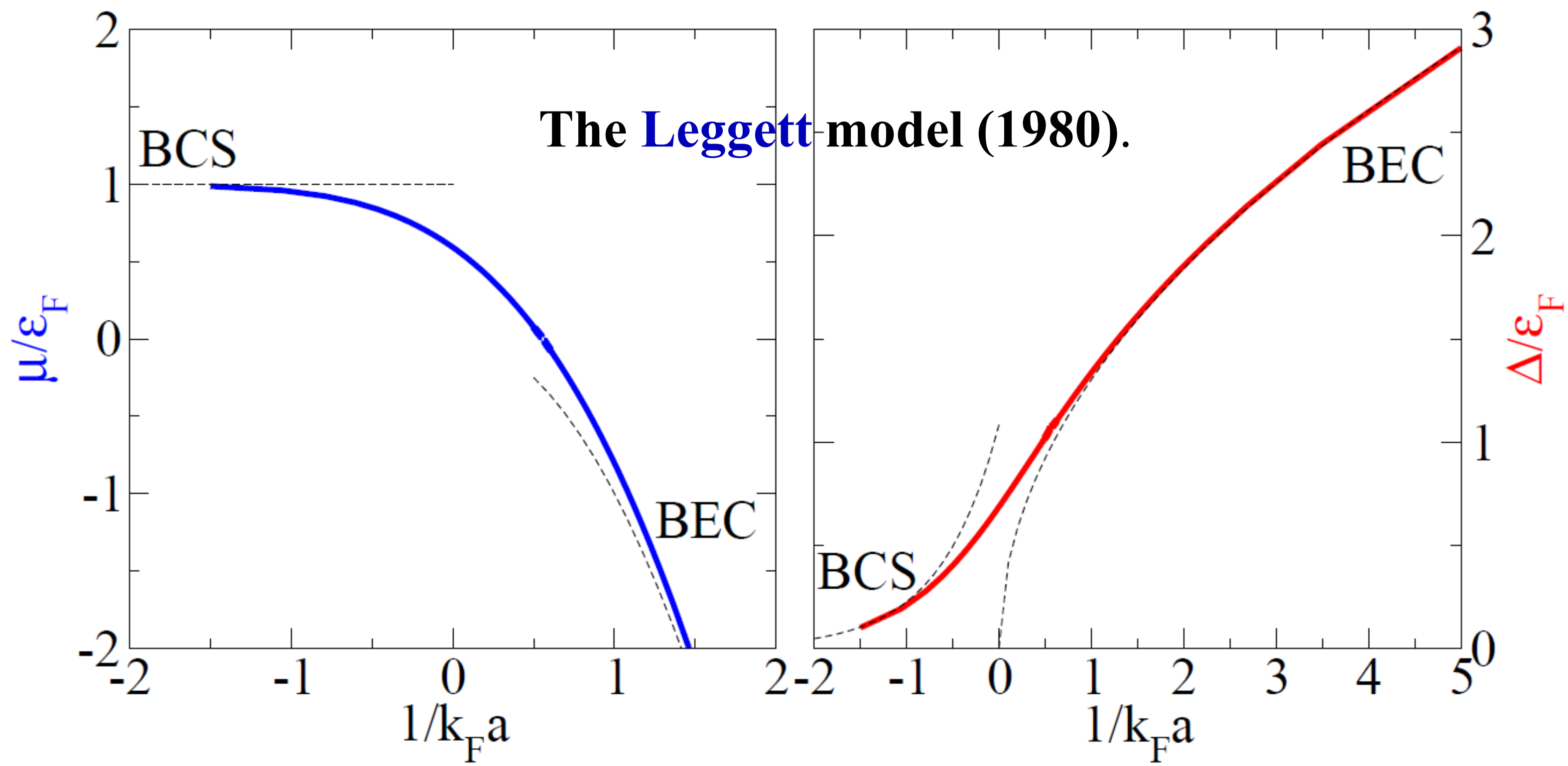
$$n = -\frac{\partial \Omega_{BCS}}{\partial \mu} = \sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

# BCS theory : standard formulation

Great! We can work out the **mean-field results** by solving the coupled gap and number equations:

$$\frac{m}{4\pi\hbar^2 a} + \sum_{\mathbf{k}} \left( \frac{1}{2E_{\mathbf{k}}} - \frac{m}{\hbar^2 \mathbf{k}^2} \right) = 0$$

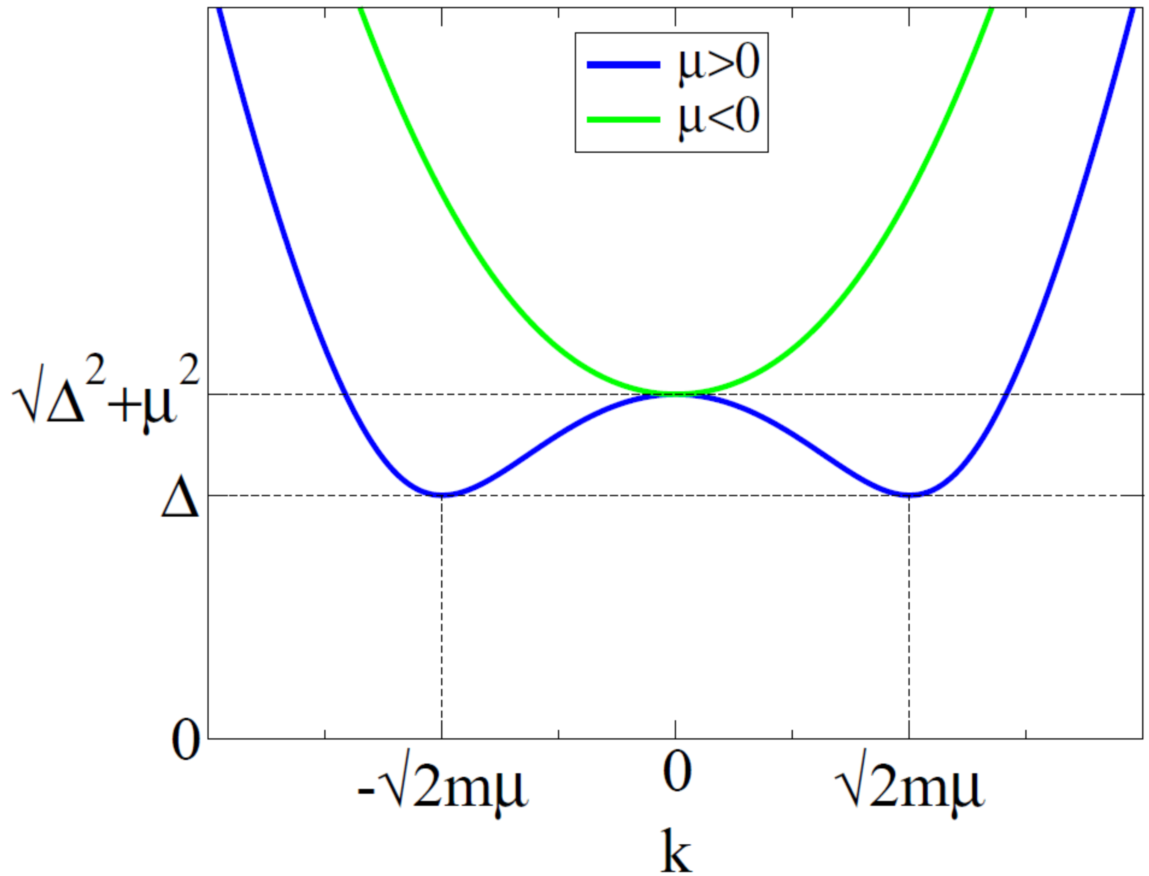
$$n = \sum_{\mathbf{k}} \left( 1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \right)$$




**The Leggett model (1980).**

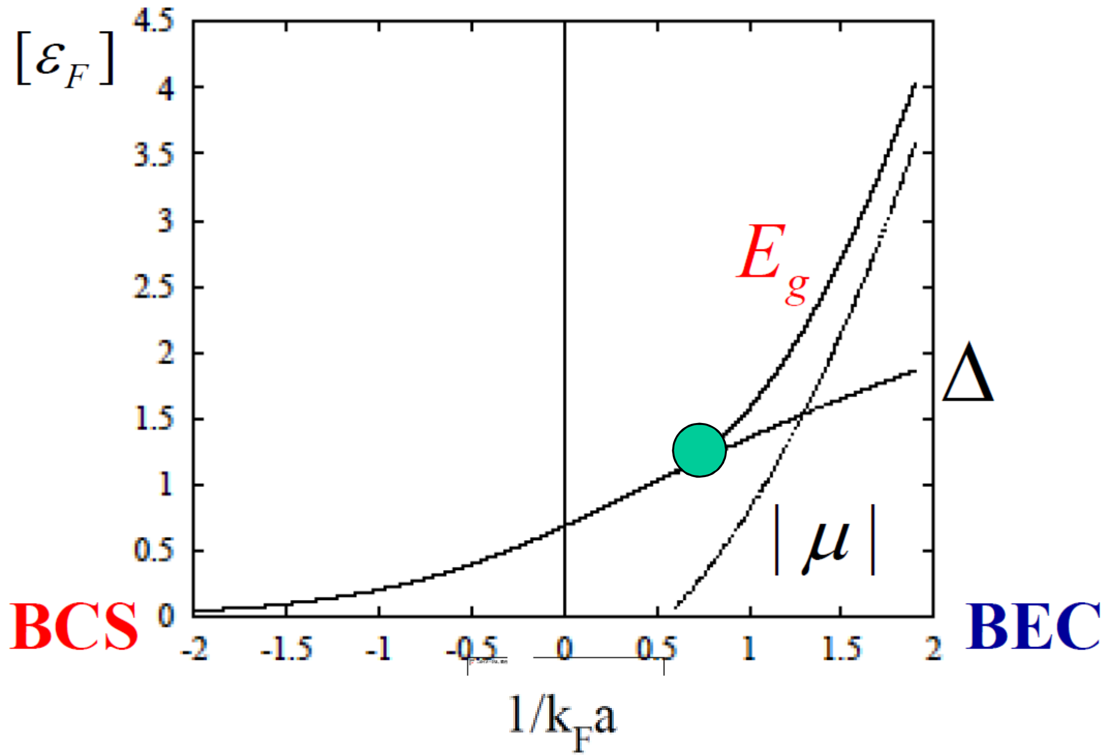
# BCS theory : standard formulation

The quasi-particle energy is given by, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$




Quasi-particle excitation spectrum versus momentum on the BCS ($\mu > 0$) and on the BEC ($\mu < 0$) side of the resonance. The spectrum changes qualitatively from one shape to the other when $\mu = 0$.

# BCS theory : standard formulation

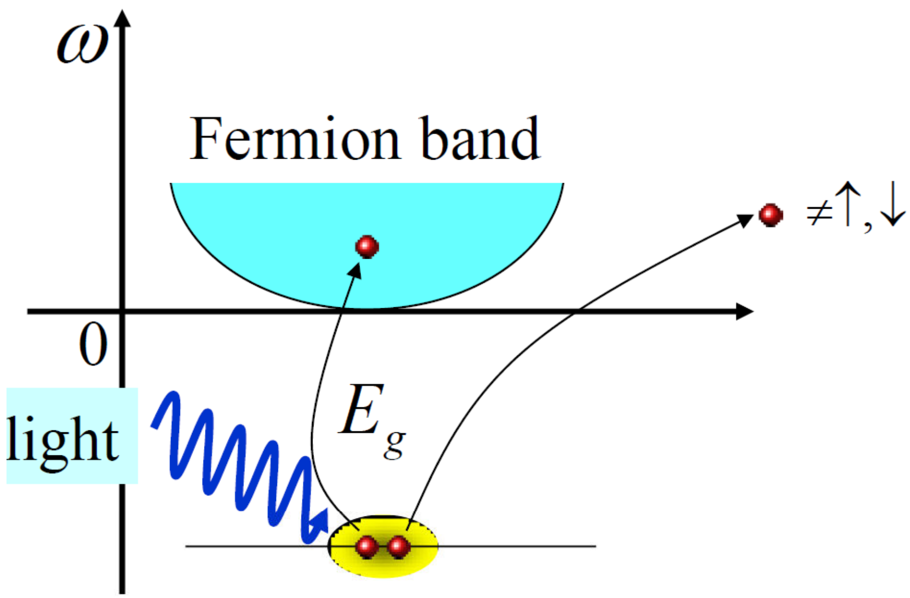



$$\text{Energy gap } E_g = \begin{cases} \Delta & \text{BCS: } \mu>0 \\ \sqrt{\mu^2+\Delta^2} & \text{BEC: } \mu<0 \\ |\mu| & \text{BEC limit } (|\mu|>>\Delta) \end{cases}$$

**Note that**, the **binding energy** of a **Cooper pair** is **$2E_g$**.

# BCS theory : standard formulation

Single-particle excitations can be observed by using the rf-tunneling current spectroscopy.




superfluid $^6Li$

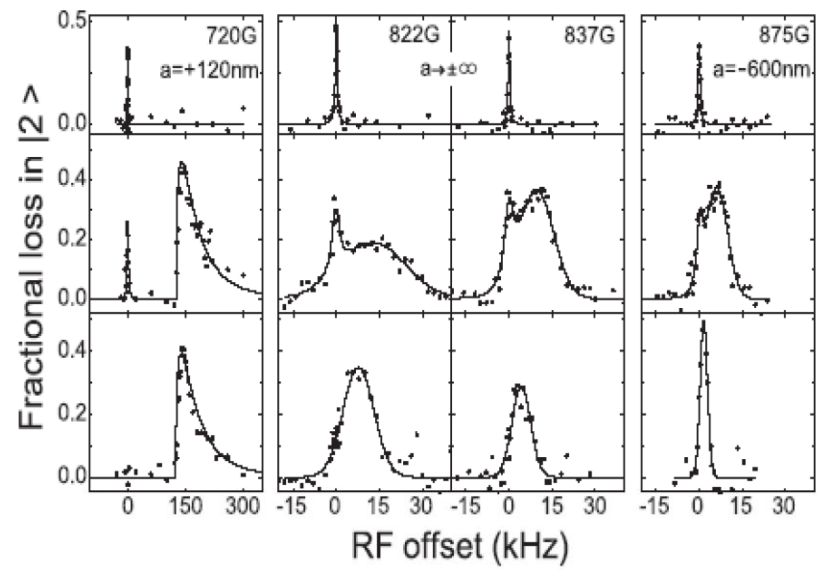



**Fig. 1.** RF spectra for various magnetic fields and different degrees of evaporative cooling. The RF offset ($k_B \times 1\ \mu K \simeq h \times 20.8$ kHz) is given relative to the atomic transition $|2\rangle \rightarrow |3\rangle$. The molecular limit is realized for $B = 720$ G (first column). The resonance regime is studied for $B = 822$ G and $B = 837$ G (second and third columns). The data at 875 G (fourth column) explore the crossover on the BCS side. Top row, signals of unpaired atoms at $T' \approx 6T_F$ ($T_F = 15\ \mu K$); middle row, signals for a mixture of unpaired and paired atoms at $T' = 0.5T_F$ ($T_F = 3.4\ \mu K$); bottom row, signals for paired atoms at $T' < 0.2T_F$ ($T_F = 1.2\ \mu K$). The true temperature $T$ of the atomic Fermi gas is below the temperature $T'$, which we measured in the BEC limit. The solid lines are introduced to guide the eye.

**C. Chin , et al. Science 305 (2004) 1128.**

**Momentum resolved rf-spectroscopy**; JILA, ***Nature*** (2008)

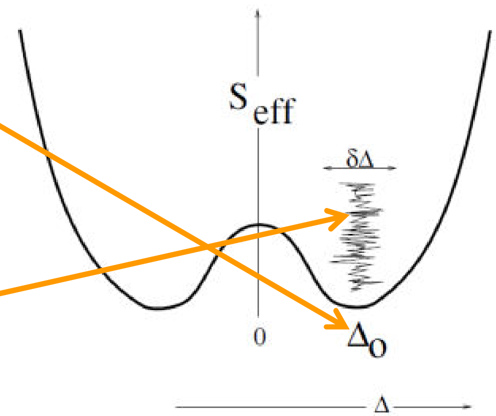
# GPF theory : the Hamiltonian

The mean-field description of the BEC-BCS crossover is **qualitative** only. How to **go beyond** the mean-field approximation? Any idea?

$$H = H_{BCS} + \widetilde{H}_{\text{int}}$$

$$H_{BCS} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{+}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \Delta \sum_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} + \Delta \sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \frac{\Delta^2}{U_0}$$

$$\widetilde{H}_{\text{int}} = U_0 \sum_{\substack{\mathbf{q}\neq 0 \\ \mathbf{kk'}}} c^{+}_{\frac{\mathbf{q}}{2}+\mathbf{k}\uparrow} c^{+}_{\frac{\mathbf{q}}{2}-\mathbf{k}\downarrow} c_{\frac{\mathbf{q}}{2}-\mathbf{k'}\downarrow} c_{\frac{\mathbf{q}}{2}+\mathbf{k'}\uparrow}$$




We may treat the BCS Hamiltonian as the "**free**", "**non-interacting**" Hamiltonian of Bogoliubov quasi-particles and then establish **new Feynman diagrammatic rules** for the **residual interaction Hamiltonian** $\widetilde{H}_{\text{int}}$ !

# GPF theory : "non-interacting" BCS Green function

Recall the BCS Hamiltonian,

$$H_{BCS} = \sum_{\mathbf{k}} (c^+_{\mathbf{k}\uparrow}, c_{-\mathbf{k}\downarrow}) \begin{bmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^+_{-\mathbf{k}\downarrow} \end{pmatrix} + E_0$$

Let us define the **2 x 2** Green function in the **Nambu spinor representation**, $\Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^+_{-\mathbf{k}\downarrow} \end{pmatrix}$

$$G^{(0)}(\mathbf{k},\tau) = -\left\langle T_\tau \Psi_{\mathbf{k}}(\tau) \Psi^+_{\mathbf{k}}(0) \right\rangle_0 = \begin{bmatrix} -\left\langle T_\tau c_{\mathbf{k}\uparrow}(\tau) c^+_{\mathbf{k}\uparrow}(0) \right\rangle_0 & -\left\langle T_\tau c_{\mathbf{k}\uparrow}(\tau) c_{-\mathbf{k}\downarrow}(0) \right\rangle_0 \\ -\left\langle T_\tau c^+_{-\mathbf{k}\downarrow}(\tau) c^+_{\mathbf{k}\uparrow}(0) \right\rangle_0 & -\left\langle T_\tau c^+_{-\mathbf{k}\downarrow}(\tau) c_{-\mathbf{k}\downarrow}(0) \right\rangle_0 \end{bmatrix}$$

From this definition, it is easy to see that, $G^{(0)}_{11}(\mathbf{k}, i\omega_m) = -G^{(0)}_{22}(-\mathbf{k}, -i\omega_m)$

**Problem**: How to obtain the "***non-interacting***" BCS Green function? We may consider the BCS Green function for the quasi-particle field operators:

$$\Lambda_{\mathbf{k}} = \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^+_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{bmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^+_{-\mathbf{k}\downarrow} \end{pmatrix} = \mathbf{A} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^+_{-\mathbf{k}\downarrow} \end{pmatrix} = \mathbf{A}\Psi_{\mathbf{k}}$$

## GPF theory : "non-interacting" BCS Green function

Here, the unitary transformation matrix **A** satisfies,

$$\mathbf{A}\begin{bmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{bmatrix}\mathbf{A}^{-1} = \begin{bmatrix} E_{\mathbf{k}} & 0 \\ 0 & -E_{\mathbf{k}} \end{bmatrix}$$

Of course, we may define the **2 x 2** Green function for the quasi-particle field operators:

$$\Pi^{(0)}(\mathbf{k},\tau) = -\left\langle T_\tau \Lambda_{\mathbf{k}}(\tau)\Lambda_{\mathbf{k}}^{+}(0)\right\rangle_0$$

And its expression with Matsubara frequency is given by,

$$\Pi^{(0)}(\mathbf{k},i\omega_m) = \begin{bmatrix} i\omega_m - E_{\mathbf{k}} & 0 \\ 0 & i\omega_m + E_{\mathbf{k}} \end{bmatrix}^{-1}$$

It is easy to see from the definition that,

$$G^{(0)}(\mathbf{k},i\omega_m) = \mathbf{A}^{-1}\Pi^{(0)}(\mathbf{k},i\omega_m)\mathbf{A} = \left(\mathbf{A}^{-1}\begin{bmatrix} i\omega_m - E_{\mathbf{k}} & 0 \\ 0 & i\omega_m + E_{\mathbf{k}} \end{bmatrix}\mathbf{A}\right)^{-1}$$

$$= \begin{bmatrix} i\omega_m - \xi_{\mathbf{k}} & -\Delta \\ -\Delta & i\omega_m + \xi_{\mathbf{k}} \end{bmatrix}^{-1}$$

# GPF theory : "non-interacting" BCS Green function

In greater detail, the four components of the BCS Green function are given by,

$$G^{(0)}(\mathbf{k}, i\omega_m) = \frac{1}{i\omega_m - \xi_{\mathbf{k}}\sigma_z - \Delta\sigma_x}$$

$$G_{11}^{(0)}(\mathbf{k}, i\omega_m) = \frac{u_{\mathbf{k}}^2}{i\omega_m - E_{\mathbf{k}}} + \frac{v_{\mathbf{k}}^2}{i\omega_m + E_{\mathbf{k}}}$$

$$G_{12}^{(0)}(\mathbf{k}, i\omega_m) = \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{i\omega_m - E_{\mathbf{k}}} - \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{i\omega_m + E_{\mathbf{k}}}$$

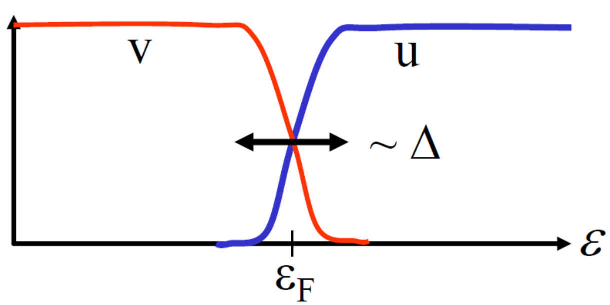
$$G_{21}^{(0)}(\mathbf{k}, i\omega_m) = \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{i\omega_m - E_{\mathbf{k}}} - \frac{u_{\mathbf{k}} v_{\mathbf{k}}}{i\omega_m + E_{\mathbf{k}}}$$

$$G_{22}^{(0)}(\mathbf{k}, i\omega_m) = \frac{v_{\mathbf{k}}^2}{i\omega_m - E_{\mathbf{k}}} + \frac{u_{\mathbf{k}}^2}{i\omega_m + E_{\mathbf{k}}}$$

where the **quasi-particle wave-functions** $u(\mathbf{k})$ and $v(\mathbf{k})$ satisfy,

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

$$v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) = n_{\mathbf{k}\sigma}$$



## GPF theory : pair fluctuations

Now, we turn to consider the interaction Hamiltonian,

$$\tilde{H}_{\rm int} = U_0 \sum_{\substack{\mathbf{q}\neq 0 \\ \mathbf{kk'}}} c^{+}_{\frac{\mathbf{q}}{2}+\mathbf{k}\uparrow} c^{+}_{\frac{\mathbf{q}}{2}-\mathbf{k}\downarrow} c_{\frac{\mathbf{q}}{2}-\mathbf{k'}\downarrow} c_{\frac{\mathbf{q}}{2}+\mathbf{k'}\uparrow} = U_0 \sum_{\substack{\mathbf{q}\neq 0 \\ \mathbf{kk'}}} \left\{ \Psi^{+}_{\mathbf{k}+\frac{\mathbf{q}}{2}} \sigma_{+} \Psi_{\mathbf{k}-\frac{\mathbf{q}}{2}} \right\} \left\{ \Psi^{+}_{\mathbf{k'}-\frac{\mathbf{q}}{2}} \sigma_{-} \Psi_{\mathbf{k'}+\frac{\mathbf{q}}{2}} \right\}$$

**Question**: In applying Wick theorem, can we connect **$\Psi$ with $\Psi$** or **$\Psi^{+}$ with $\Psi^{+}$** ?

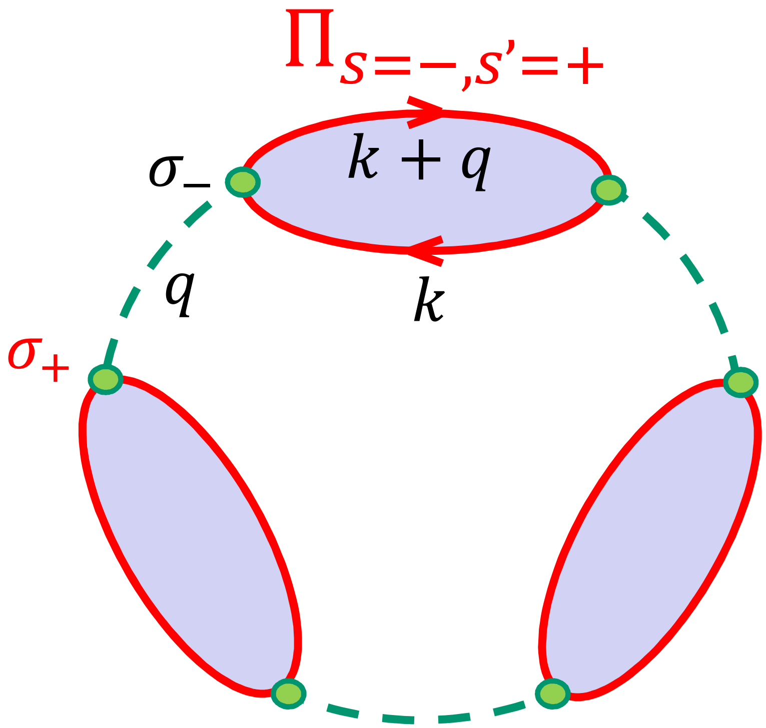
**Answer**: No, you can't. This is the convenience of the use of **Nambu spinor representation**!

You may wish to establish the Feynman rules by yourself. The rules are actually very similar to what have learned before, but with special care on the Pauli matrices.

**Now, let us consider the thermodynamic potential…**

# GPF theory : pair fluctuations

**Here, let us consider the thermodynamic potential…**




For the $n$-th order ring (bubble) diagram, the result is,

$$\frac{V}{2}\sum_q \frac{1}{n}(-1)^{n+1}U_0^n \sum_{s_1,s_1'=s_2,\ldots,s_n,s_n'=s_1}[\Pi(q)]_{s_1 s_1'}\cdots[\Pi(q)]_{s_n s_n'} = \frac{V}{2}\sum_q \frac{1}{n}(-1)^{n+1}U_0^n \mathrm{Tr}\begin{bmatrix}\Pi(q)^{-+} & \Pi(q)^{--}\\ \Pi(q)^{++} & \Pi(q)^{+-}\end{bmatrix}^n$$

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# GPF theory : pair fluctuations

After taking the summation over "*n*", the fluctuation contribution to the thermodynamic potential is given by,

$$\Omega_{GPF} = \frac{V}{2}\sum_q \mathrm{Tr}\ln\left(1 + U_0 \begin{bmatrix} \Pi(q)^{-+} & \Pi(q)^{--} \\ \Pi(q)^{++} & \Pi(q)^{+-} \end{bmatrix}\right) = \frac{V}{2}\sum_q \ln\det\left(\frac{1}{U_0} + \begin{bmatrix} \Pi(q)^{-+} & \Pi(q)^{--} \\ \Pi(q)^{++} & \Pi(q)^{+-} \end{bmatrix}\right)$$

Here, the matrix elements (within ladder diagrams) are ($ij$ = +, –),

$$\Pi(q)^{ij} = \sum_k \mathrm{Tr}\left[\sigma_i G^{(0)}(k + \frac{q}{2})\sigma_j G^{(0)}(k - \frac{q}{2})\right]$$

These elements can be calculated by inserting the BCS Green function and taking the Matsubara frequency summation. We have, for example, at ***zero*** temperature:

$$\Pi(\mathbf{q}, i\nu_n)^{-+} = \sum_{\mathbf{k}}\left[\frac{u_+^2 u_-^2}{i\nu_n - E_+ - E_-} - \frac{v_+^2 v_-^2}{i\nu_n - E_+ - E_-}\right] = \left[\Pi(\mathbf{q}, i\nu_n)^{+-}\right]^*$$

$$\Pi(\mathbf{q}, i\nu_n)^{--} = \sum_{\mathbf{k}}\left[\frac{u_+ v_+ u_- v_-}{i\nu_n - E_+ - E_-} - \frac{u_+ v_+ u_- v_-}{i\nu_n - E_+ - E_-}\right] = \left[\Pi(\mathbf{q}, i\nu_n)^{++}\right]^*$$

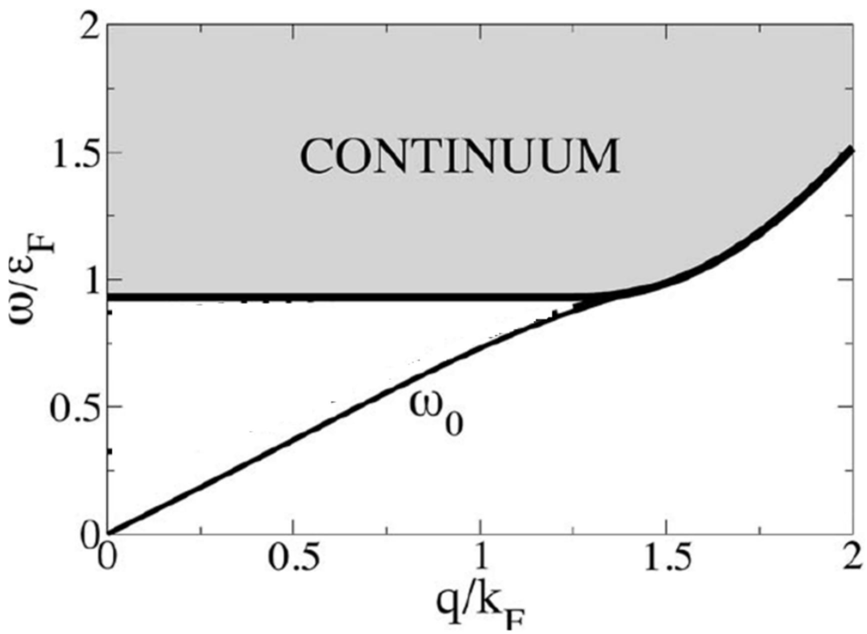
where $E_+ = E_{\frac{\mathbf{q}}{2}+\mathbf{k}}$ and $E_- = E_{\frac{\mathbf{q}}{2}-\mathbf{k}}$

# GPF theory : pair fluctuations

Of course, we may define a vertex function,

$$\Gamma(q) = -\left(\frac{1}{U_0} + \begin{bmatrix} \Pi(q)^{-+} & \Pi(q)^{--} \\ \Pi(q)^{++} & \Pi(q)^{+-} \end{bmatrix}\right)^{-1}$$

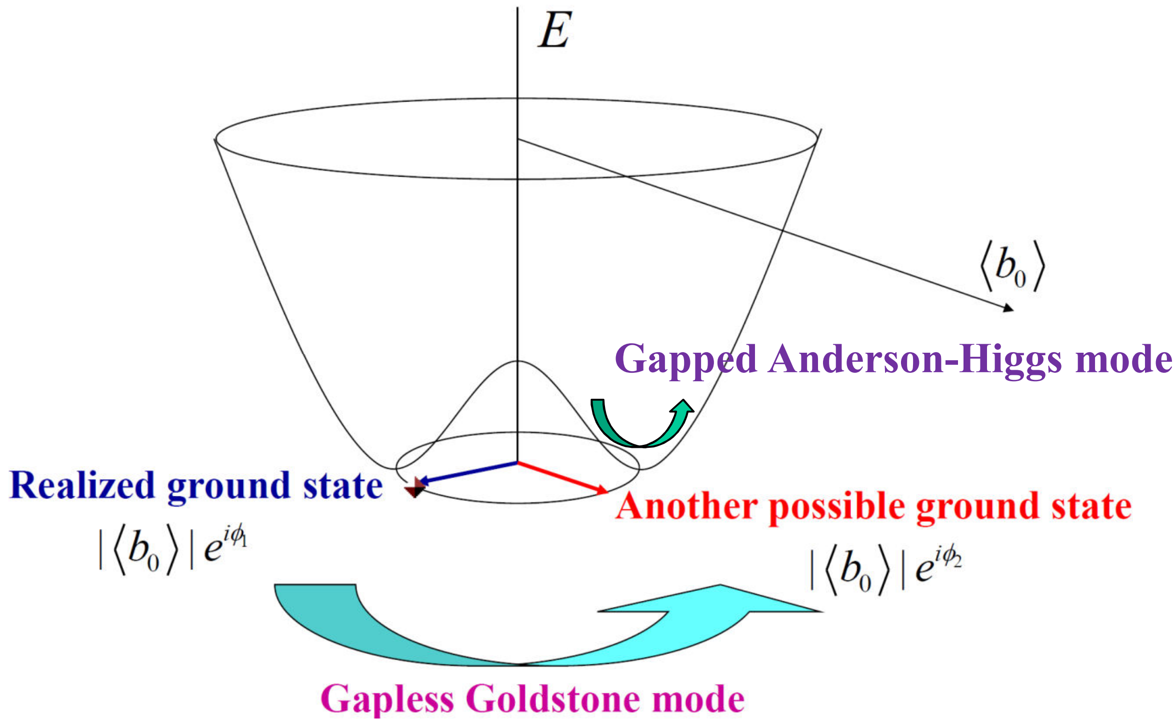
which is basically the Green function of Cooper pairs in the condensed phase.
**What is the spectral function of this Green function**? i.e., $-(1/\pi)\mathrm{Im}\Gamma(q)$ ? It looks like:



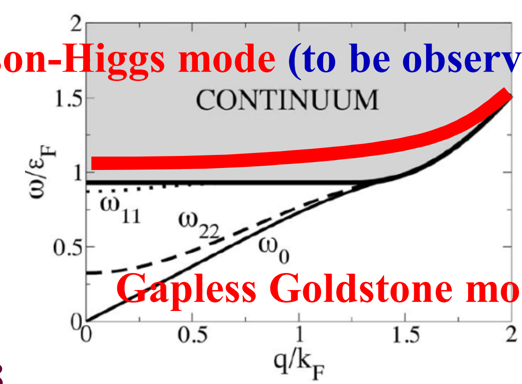

**Why there is a gapless "phonon" mode?**

# GPF theory : pair fluctuations

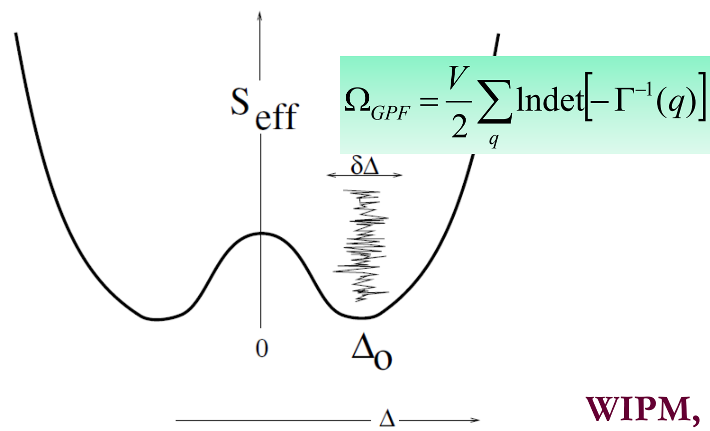
Actually, we have the following picture:

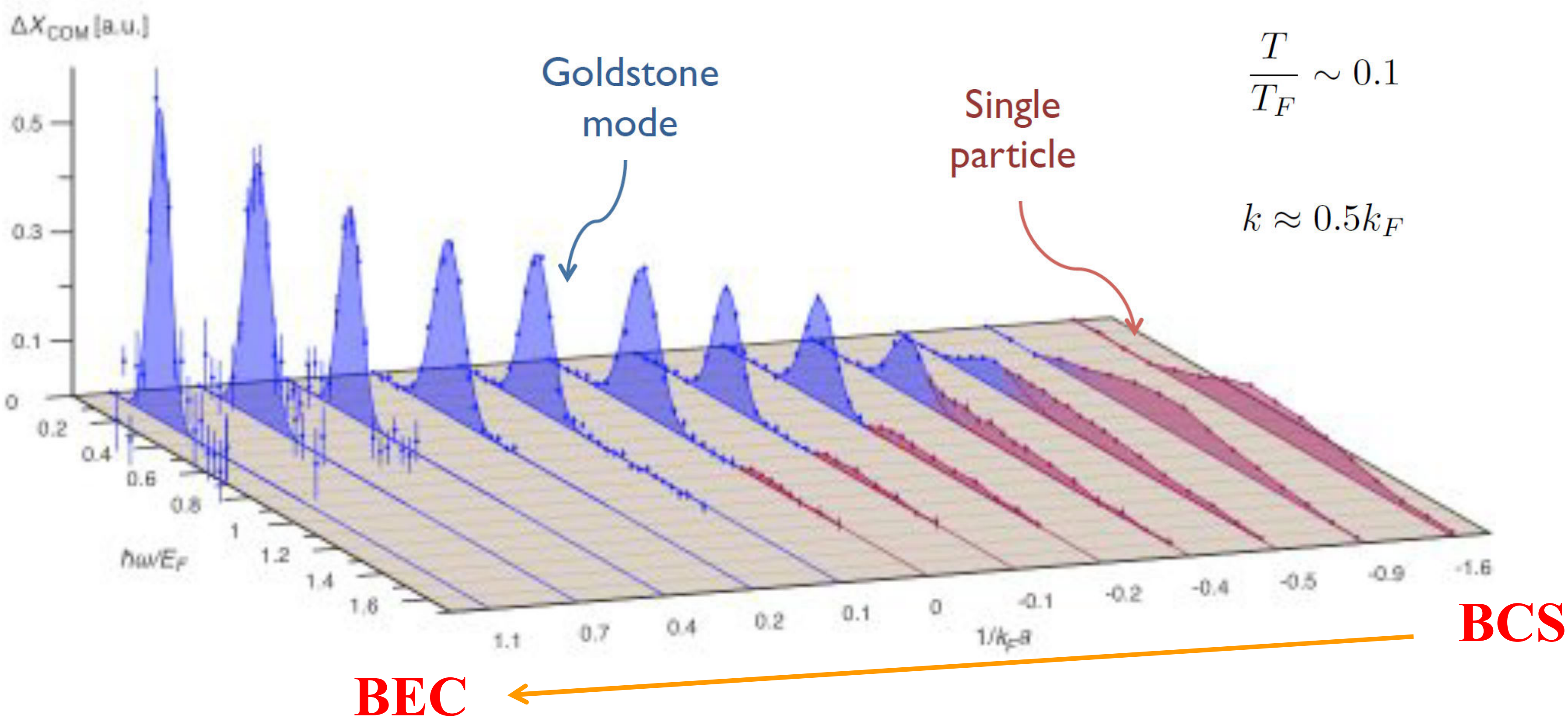



Gapped Anderson-Higgs mode (to be observed!)




Gapless Goldstone mode






**The gapless Goldstone mode of Fermi superfluids can be experimentally detected by Bragg spectroscopy.**

# GPF theory : pair fluctuations

EUROPHYSICS LETTERS 15 May 2006

*Europhys. Lett.*, **74** (4), pp. 574–580 (2006)
DOI: 10.1209/epl/i2006-10023-y

## Equation of state of a superfluid Fermi gas in the BCS-BEC crossover

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received 30 January 2006; accepted in final form 22 March 2006
published online 12 April 2006

PACS. 03.75.Hh – Static properties of condensates; thermodynamical, statistical, and structural properties.
PACS. 03.75.Ss – Degenerate Fermi gases.
PACS. 05.30.Fk – Fermion systems and electron gas.

**Abstract.** – We present a theory for a superfluid Fermi gas near the BCS-BEC crossover, including pairing fluctuation contributions to the free energy similar to that considered by Nozières and Schmitt-Rink for the normal phase. In the strong coupling limit, our theory is able to recover the Bogoliubov theory of a weakly interacting Bose gas with a molecular scattering length very close to the known exact result. We compare our results with recent Quantum Monte Carlo simulations both for the ground state and at finite temperature. Excellent agreement is found for all interaction strengths where simulation results are available.

# GPF theory : Functional path-integral reformulation

PHYSICAL REVIEW A **77**, 023626 (2008)

## Quantum fluctuations in the superfluid state of the BCS-BEC crossover

Roberto B. Diener, Rajdeep Sensarma, and Mohit Randeria
*Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA*
(Received 17 September 2007; published 29 February 2008)

We determine the effects of quantum fluctuations about the $T=0$ mean-field solution of the BCS-BEC crossover in a dilute Fermi gas using the functional integral method. These fluctuations are described in terms of the zero-point motion of collective modes and the virtual scattering of gapped quasiparticles. We calculate their effects on various measurable properties, including chemical potential, ground-state energy, the gap, the speed of sound and the Landau critical velocity. At unitarity, we find excellent agreement with quantum Monte Carlo and experimental results. In the BCS limit, we show analytically that we obtain Fermi liquid interaction corrections to thermodynamics including the Hartree shift. In the Bose-Einstein condensation (BEC) limit, we show that the theory leads to an approximate description of the reduction of the scattering length for bosonic molecules and also obtain quantum depletion of the Lee-Yang form. At the end of the paper, we describe a method to include feedback of quantum fluctuations into the gap equation, and discuss the problems of self-consistent calculations in satisfying Goldstone's theorem and obtaining ultraviolet finite results at unitarity.

DOI: 10.1103/PhysRevA.77.023626 PACS number(s): 03.75.Ss, 05.30.Fk

# *EoS*: comparison to QMC ($T$=0)

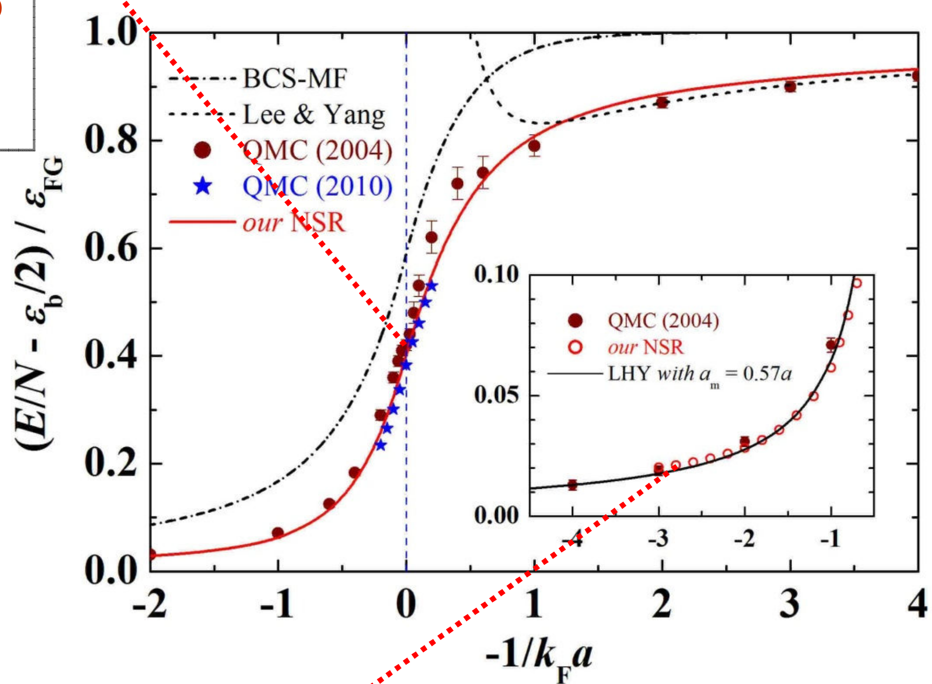
Unitarity limit:

$$\mathcal{E} = \frac{\hbar^2 k_F^5}{10\pi^2 m}\left(\xi - \frac{\zeta}{k_F a} + \ldots\right)$$

QMC (2004): G. E. Astrakharchik *et al.*, *PRL* **93**, 200404 (2004).
QMC (2011): S. Gandolfi *et al.*, *Phys. Rev. A* **83**, 041601(R) (2011)
*Our* GPF: Hu, Liu & Drummond, *Europhys. Lett.* **74**, 574 (2006).

**QMC (2004) ξ = 0.42(.01)**
**QMC (2011) ξ = 0.383 (.001)**
*Our* **GPF ξ= 0.401**

**MIT Expt (2012) ξ = 0.376**



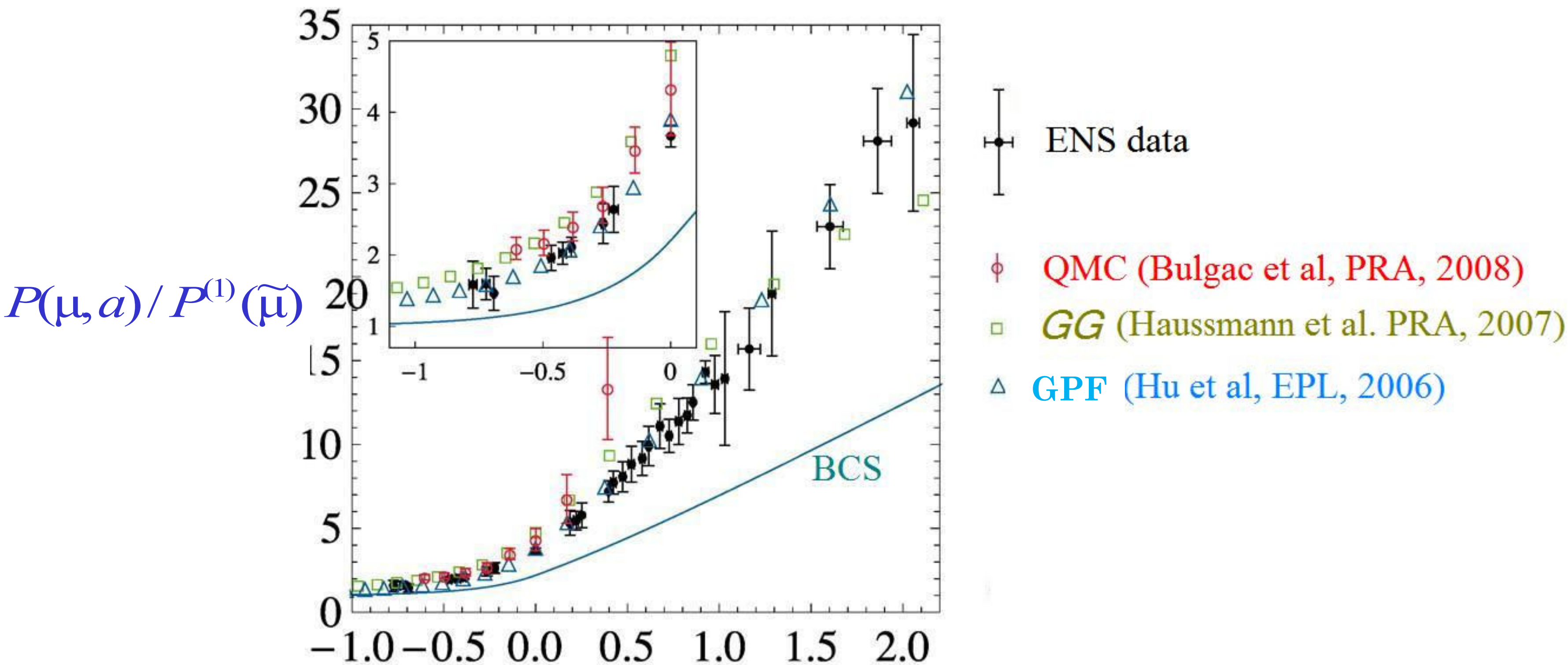

Lee, Huang & Yang (BEC):

$$\mathcal{E} = -\frac{\hbar^2 n}{2ma^2} + \frac{\pi\hbar^2 n^2 a_{dd}}{4m}\left(1 + \frac{128}{15}\sqrt{n a_{dd}^3/2\pi} + \ldots\right)$$

$= \varepsilon_b/2$

Exact 4-body calculation: $a_{dd}\approx 0.60a$;
*Our* GPF predicts: $a_{dd}\approx 0.57a$!

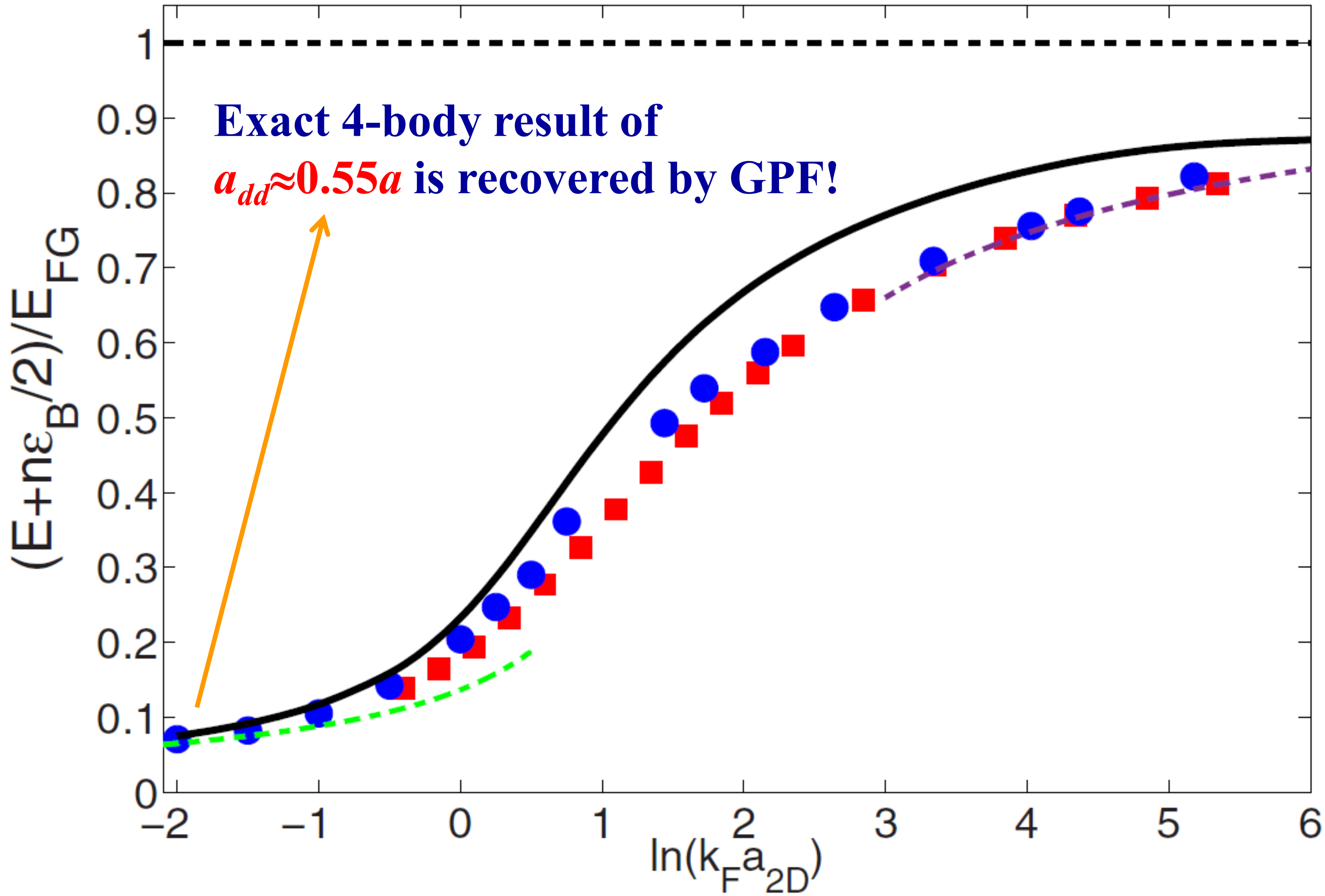
# *EoS*: comparison to the ENS measurement (*T*=0)




$\tilde{\delta} \equiv \hbar/\sqrt{2m\tilde{\mu}a}$ where $\tilde{\mu} = \mu - \varepsilon_b/2$

ENS *low temperature EoS*: **Navon *et al.*, *Science*, 5 May 2010.**

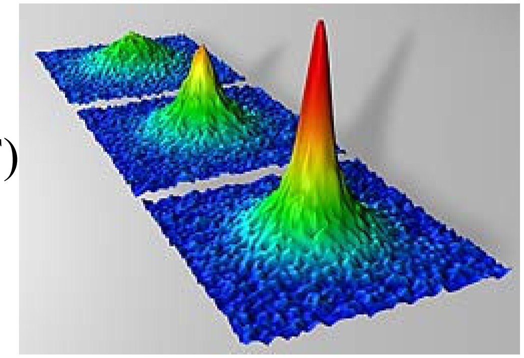
# Reliability of GPF in 2D (T=0)




Lianyi He *et al.*, PRA 2015, compared with QMC (PRA 2015).

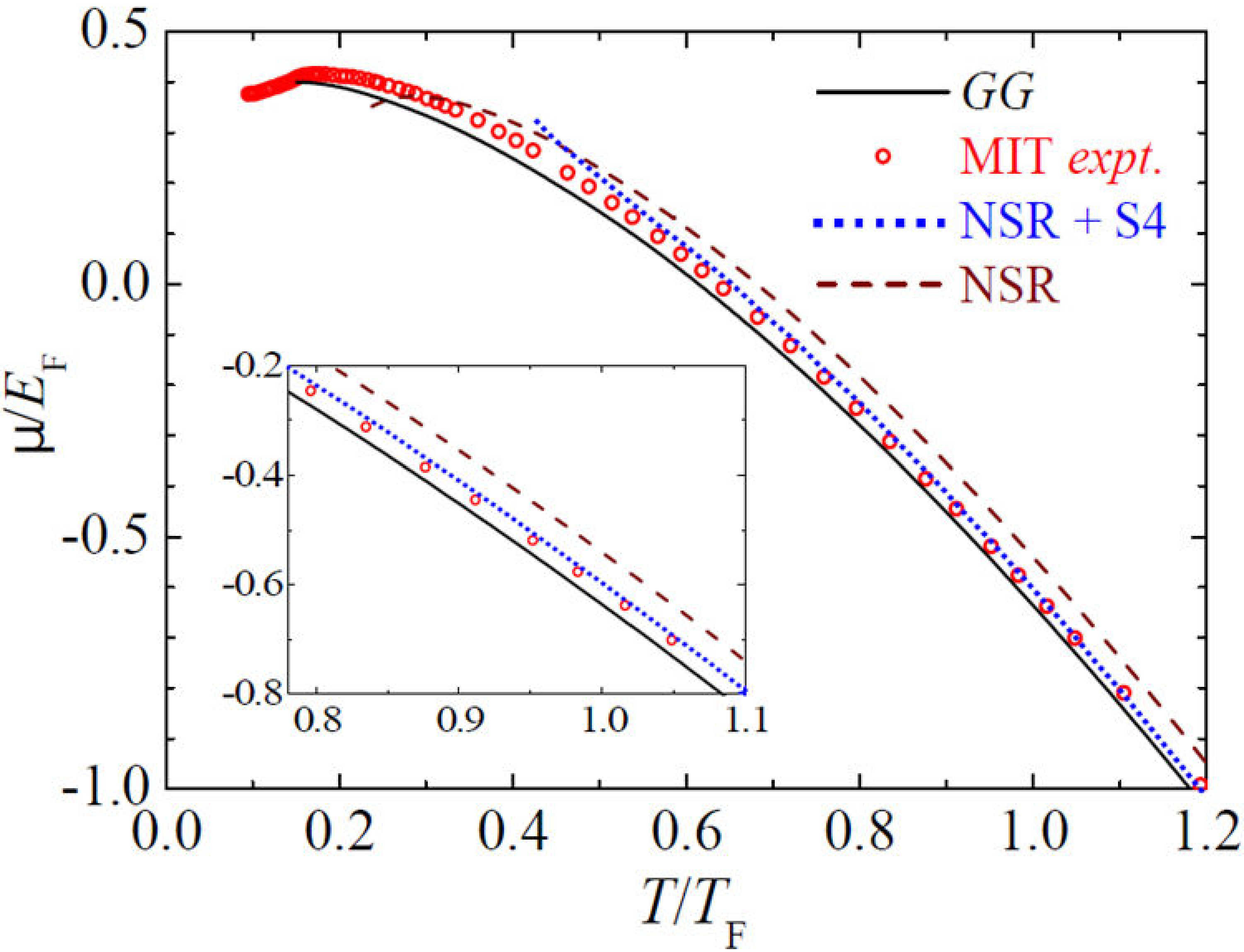
# Application 4:
# BEC-BCS crossover (beyond-GPF)

How can we go beyond the Gaussian pair fluctuation (GPF) theory? It is extremely difficult. But ….



**We may have some ideas, inspired by the ε-expansion theory (Nishida & Son, 2006)**

## Beyond NSR or GPF: still a long way to go…



## Functional path-integral approach

To begin with we have the thermodynamic potential found through the partition function

$$\Omega = -\beta^{-1} \ln \mathcal{Z},$$

where the partition function is given by,

$$\mathcal{Z} = \int \mathcal{D}\left[\psi, \bar{\psi}\right] e^{-S[\psi, \bar{\psi}]},$$

and the action defined by a Hamiltonian is

$$S = \int_0^{\hbar\beta} d\tau \left[ \int d\mathbf{r} \sum_\sigma \bar{\psi}_\sigma(x) \partial_\tau \psi_\sigma(x) + H \right],$$

Decouple the Fermi fields and write the action as Bose fields to Hubbard-Stratonovich

$$\mathcal{S}_{\text{eff}}\left[\Delta, \Delta^*\right] = \int dx \left[ \frac{|\Delta(x)|^2}{U_0} - \operatorname{Tr}\ln\left[-G^{-1}\right] \right].$$

This is true for general dimension, where $\int dx = \int d^d\mathbf{r} d\tau$ and $U_0$ is regularised appropriately.

# Path-integral formalism

In order to solve the thermodynamic potential we expand the Bose field $\Delta(\mathbf{r},t)$ about its saddle point $\Delta_0$,

$$\Delta(\mathbf{r},t) = \Delta_0 + \varphi(\mathbf{r},t),$$

The action becomes

$$\mathcal{S}_{\text{eff}}\left[\Delta,\Delta^*\right] = \mathcal{S}_{\text{MF}}^{(0)} + \mathcal{S}_{\text{GF}}^{(2)} + \mathcal{S}^{(3)} + \mathcal{S}^{(4)} + \ldots,$$

where $\mathcal{S}_{\text{MF}}^{(0)}$ is the mean field contribution and $\mathcal{S}_{\text{GF}}^{(2)}$ is the guassian contribution. Most perturbation theories terminate the expansion here. The thermodynamic potential is then

$$\Omega = \Omega_{\text{MF}} + \Omega_{\text{GF}}$$

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C. A. R. Sa de Melo, M. Randeria, & J. R. Engelbrecht, PRL (1993).
H. Hu, X.-J. Liu, & P. D. Drummond, Europhys. Lett. (2006).
R. B. Diener, R. Sensarma, & M. Randeria, PRA (2008).
L. He *et al.*, PRA (2015).

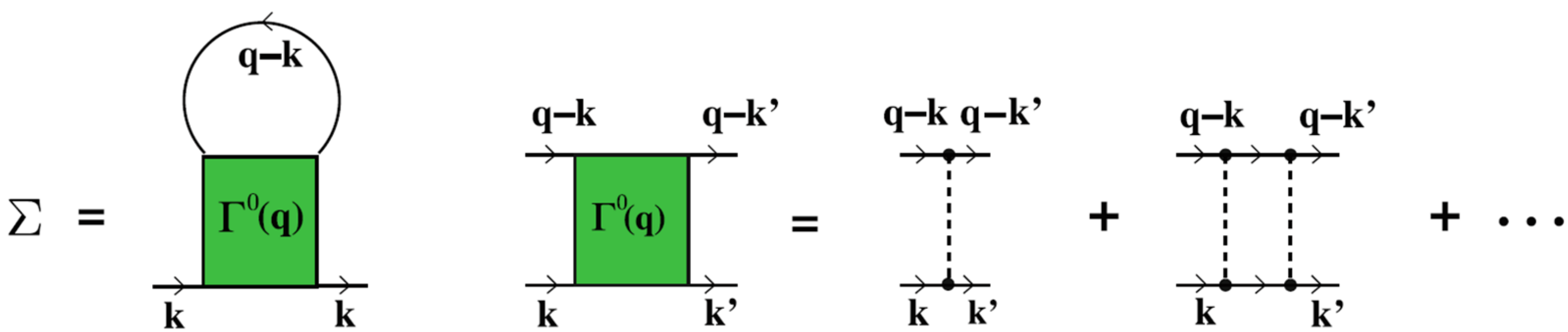
# Path-integral formalism and NSR

For $T > T_c$ the gaussian, $\mathcal{S}_{\mathrm{GF}}^{(2)}$ contribution to the thermodynamic potential can be written as the standard NSR theory

$$\mathcal{S}_{\mathrm{GF}}^{(2)} = \frac{1}{2}\sum_{q} \left(\varphi_q^* \;\; \varphi_{-q}\right) \left[-\Gamma^{-1}(q)\right] \begin{pmatrix} \varphi_q \\ \varphi_{-q} \end{pmatrix},$$

and $\Omega_{\mathrm{GF}}^{(2)} = \sum_q \ln\left(-\Gamma^{-1}(q)\right)$. Here we define the vertex function

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = \frac{1}{U_0} + k_{\mathrm{B}}T \sum_{\mathbf{k},\omega_m} G_0\left(\frac{\mathbf{q}}{2} - \mathbf{k}, i\nu_n - i\omega_m\right) G_0\left(\frac{\mathbf{q}}{2} + \mathbf{k}, i\omega_m\right)$$



# Path-integral formalism

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The action becomes

$$\mathcal{S}_{\text{eff}}\left[\Delta, \Delta^*\right] = \mathcal{S}_{\text{MF}}^{(0)} + \mathcal{S}_{\text{GF}}^{(2)} + \mathcal{S}^{(3)} + \mathcal{S}^{(4)} + \ldots,$$

where $\mathcal{S}_{\text{MF}}^{(0)}$ is the mean field contribution and $\mathcal{S}_{\text{GF}}^{(2)}$ is the guassian contribution. Most perturbation theories terminate the expansion here. The thermodynamic potential is then

$$\Omega = \Omega_{\text{MF}} + \Omega_{\text{GF}}$$

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C. A. R. Sa de Melo, M. Randeria, & J. R. Engelbrecht, PRL (1993).
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R. B. Diener, R. Sensarma, & M. Randeria, PRA (2008).
L. He *et al.*, PRA (2015).

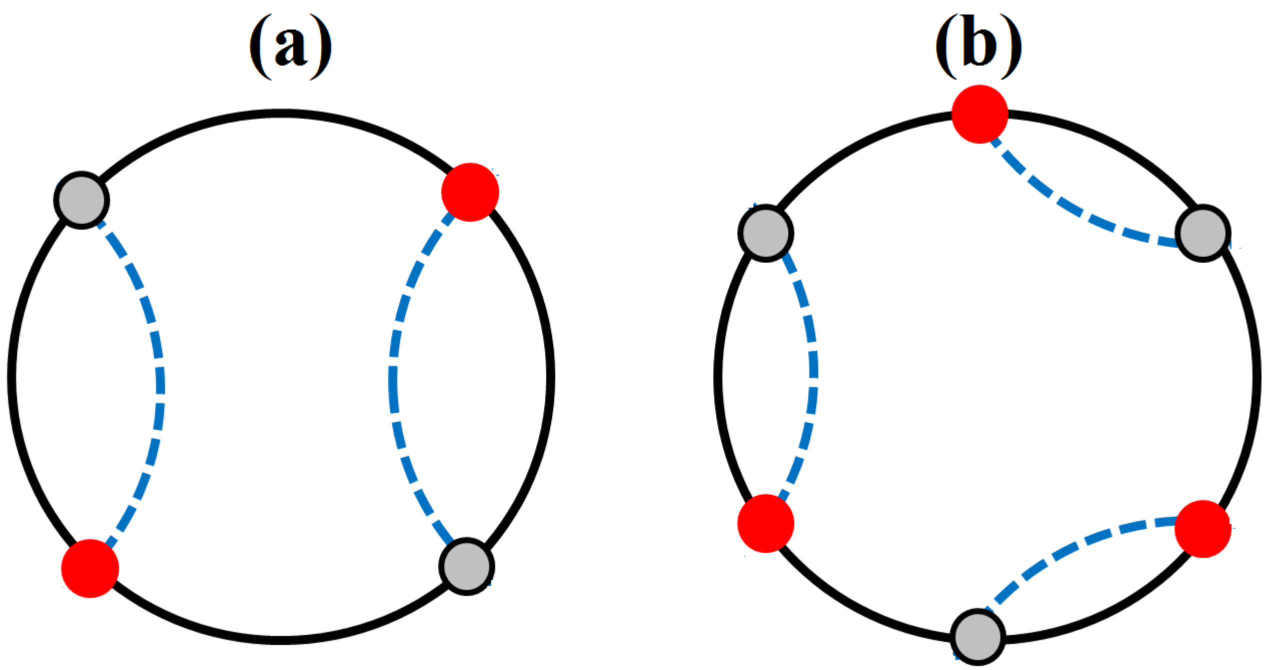
# New expansion in terms of the vertex function

Denote some higher order terms

$$\hat{V} = \mathcal{S}^{(3)} + \mathcal{S}^{(4)} + \ldots,$$

we have for the partition function

$$\mathcal{Z} = e^{-S_{\rm MF}^{(0)}} \int \mathcal{D}[\Delta, \Delta^*]\, e^{-S_{\rm GF}^{(2)} + \hat{V}} = e^{-S_{\rm MF}^{(0)}} \int \mathcal{D}[\Delta, \Delta^*]\, e^{-S_{\rm GF}^{(2)}} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left\langle \hat{V}_1 \hat{V}_2 \ldots \hat{V}_n \right\rangle,$$



# Epsilon expansion

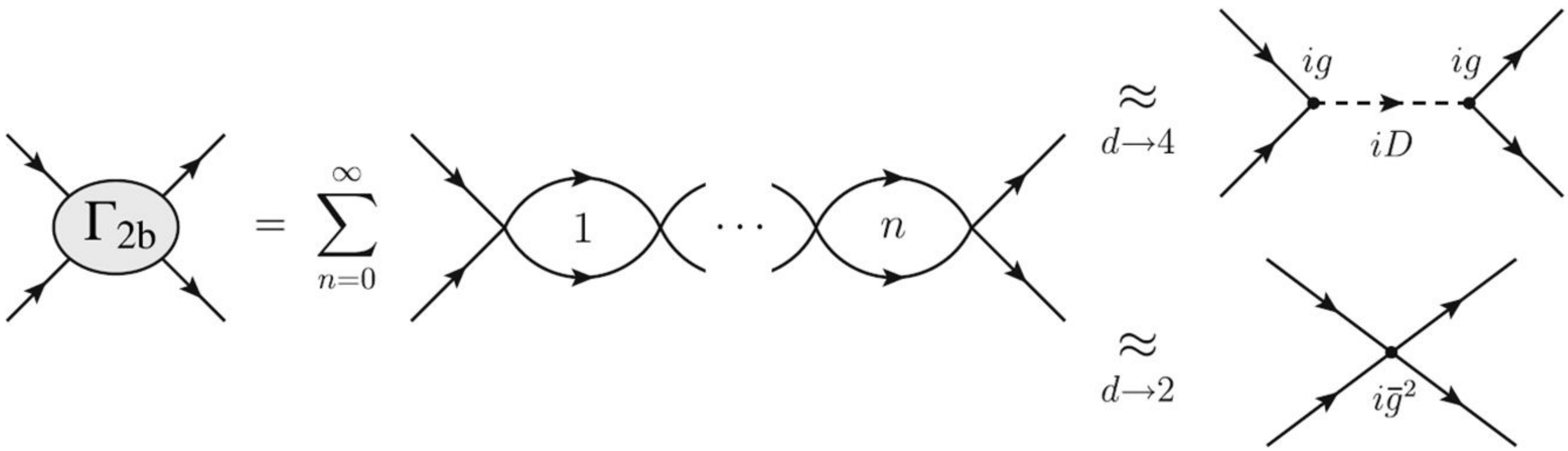
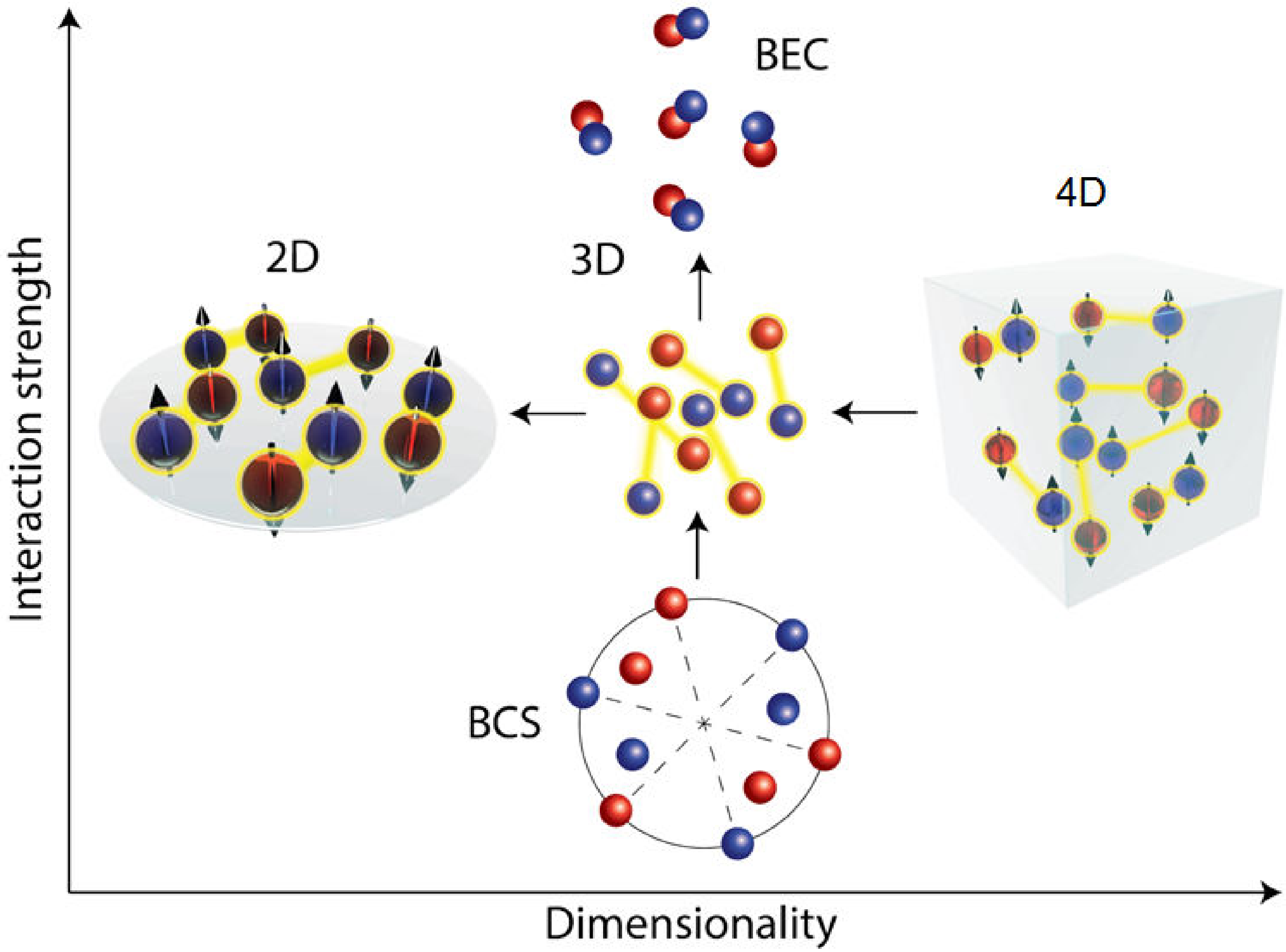



Figure : Two-body T-matrix near four and two dimension.

Y. Nishida and D. T. Son, Phys. Rev. Lett. **97**, 050403 (2006).

# Epsilon expansion




BEC-BCS crossover driven by dimensionality

# Epsilon expansion: the key point

We can write $\Gamma^{-1}(\mathbf{q}, i\nu_n) = \Gamma^{-1}_{2b}(\mathbf{q}, i\nu_n) + \Gamma^{-1}_{mb}(\mathbf{q}, i\nu_n)$ and taking $d = 4 - \epsilon$

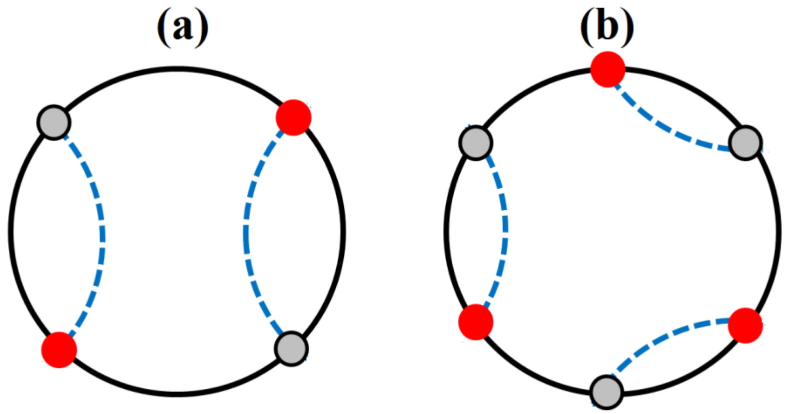
$$\Gamma^{-1}_{2b}(\mathbf{q}, i\nu_n) = \left(\frac{m}{4\pi\hbar^2}\right)^{2-\epsilon/2} \Gamma\left(-1 + \frac{\epsilon}{2}\right) \left(-i\nu_n + \varepsilon_{\mathbf{q}}/2 - 2\mu\right)^{1-\epsilon/2}$$
$$\simeq \left(\frac{m}{4\pi\hbar^2}\right)^{2} \frac{2}{\epsilon} \left(i\nu_n - \varepsilon_{\mathbf{q}}/2 + 2\mu\right),$$

In 4 dimensions $\Gamma^{-1}_{2b}(\mathbf{q}, i\nu_n)$ dominates as there is a pole.

$$\Gamma_{2b}(\mathbf{q}, i\nu_n) = \frac{8\pi^2\hbar^4}{m^2}\epsilon \left[\frac{1}{i\nu_n - \varepsilon_{\mathbf{q}}/2 + 2\mu}\right] + \mathcal{O}\left(\epsilon^2\right) = g^2 D(\mathbf{q}, i\nu_n) + \mathcal{O}\left(\epsilon^2\right),$$

where $D(\mathbf{q}, i\nu_n)$ is a bosonic propagator of a mass of $2M$. Near four (or two) dimensions the vertex function, $\Gamma(q)$ is a small quantity of order $\epsilon$.

# How to use the epsilon expansion idea?




**What can we do, if we use ε = 4 – *d* as an artificial small parameter?**

(i) select some beyond GPF diagrams, such as (a) and (b);
(ii) determine the order of the diagram, in power of ε;
(iii) keep all the low-order diagrams, with order ≤ *n* ;
(iv) calculate these diagrams in **three** dimensions;
(v) may check the accuracy by increasing *n*.

# Epsilon expansion and GPF/NSR

The thermodynamic potential $\Omega$ expanded in $NSR$ theory

$$\Omega = -\frac{2}{\beta}\sum_{\mathbf{k}} \ln\left(1+e^{-\beta\xi_{\mathbf{k}}}\right) + \frac{1}{\beta}\sum_{\mathbf{q},i\nu_n} \ln\left(-\Gamma^{-1}(\mathbf{q},i\nu_n)\right)$$

where $\Gamma^{-1}(\mathbf{q},i\nu_n) = \Gamma_{2b}^{-1}(\mathbf{q},i\nu_n) + \Gamma_{mb}^{-1}(\mathbf{q},i\nu_n)$ and we can Taylor expand the logarithm, giving

$$\frac{1}{\beta}\sum_{\mathbf{q}i\nu_n} \ln\left(1+\Gamma_{2b}(\mathbf{q},i\nu_n)/\Gamma_{mb}(\mathbf{q},i\nu_n)\right) = \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{\mathbf{q},i\nu_n}\left[\Gamma_{2b}(\mathbf{q},i\nu_n)\Gamma_{mb}^{-1}(\mathbf{q},i\nu_n)\right]^n$$

$$\Omega = \Omega^{(0)} + \Omega^{(1)} + \mathcal{O}(\epsilon^2),$$

and solve the corresponding number equations $n = -\partial\Omega/\partial\mu = n^{(0)} + n^{(1)} + \mathcal{O}(\epsilon^2)$.

Epsilon expansion can be understood in the framework of GPF/NSR.

# Appendix A

## Large orders in the $\epsilon = 4 - d$ expansion

In this Appendix, we show that there exists a type of diagrams which grows as $n!$ by itself at order $\epsilon^n$ of the $\epsilon = 4 - d$ expansion. Such a factorial contribution originates from the large momentum region of the loop integrals which resembles the *ultraviolet renormalon* in relativistic field theories [137, 138, 139]. An example of the $n+1$-loop diagram contributing to the effective potential as $n!$ at $O(\epsilon^n)$ is depicted in Fig. A.1, which can be written as

$$V_n = \frac{i}{n} \int \frac{dk}{(2\pi)^{d+1}} \left[ (\Pi_0(k) + \Pi_a(k)) \, D(k) \right]^n , \qquad \text{(A.1)}$$

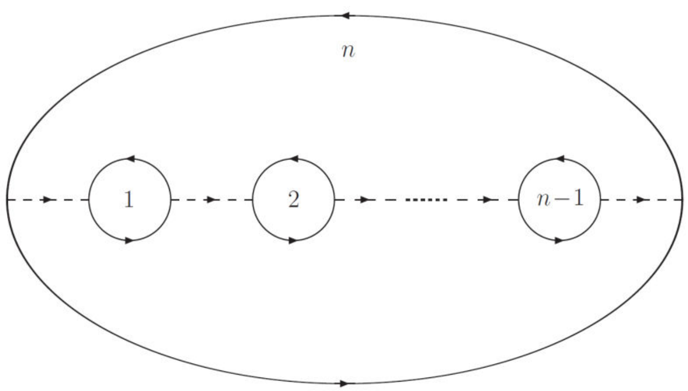



Figure A.1: A $n$-th order diagram at $d = 4$ which contributes to the effective potential as $n!$ by itself. The counter vertex $-i\Pi_0$ for each bubble diagram is understood implicitly.

# Epsilon expansion and GPF

We may obtain **NLO epsilon-expansion** at finite temperature for a unitary Fermi gas from the NSR calculation near four dimensions.

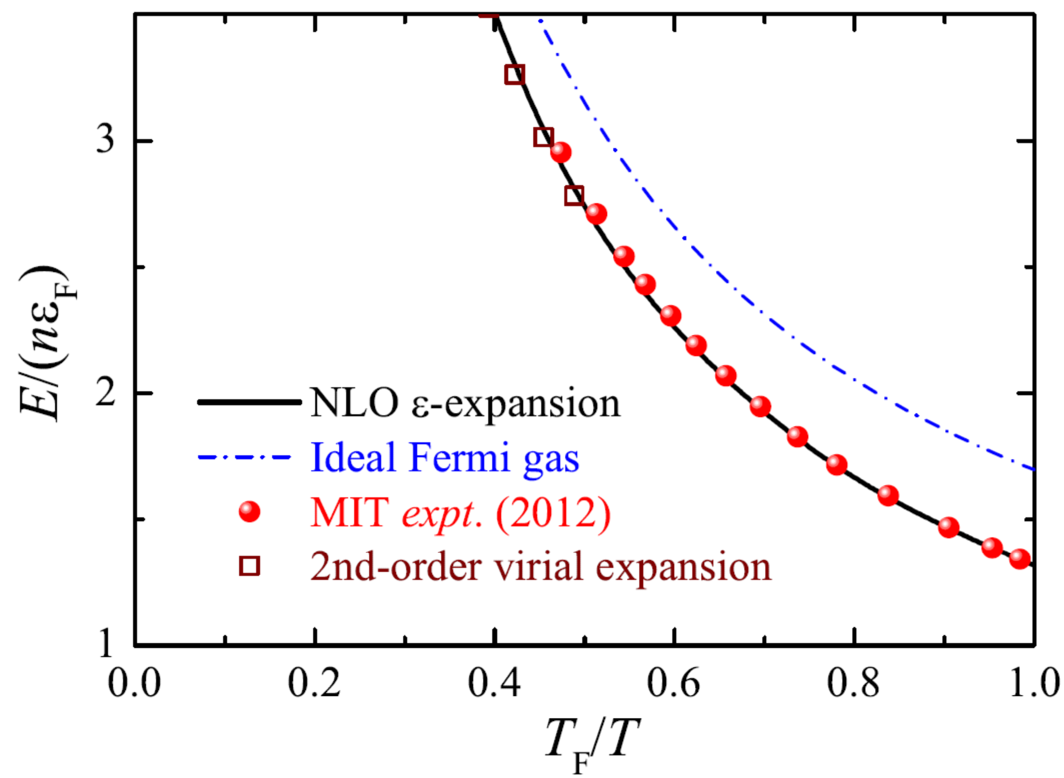



FIG. 3: (Color online) Temperature dependence of the total energy of a unitary Fermi gas predicted by the dimensional $\epsilon$ expansion theory. As the same as in Fig. 2, the next-to-leading-order (NLO) $\epsilon$ expansion results (solid line) are contrasted with the MIT data (solid circles), as well as the 2nd-order virial expansion (empty squares).

# New expansion in terms of the vertex function

Denote some higher order terms

$$\hat{V} = \mathcal{S}^{(3)} + \mathcal{S}^{(4)} + ...,$$

we have for the partition function

$$\mathcal{Z} = e^{-S_{\mathrm{MF}}^{(0)}} \int \mathcal{D}[\Delta, \Delta^*] \, e^{-S_{\mathrm{GF}}^{(2)} + \hat{V}} = e^{-S_{\mathrm{MF}}^{(0)}} \int \mathcal{D}[\Delta, \Delta^*] \, e^{-S_{\mathrm{GF}}^{(2)}} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left\langle \hat{V}_1 \hat{V}_2 \ldots \hat{V}_n \right\rangle ,$$

We define a new perturbation expansion with respect to the $\mathcal{O}(\epsilon)$ action, $S_{\mathrm{GF}}^{(2)}$. What this amounts to is letting us write down the thermodynamic potential per volume where the first order contribution is,

$$\Omega = \Omega_{\mathrm{MF}}^{(0)} + \Omega_{\mathrm{GF}}^{(2)} + \frac{1}{\beta V} \left\langle \mathcal{S}^{(4)} \right\rangle .$$

**@ NNLO order ($\varepsilon^2$)**

$$\frac{1}{\beta V} \left\langle \mathcal{S}^{(4)} \right\rangle = \sum_k \left[ G_0(k) \Sigma_0(k) \right]^2 ,$$

where

$$\Sigma_0(k) = \sum_q G_0(q-k) \Gamma(q).$$

# Results: A 3D unitary Fermi gas

We can self-consistently solve for a given chemical potential and compare to experiment and other theoretical methods. In particular the MIT results for a unitary gas, the fully self-consistent *GG* T-matrix theory and *NSR* theories.

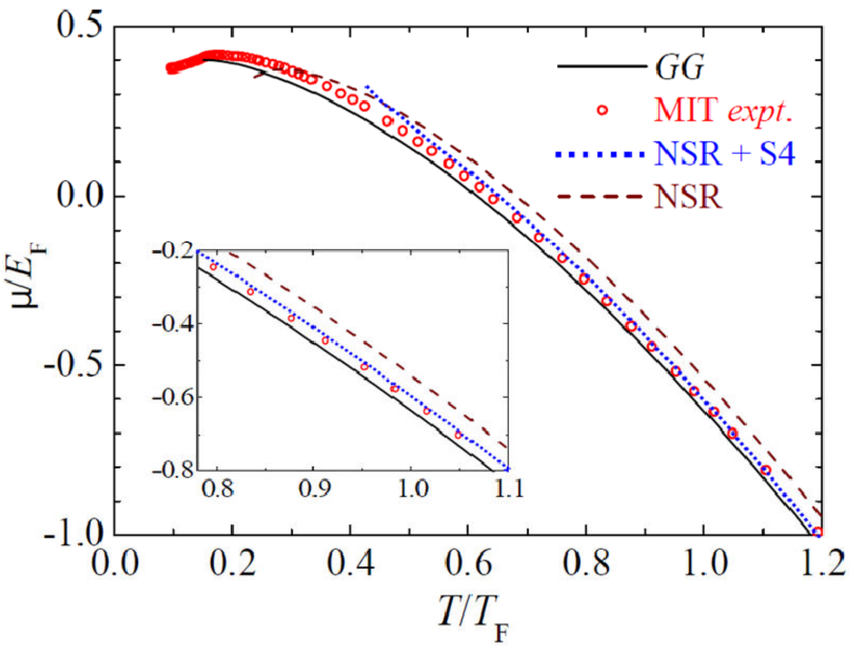



Figure : Experimental results from Mark J. H. Ku, Ariel T. Sommer, Lawrence W. Cheuk, Martin W. Zwierlein Science 335, 563-567 (2012).

# Additional work along the line

Further work to be done

- Explore the system below $T_c$, the inclusion of the superfluid parameter $\Delta_0$
- $T = 0$ calculations, these are considerably simpler than the full below $T_c$ calculation
- What is the behaviour in the deep BEC side and the contribution to the dimer-dimer scattering length
- Find the below $T_c$ behaviour of the two-dimensional gas

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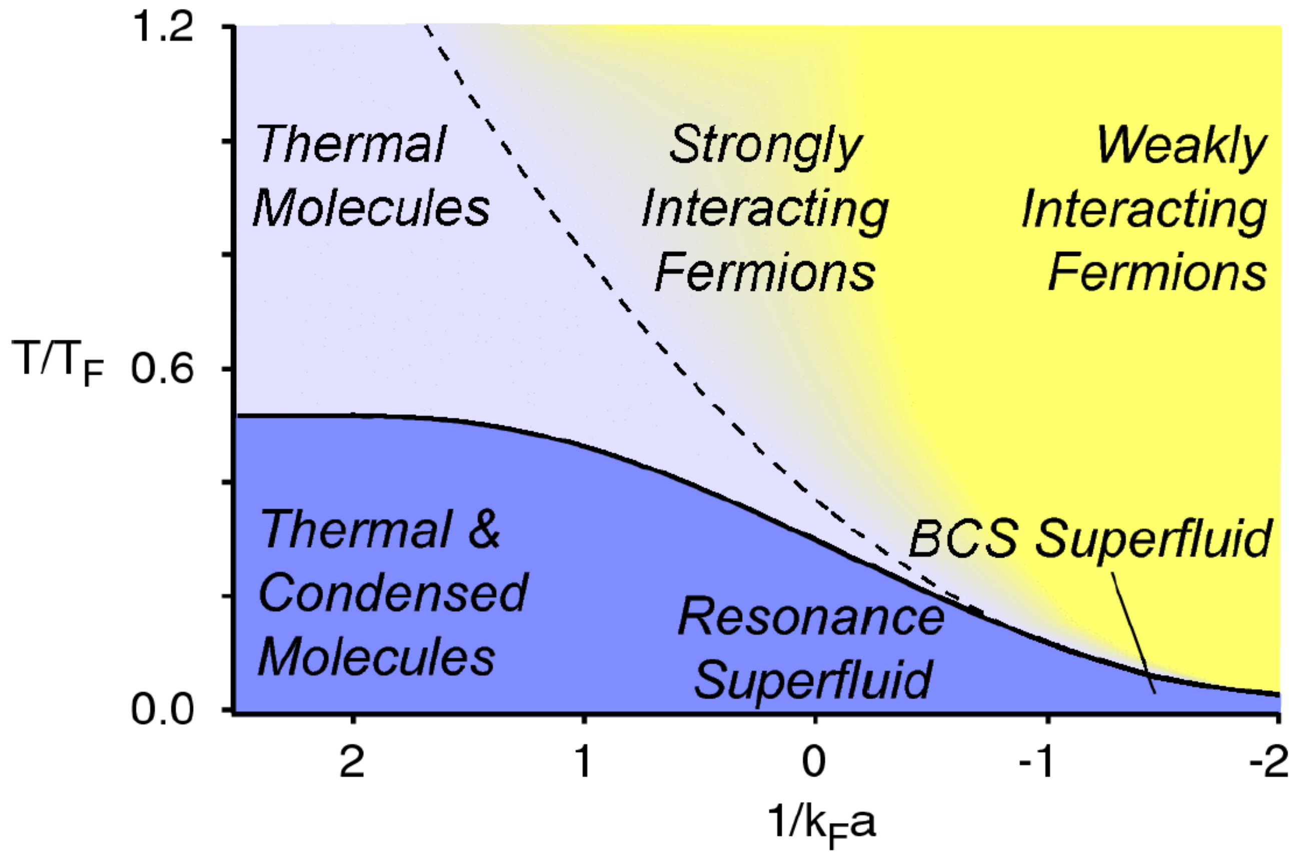
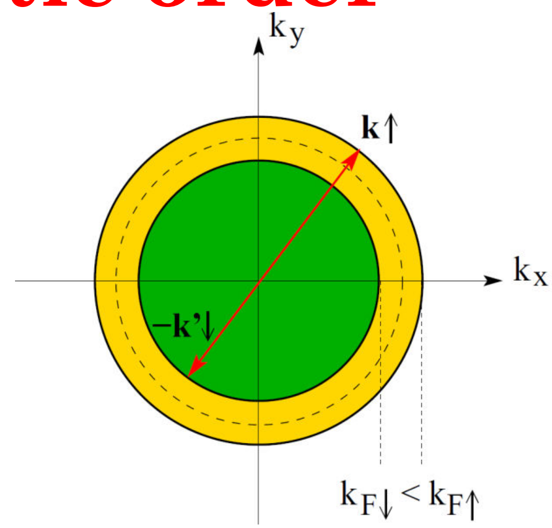
# Summary of the 2nd lecture




Leggett, Nozieres & Schmitt-Rink, Sa de Melo & Randeria, Griffin & Ohashi, Strinati, Haussmann, Levin, Combescot, Nishida & Son, Nikolic & Sachedev, Veillette, Sheehy & Radzihovsky, …

# Any unsolved challenge?
# Superfluidity vs. Magnetic order

How can Feynman diagrams help us?

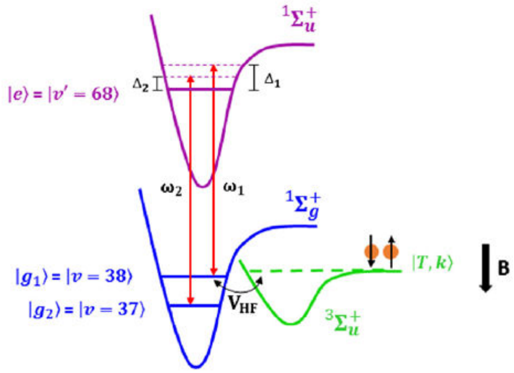



**We may understand the Cooper pairing, beyond the BCS framework!**

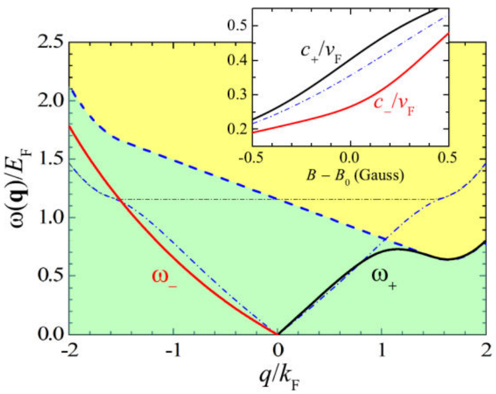
# Outline of this part

- An overview of FFLO physics

- The dark-state control of Feshbach resonances




- A new routine to observing FF superfluids



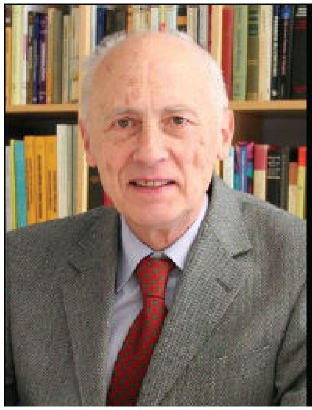

- Taking home message and outlook

# Part I: An overview of FFLO physics

How to observe a FF(LO) superfluid proposed in 1960s?

$$\Delta(\mathbf{x}) \propto e^{i\mathbf{Q}\cdot\mathbf{x}}$$

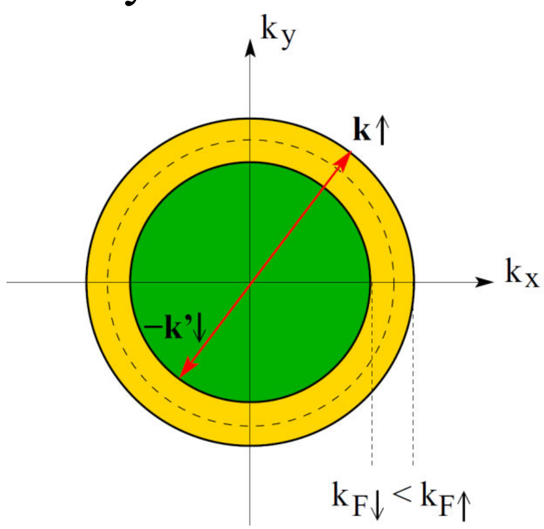
**Fulde**



**Ferrell**

# FFLO pairing by Fermi surface mismatch

- BCS Cooper pairs have zero momentum
- Population imbalance leads to finite-momentum pairs (FF 1964, see also LO)
- Fulde-Ferrell-Larkin-Ovichinnikov (FFLO) instability results in textured states
- Spontaneously breaks translational symmetry



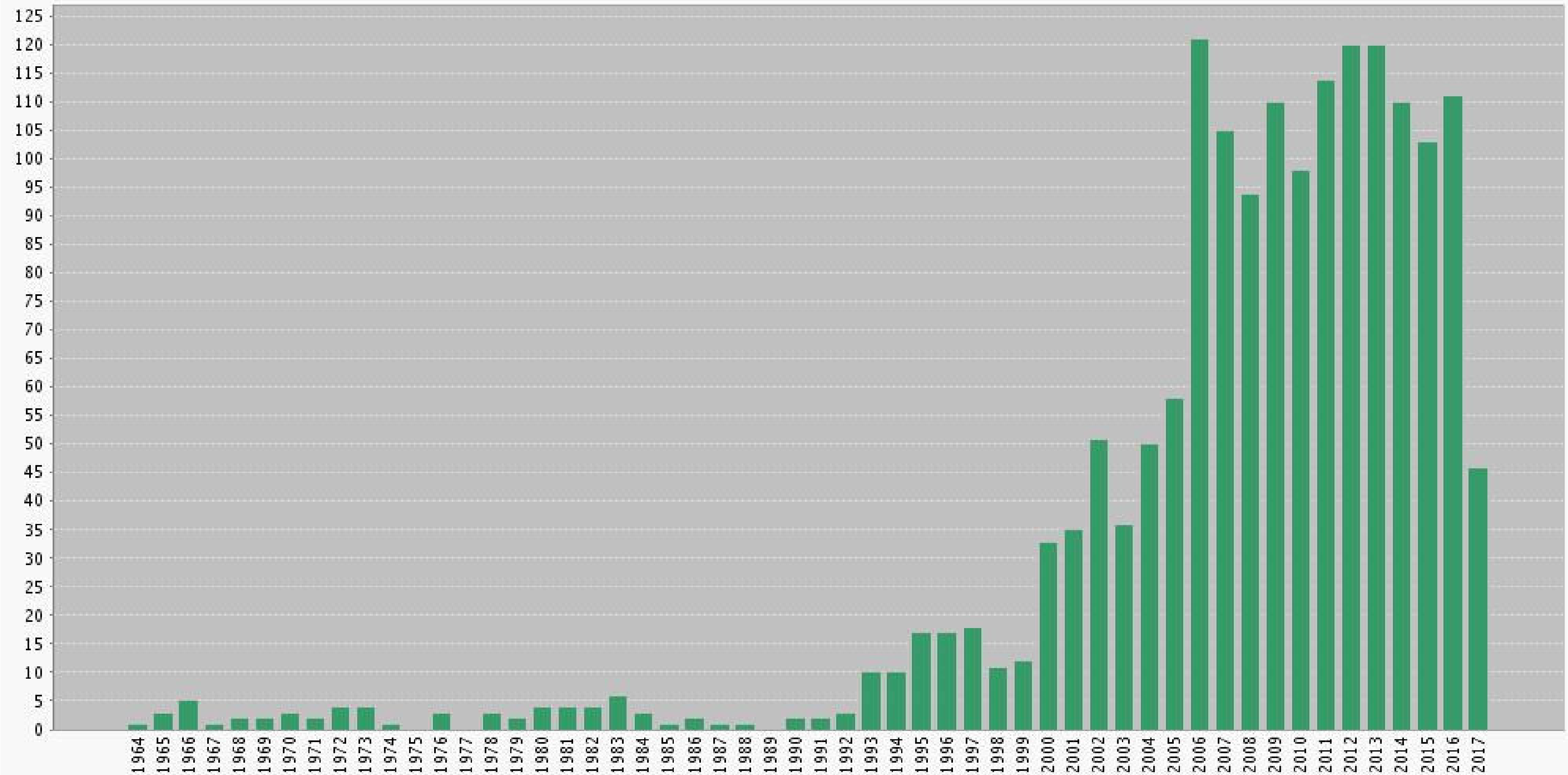

**Larkin** **Ovichinnikov**


$$Q \propto k_{F\uparrow} - k_{F\downarrow}$$

$$\Delta(\mathbf{x}) \propto e^{i\mathbf{Q}\cdot\mathbf{x}} \xrightarrow{e^{i\mathbf{Q}\cdot\mathbf{x}} + e^{-i\mathbf{Q}\cdot\mathbf{x}}} \Delta(\mathbf{x}) \propto \cos(\mathbf{Q}\cdot\mathbf{x})$$

**FF** superfluid　　　　**LO** superfluid

# Research activity on FFLO




| | 2013 | 2014 | 2015 | 2016 | 2017 | Total | Average Citations per Year |
|---|---|---|---|---|---|---|---|
| Use the checkboxes to remove individual items from this Citation Report or restrict to items published between 1900 and 2017 Go | **120** | **110** | **103** | **111** | **46** | **1679** | **31.09** |
| 1. **SUPERCONDUCTIVITY IN STRONG SPIN-EXCHANGE FIELD** By: FULDE, P; FERRELL, RA PHYSICAL REVIEW Volume: 135 Issue: 3A Pages: A550-+ Published: 1964 | 120 | 110 | 103 | 111 | 46 | 1679 | 31.09 |

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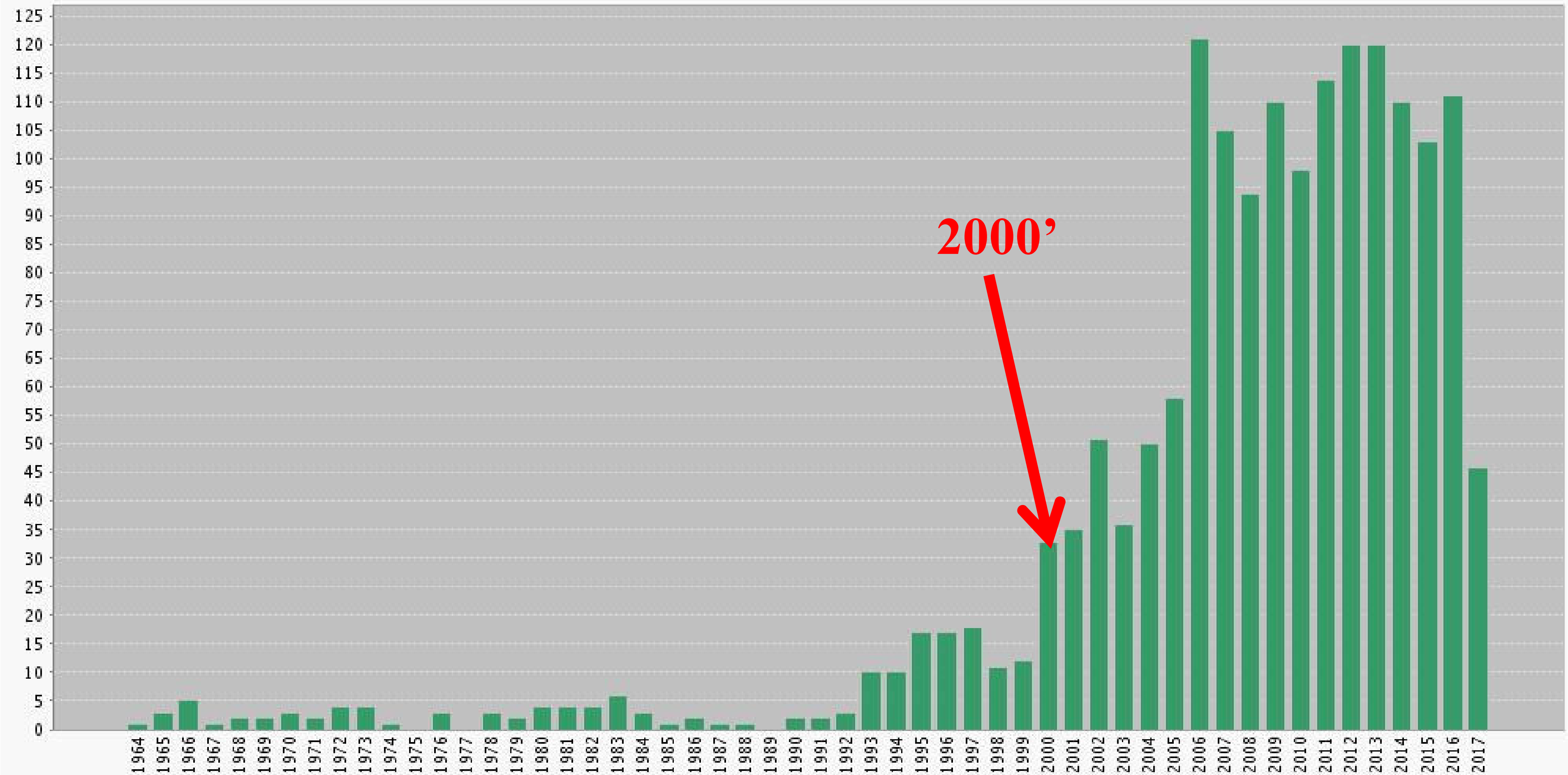
# The first burst in research activity

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| | 2013 | 2014 | 2015 | 2016 | 2017 | Total | Average Citations per Year |
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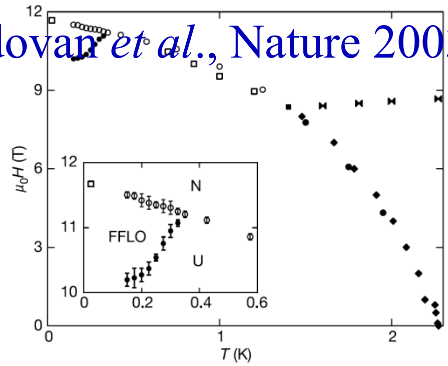
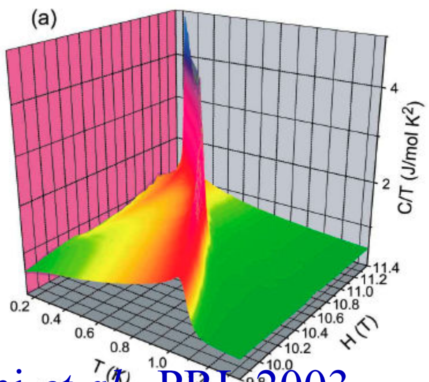
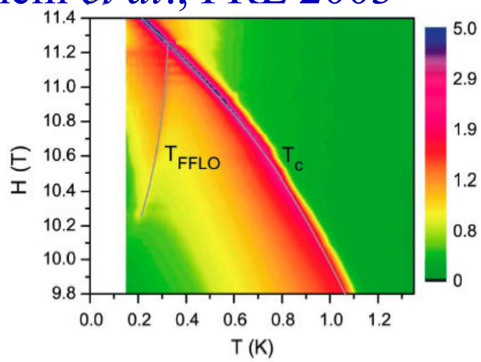
# Overview: FFLO in condensed matter physics

Radovan *et al.*, Nature 2003




Bianchi *et al.*, PRL 2003




CeCoIn5 --- **heavy-fermion SC**

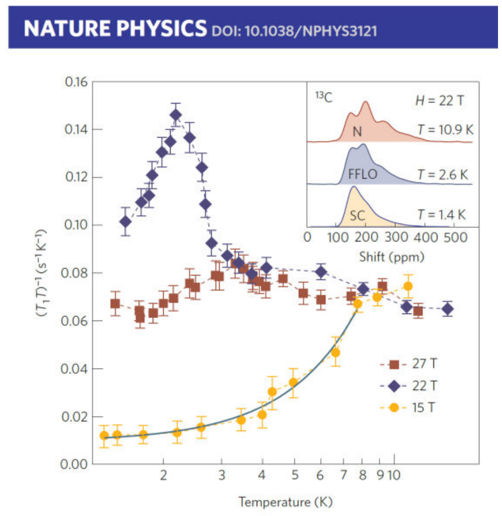
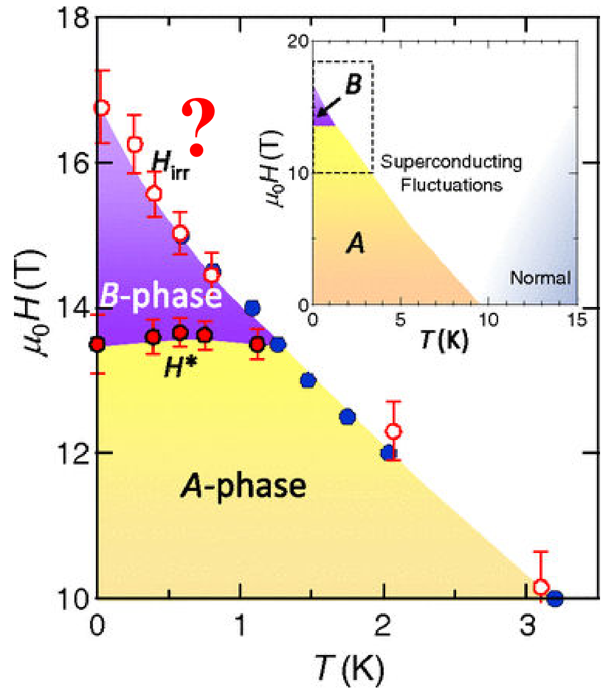



**Figure 2 | NMR relaxation rate in the normal and superconducting states.**

κ-$(ET)_2X$--- **organic SC**

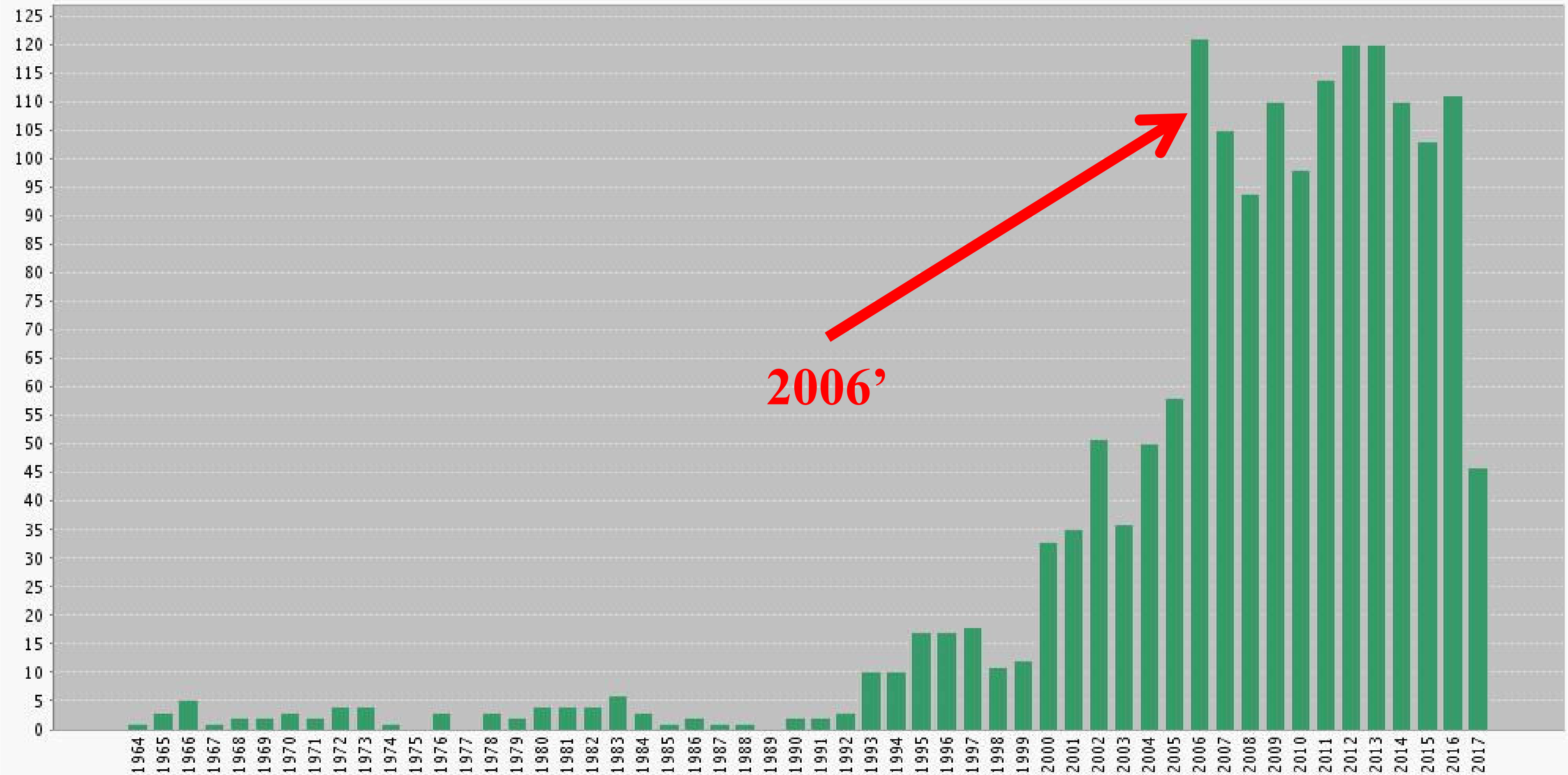
Mayaffre *et al.*, NPHYS 2014

Kasaharaa *et al.*, PNAS 2014




FeSe --- **iron-based SC**

# The second burst in research activity




| | 2013 ◀ | 2014 | 2015 | 2016 | 2017 ▶ | Total | Average Citations per Year |
|---|---|---|---|---|---|---|---|
| Use the checkboxes to remove individual items from this Citation Report or restrict to items published between 1900 and 2017 Go | 120 | 110 | 103 | 111 | 46 | 1679 | 31.09 |
| ☐ 1. **SUPERCONDUCTIVITY IN STRONG SPIN-EXCHANGE FIELD** By: FULDE, P; FERRELL, RA PHYSICAL REVIEW Volume: 135 Issue: 3A Pages: A550-+ Published: 1964 | 120 | 110 | 103 | 111 | 46 | 1679 | 31.09 |

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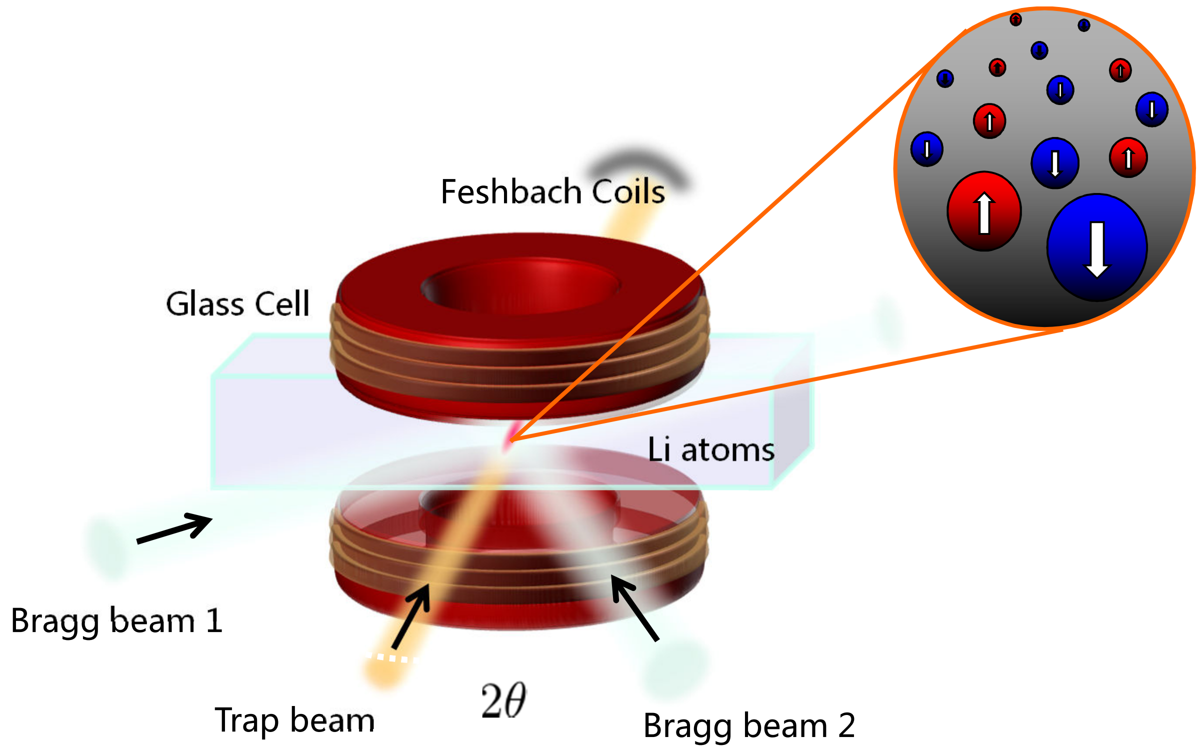
# Overview: cold-atoms come into play




**Ultracold atoms is an ideal platform to emulate FFLO physics**

**<u>Toolbox</u>: magnetic Feshbach resonance (MFR) + optical lattice + disorder + spin-orbit coupling (SOC) + optical control of MFR**

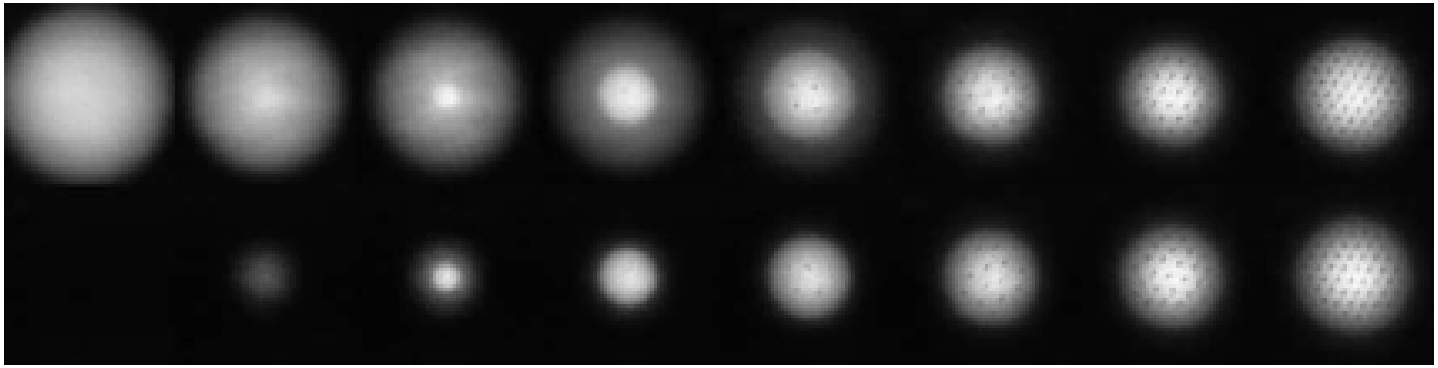
# Overview: cold-atoms experiments

- Rice (Hulet Group)
  - Science **311**, 503 (2006)
  - PRL **97**, 190407 (2006)
  - Nuclear Phys. A **790**, 88c (2007)
  - JLTP. **148**, 323 (2007)
  - Nature **467**, 567 (2010) @ **1D**
  - PRA **92**, 063616 (2015)
  - PRL **117**, 235301(2016)
- MIT (Ketterle Group)
  - Science **311**, 492 (2006)
  - Nature **442**, 54 (2006)
  - PRL **97**, 030401 (2006)
  - Science **316**, 867 (2007)
  - Nature **451**, 689 (2008)
- ENS (Salomon Group)
  - PRL **103**, 170402 (2009)
- NCSU (Thomas Group)
  - PRL **114**, 110403 (2015) @ **2D**
- Princeton (Bakr Group)
  - PRL **117**, 093601 (2016) @ **2D**

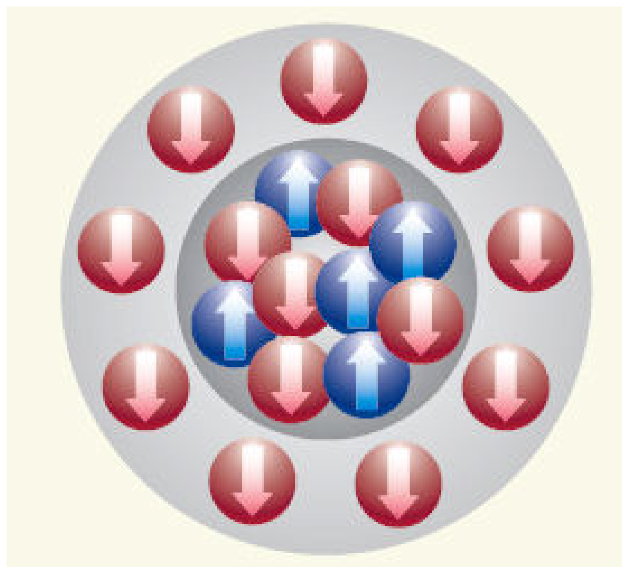
# Overview: cold-atoms experiments

$n_\uparrow$

$n_\downarrow$



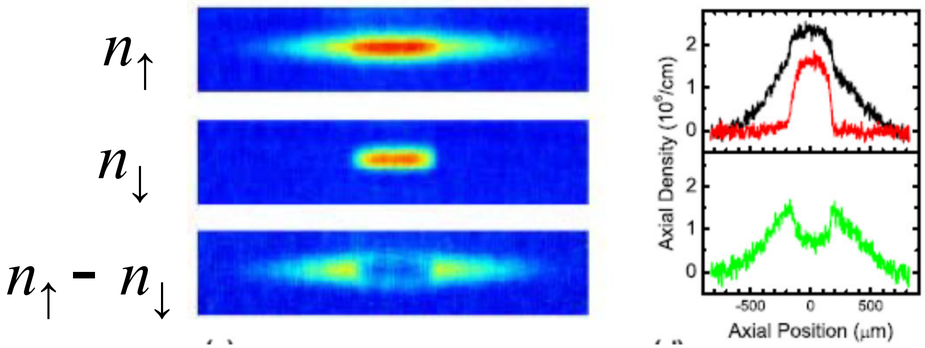
M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Science **311**, 492 (2006)



3D trapped Fermi gas: superfluid core with polarized halo…

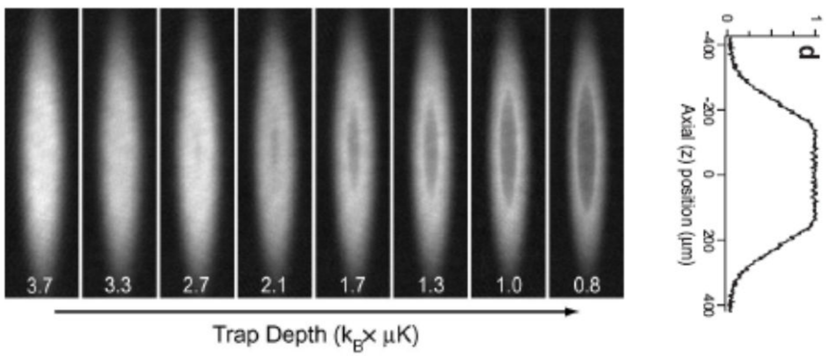
# Overview: cold-atoms experiments

## Hulet (Rice University)

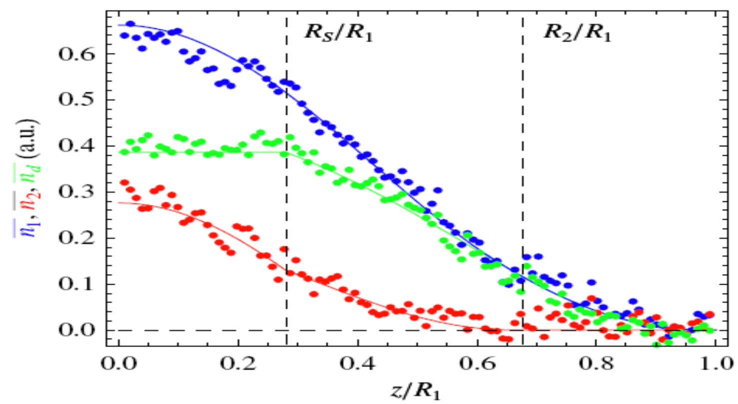



Low $T$

## Ketterle (MIT)

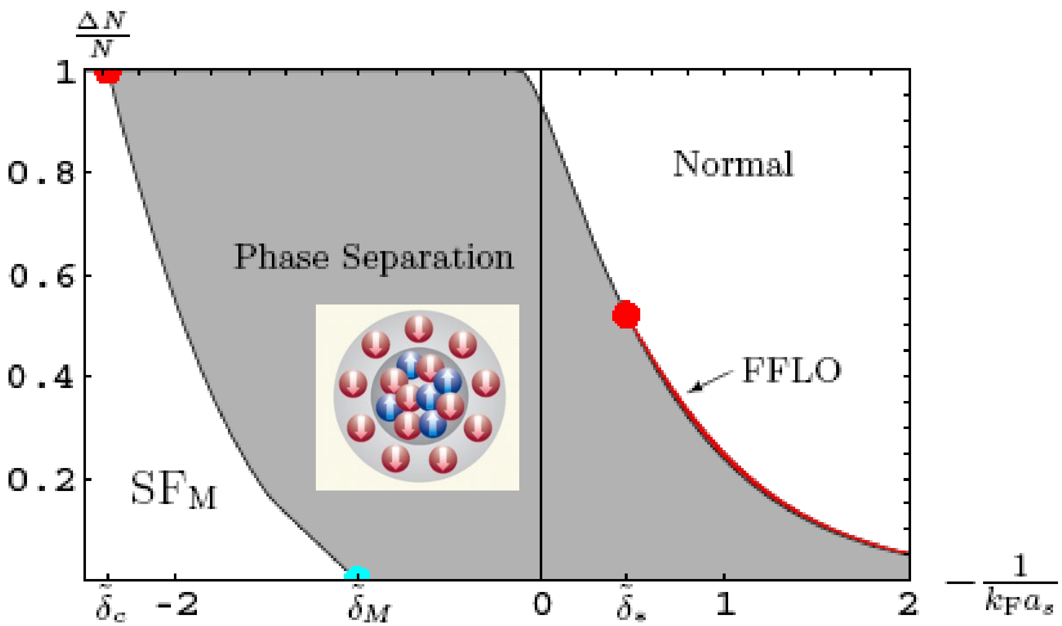



## Salomon (ENS, 2009)




MIT/Paris data are consistent with Local Density Approximation (LDA)
Rice data (low $T$) strongly violates LDA.

# Overview: cold-atoms theories at $T$=0



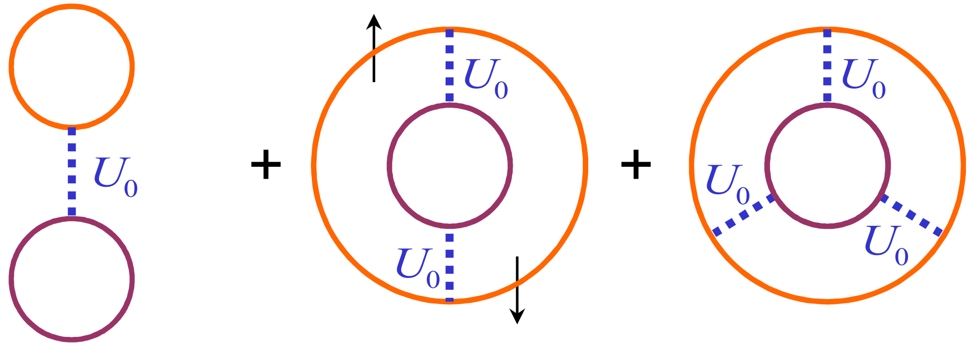

**FFLO is not favored in 3D.**

Sheehy and Radzihovsky (**mean-field**)
PRL (2006); Ann. Phys. (2007)

# Overview: cold-atoms theories at nonzero $T$

What happens if the spin population of a two-component Fermi gas is not equal? *i.e.*, $\delta\mu = \mu_\uparrow - \mu_\downarrow > 0$.
Novel spatially inhomogeneous **FFLO** superfluidity?

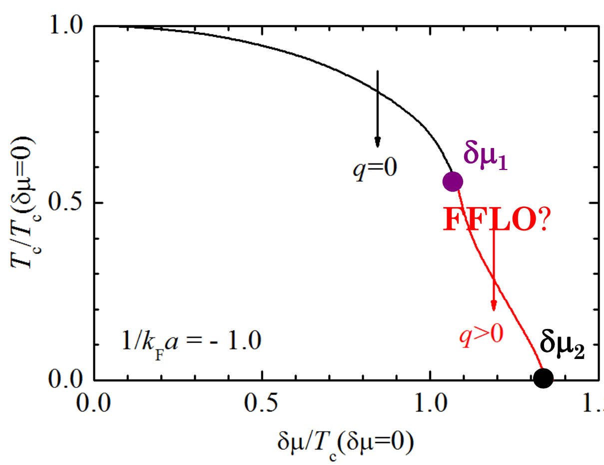
$\Omega - \Omega^{(1)} =$

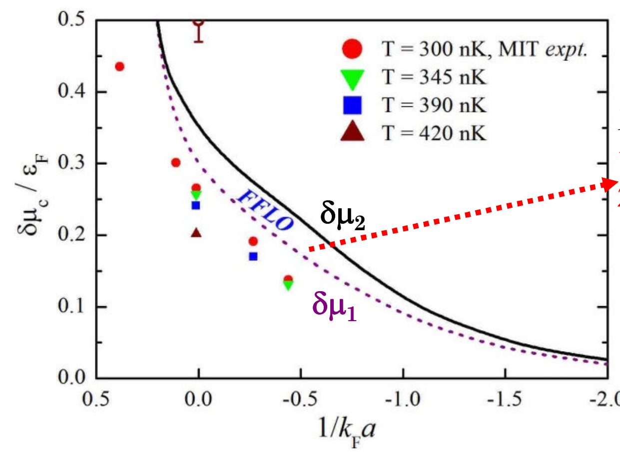



**Infinite diagrams!**

Here, (orange line) $= \dfrac{1}{i\omega_m - (\varepsilon_k - \mu_\uparrow)}$ and (purple line) $= \dfrac{1}{i\omega_m - (\varepsilon_k - \mu_\downarrow)}$

$$\max_q [\Gamma^{-1}(\mathbf{q},0)]_{T=T_c} = 0$$




*Require*:
1. fine tune of imbalance;
2. sufficient low $T$

Hu & Liu, *Europhys. Lett.* **75**, 364 (2006).

# Overview: cold-atoms theories (1D FFLO)

PRL **98,** 070402 (2007) PHYSICAL REVIEW LETTERS week ending 16 FEBRUARY 2007

### Attractive Fermi Gases with Unequal Spin Populations in Highly Elongated Traps

G. Orso[1,2]
[1]*BEC-INFM and Dipartimento di Fisica, Universitá di Trento, 1-38050 Povo, Italy*
[2]*LENS and Dipartimento di Fisica, Universitá di Firenze, via N. Carrara 1, 50019 Sesto Fiorentino, Italy*
(Received 16 October 2006; published 14 February 2007)

We investigate two-component attractive Fermi gases with imbalanced spin populations in trapped one-dimensional configurations. The ground state properties are determined with the local density approximation, starting from the exact Bethe-ansatz equations for the homogeneous case. We predict that the atoms are distributed according to a *two-shell* structure: a partially polarized phase in the center of the trap and either a fully paired or a fully polarized phase in the wings. The partially polarized core is expected to be a superfluid of the Fulde-Ferrell-Larkin-Ovchinnikov type. The size of the cloud as well as the critical spin polarization needed to suppress the fully paired shell are calculated as a function of the coupling strength.

PRL **98,** 070403 (2007) PHYSICAL REVIEW LETTERS week ending 16 FEBRUARY 2007

### Phase Diagram of a Strongly Interacting Polarized Fermi Gas in One Dimension

Hui Hu,[1,2] Xia-Ji Liu,[2] and Peter D. Drummond[2]
[1]*Department of Physics, Renmin University of China, Beijing 100872, China*
[2]*ARC Centre of Excellence for Quantum-Atom Optics, Department of Physics, University of Queensland, Brisbane, Queensland 4072, Australia*
(Received 17 October 2006; published 14 February 2007)

Based on the integrable Gaudin model and local density approximation, we discuss the ground state of a one-dimensional trapped Fermi gas with imbalanced spin population, for an arbitrary attractive interaction. A phase separation state, with a polarized superfluid core immersed in an unpolarized superfluid shell, emerges below a critical spin polarization. Above it, coexistence of polarized superfluid matter and a fully polarized normal gas is favored. These two exotic states could be realized experimentally in highly elongated atomic traps, and diagnosed by measuring the lowest density compressional mode. We identify the polarized superfluid as having an Fulde-Ferrell-Larkin-Ovchinnikov structure, and predict the resulting mode frequency as a function of the spin polarization.

### Phase transitions and pairing signature in strongly attractive Fermi atomic gases

X. W. Guan,[1] M. T. Batchelor,[1,2] C. Lee,[3] and M. Bortz[1]
[1]*Department of Theoretical Physics, Research School of Physical Sciences and Engineering, Australian National University, Canberra, Australian Capital Territory 0200, Australia*
[2]*Mathematical Sciences Institute, Australian National University, Canberra, Australian Capital Territory 0200, Australia*
[3]*Nonlinear Physics Centre and ARC Centre of Excellence for Quantum-Atom Optics, Research School of Physical Sciences and Engineering, Australian National University, Canberra, Australian Capital Territory 0200, Australia*
(Received 14 May 2007; published 16 August 2007)

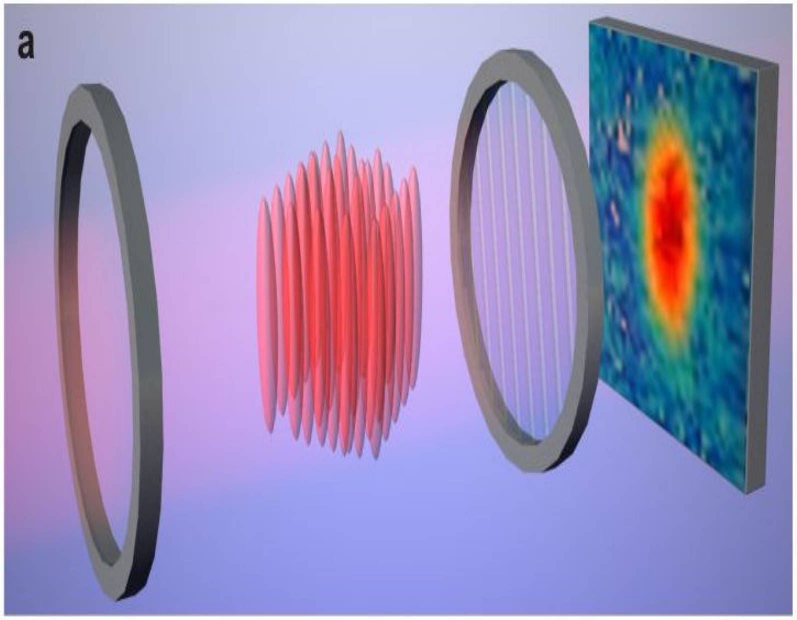
We investigate pairing and quantum phase transitions in the one-dimensional two-component Fermi atomic gas in an external field. The phase diagram, critical fields, magnetization, and local pairing correlation are obtained analytically via the exact thermodynamic Bethe ansatz solution. At zero temperature, bound pairs of fermions with opposite spin states form a singlet ground state when the external field $H<H_{c1}$. A completely ferromagnetic phase without pairing occurs when the external field $H>H_{c2}$. In the region $H_{c1}<H<H_{c2}$, we observe a mixed phase of matter in which paired and unpaired atoms coexist. The phase diagram is reminiscent of that of type II superconductors. For temperatures below the degenerate temperature and in the absence of an external field, the bound pairs of fermions form hard-core bosons obeying generalized exclusion statistics.

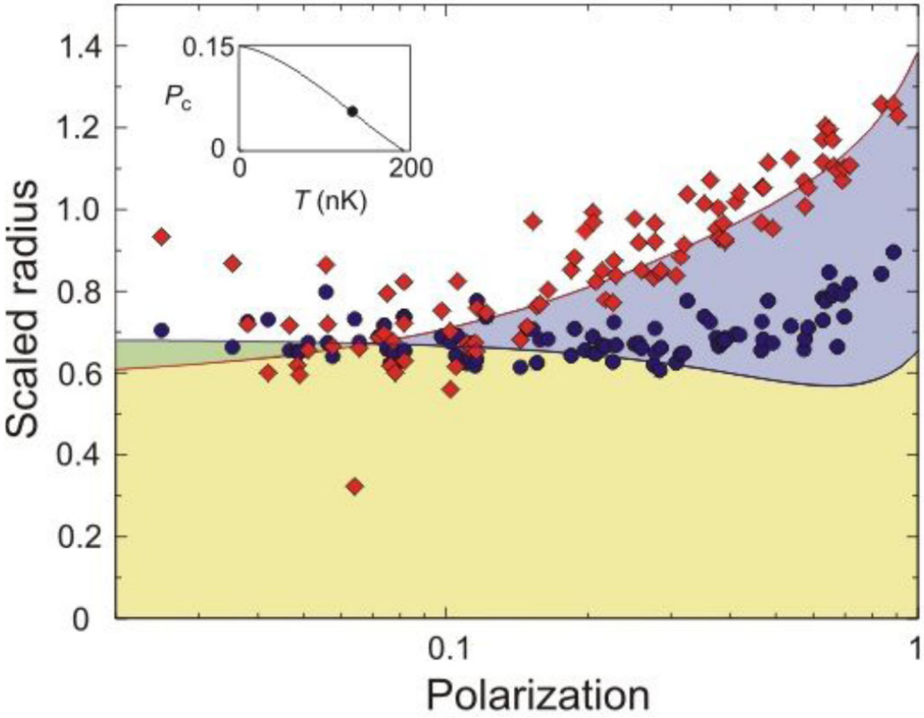
DOI: 10.1103/PhysRevB.76.085120 PACS number(s): 05.30.Fk, 03.75.Hh, 03.75.Ss, 05.30.Pr

and many others…

# Overview: cold-atoms experiments (1D FFLO)

## 2D deep optical lattice




Liao *et al*., Nature **467**, 567 (2010).
**Hulet Group**

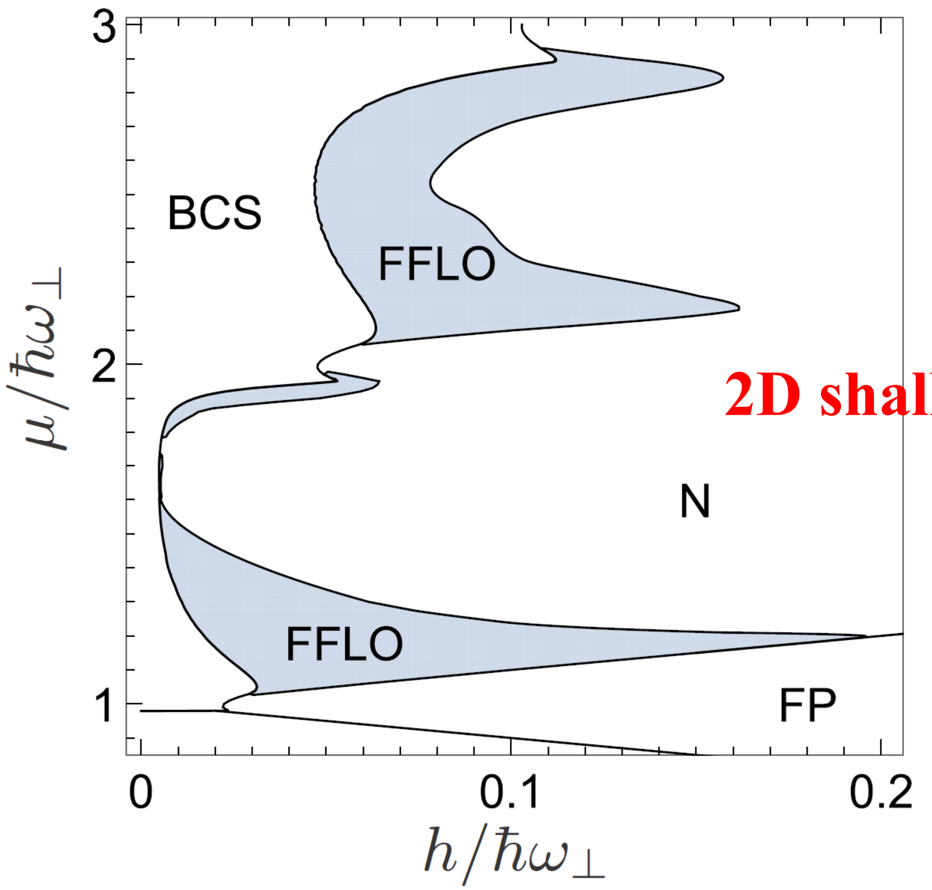
# Overview: cold-atoms theories (possible FFLO in 2D & 3D)

PHYSICAL REVIEW A **94**, 063627 (2016)

## Dimensional crossover in a spin-imbalanced Fermi gas

Shovan Dutta and Erich J. Mueller
*Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853, USA*

(Received 13 August 2015; published 21 December 2016)



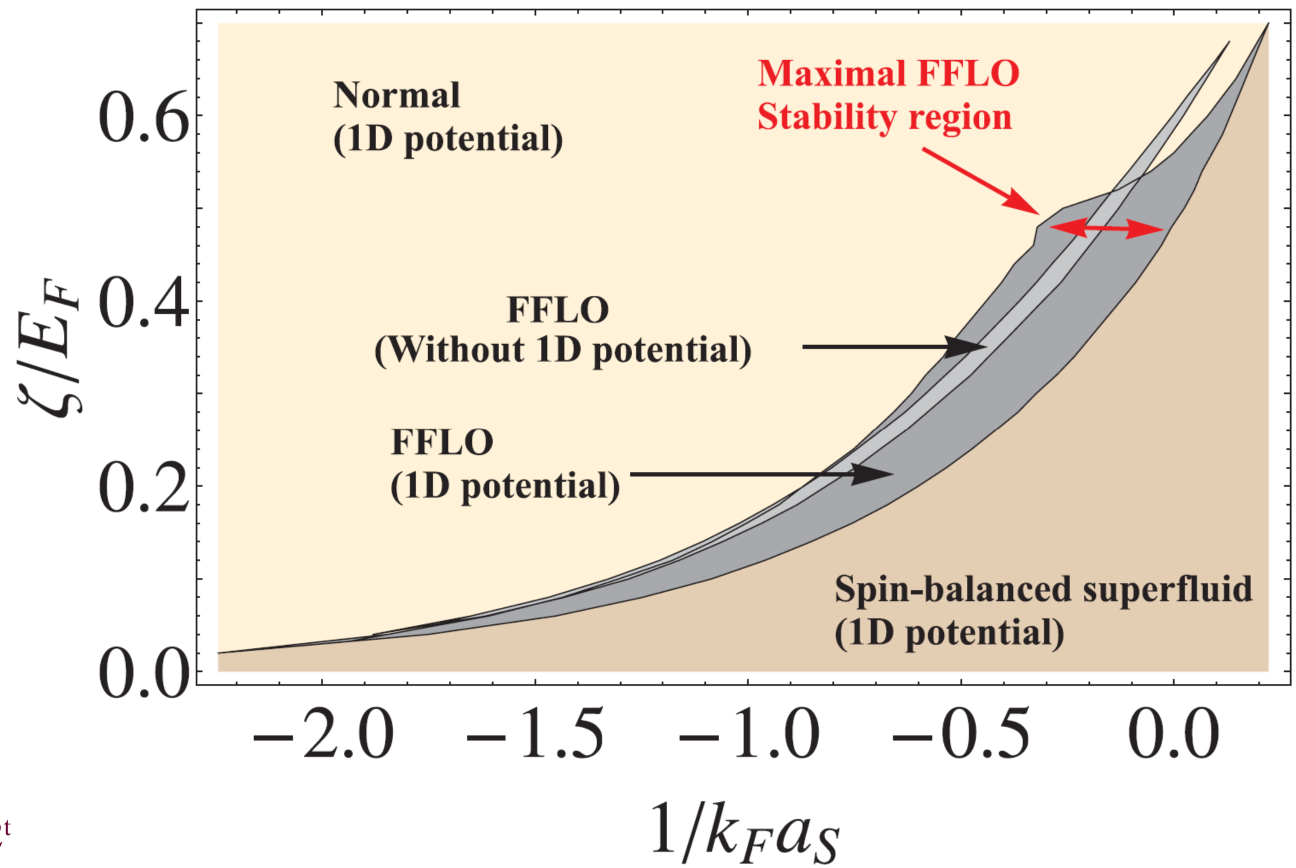

**2D shallow optical lattice**

# Overview: cold-atoms theories (possible FFLO in 2D & 3D)

PHYSICAL REVIEW A **83**, 013606 (2011)

**Resonant enhancement of the Fulde-Ferell-Larkin-Ovchinnikov state in three dimensions by a one-dimensional optical potential**

Jeroen P. A. Devreese,[1] Sergei N. Klimin,[1,*] and Jacques Tempere[1,2]
[1] *Theorie van Kwantumsystemen en Complexe Systemen (TQC), Universiteit Antwerpen, B-2020 Antwerpen, Belgium*
[2] *Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
(Received 27 August 2010; published 14 January 2011)




**1D shallow optical lattice**

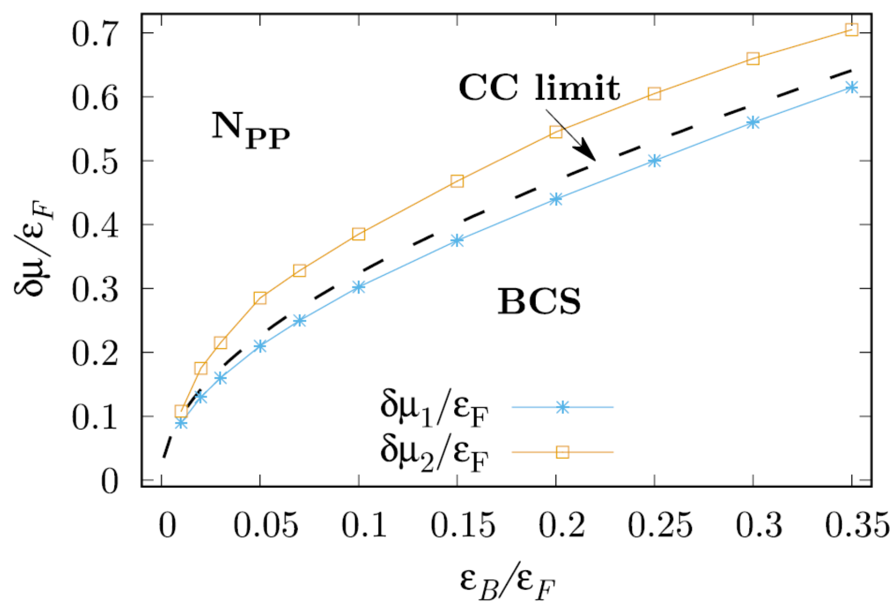
# Overview: cold-atoms theories (possible FFLO in 2D & 3D)

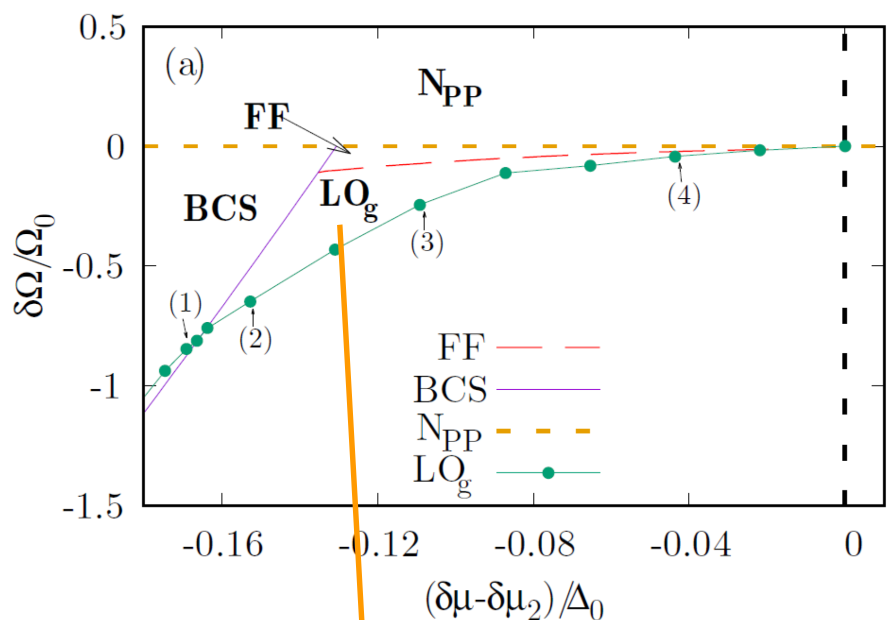
## 1D deep optical lattice

PHYSICAL REVIEW A **95**, 013603 (2017)

### Larkin-Ovchinnikov superfluidity in a two-dimensional imbalanced atomic Fermi gas

Umberto Toniolo,* Brendan Mulkerin, Xia-Ji Liu, and Hui Hu
*Centre for Quantum and Optical Science, Swinburne University of Technology, Hawthorn 3122 VIC, Australia*
(Received 22 September 2016; published 4 January 2017)



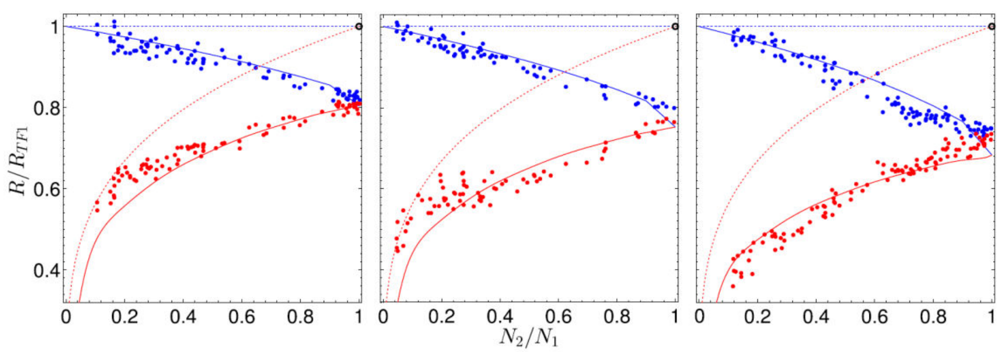

**Enlarged window for FFLO in 2D:**
**$(\delta\mu_1 - \delta\mu_2)/\varepsilon_F \sim 0.1 - 0.2$** larger than that (i.e., **$\sim 0.05$**) in 3D

**$LO_g$**: generalized LO state with many harmonics in one direction

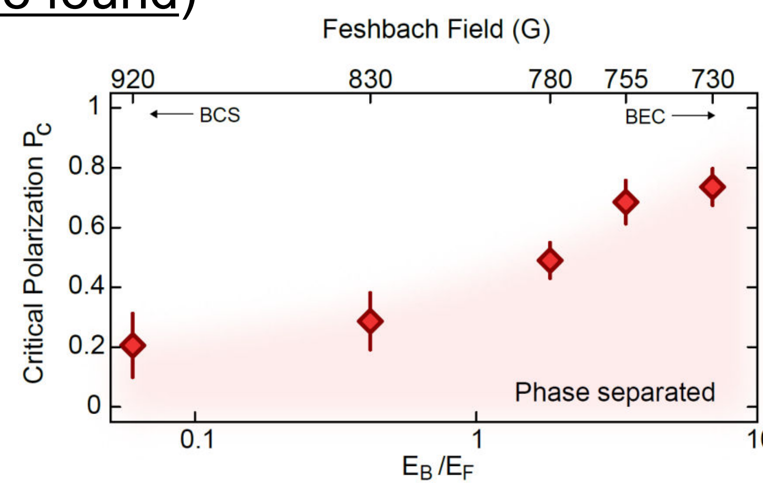
# Overview: latest cold-atom progress


Revelle *et al.*, PRL **117**, 235301(2016).
**Hulet Group**

## 2D experiments: (<u>phase-separation phase found</u>)



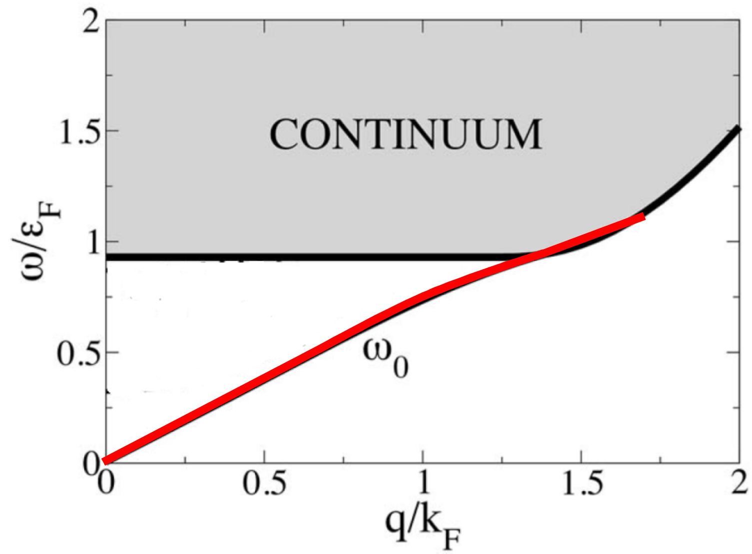

Ong *et al.*, PRL **114**, 110403 (2015).
**Thomas Group**



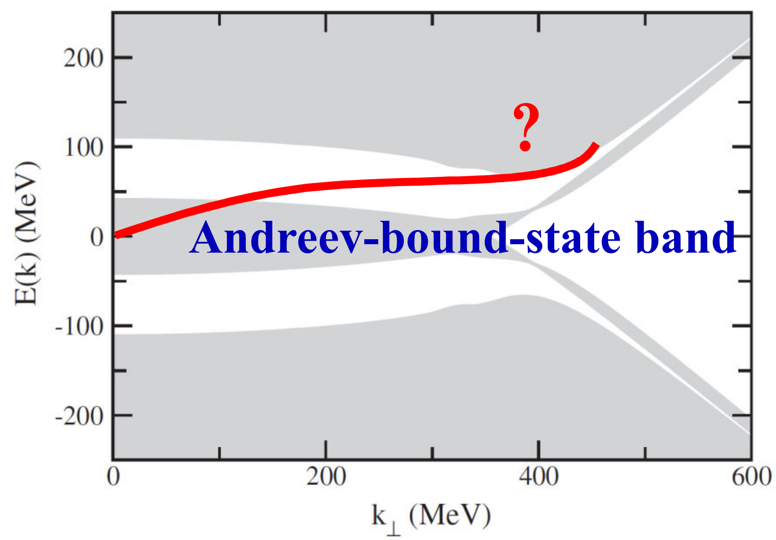

Mitra *et al.*, PRL **117**, 093601 (2016) .
**Bakr Group**

# Overviews: FFLO is not so happy with spin-imbalance ☹

- Exact configuration of the FFLO phase (without **GL** approximation)?
- Strong pair fluctuations (**GPF** considered by HH *et al*. & M. Randeria *et al*. )?
- Interplay between **Andreev bound states** and **phonons** (low-energy physics?):




Ordinary BCS superfluids

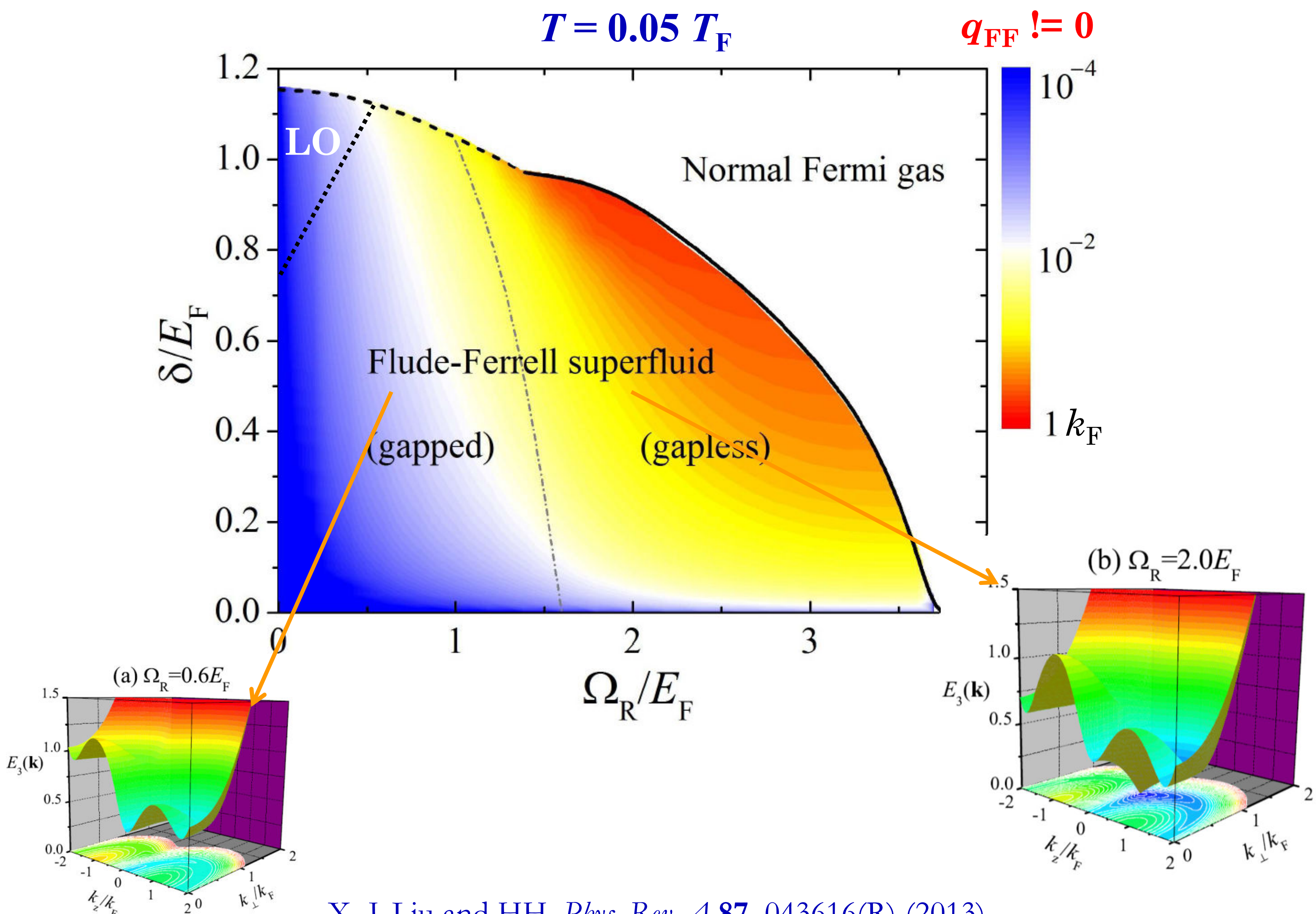



FFLO superfluids:
**roton structure, multi-phonon modes**?

# Latest conference: "FFLO16" in Dresden

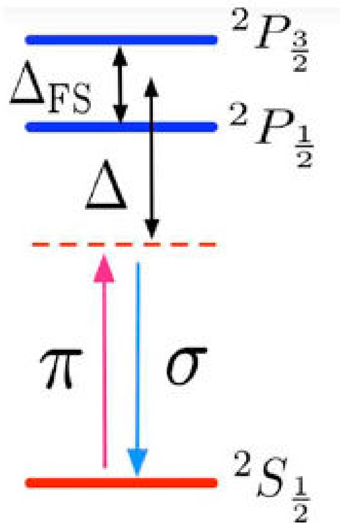
Peter Fulde Yuri Ovichinnikov

# FF phase diagram: ERDSOC




X.-J. Liu and HH, *Phys. Rev. A* **87**, 043616(R) (2013).
see also the works by C. Zhang, W. Yi, H. Pu, *et al.*…

# The heating problem with Raman SOC ☹




spin-dependent Raman coupling ~ $W\frac{\Delta_{FS}}{\Delta^2}$, not $\frac{1}{\Delta}$

heating from spontaneous emission ~ $W\frac{\Gamma}{\Delta^2}$

smaller **fine structure splitting** (or **mass**) means more heating

In reality:

- Li-6: unable to reach equilibrium
- K-40: the lowest temperature is about $0.5T_F$

- Rb-87: okay, but bosonic
- **Dy-161: excellent candidate, but dipolar (PRX 2016)**

## What have we learned from SOC?

It is important to violate **Galilean invariance**!

- **In the case of spin-imbalance, the driving force towards FFLO is weak: ~ $q^2$ at small $q$ and energy barrier, because of Galilean invariance**. A number of related problems, such as the unknown structure of the FFLO pairing order parameter: LO – $\cos(qx)$? $LO_2$ – $\cos(qx)+\cos(qy)$? or $LO_3$?

- **In the presence of spin-orbit coupling, which violates Galilean invariance, the internal driving force ~ $q$ at small $q$**. The mean-field structure of the FFLO pairing order parameter is determined.

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# Part II: Dark state control of magnetic FR

- Optical control of magnetic FR
- Dark state control of magnetic FR
- Center-of-mass dependent interatomic interaction

PRL **116,** 075301 (2016) PHYSICAL REVIEW LETTERS week ending 19 FEBRUARY 2016

**Optical Control of Magnetic Feshbach Resonances by Closed-Channel Electromagnetically Induced Transparency**

A. Jagannathan,[1,2] N. Arunkumar,[1] J. A. Joseph,[1] and J. E. Thomas[1*]
[1]*Department of Physics, North Carolina State University, Raleigh, North Carolina 27695, USA*
[2]*Department of Physics, Duke University, Durham, North Carolina 27708, USA*
(Received 4 November 2015; published 17 February 2016)

We control magnetic Feshbach resonances in an optically trapped mixture of the two lowest hyperfine states of a $^6$Li Fermi gas, using two optical fields to create a dark state in the closed molecular channel. In the experiments, the narrow Feshbach resonance is tuned by up to 3 G. For the broad resonance, the spontaneous lifetime is increased to 0.4 s at the dark-state resonance, compared to 0.5 ms for single-field tuning. We present a new model of light-induced loss spectra, employing continuum-dressed basis states, which agrees in shape and magnitude with loss measurements for both broad and narrow resonances. Using this model, we predict the trade-off between tunability and loss for the broad resonance in $^6$Li, showing that our two-field method substantially reduces the two-body loss rate compared to single-field methods for the same tuning range.

DOI: 10.1103/PhysRevLett.116.075301

# A precise optical control of Feshbach resonances in Rb-87 (2009)

nature physics LETTERS
PUBLISHED ONLINE: 6 APRIL 2009 | DOI: 10.1038/NPHYS1232

# Control of a magnetic Feshbach resonance with laser light

Dominik M. Bauer, Matthias Lettner, Christoph Vo, Gerhard Rempe and Stephan Dürr*

**The capability to tune the strength of the elastic interparticle interaction is crucial for many experiments with ultracold gases. Magnetic Feshbach resonances[1,2] are widely harnessed for this purpose, but future experiments[3-8] would benefit from extra flexibility, in particular from the capability to spatially modulate the interaction strength on short length scales. Optical Feshbach resonances[9-15] do offer this possibility in principle, but in alkali atoms they induce rapid loss of particles due to light-induced inelastic collisions. Here, we report experiments that demonstrate that light near-resonant with a molecular bound-to-bound transition in $^{87}$Rb can be used to shift the magnetic field at which a magnetic Feshbach resonance occurs. This enables us to tune the interaction**

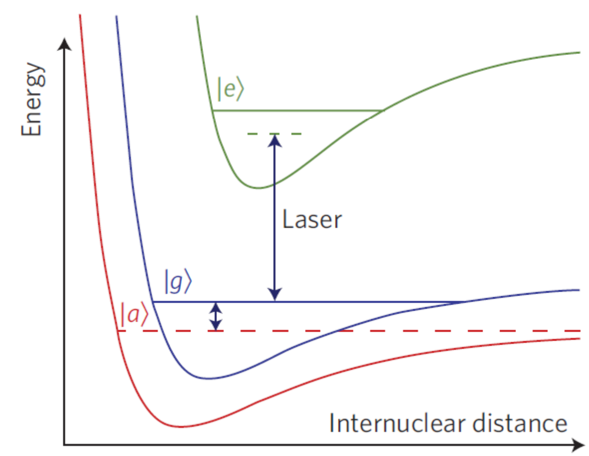



**Figure 1 | Level scheme of the experiment.** The Feshbach resonance

# Optical control of MFR for atomic Fermi gases

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **88**, 041601(R) (2013)

## Optical control of a magnetic Feshbach resonance in an ultracold Fermi gas

Zhengkun Fu,[1] Pengjun Wang,[1] Lianghui Huang,[1] Zengming Meng,[1] Hui Hu,[2] and Jing Zhang[1,*]

[1] *State Key Laboratory of Quantum Optics and Quantum Optics Devices, Institute of Opto-Electronics, Shanxi University, Taiyuan 030006, People's Republic of China*

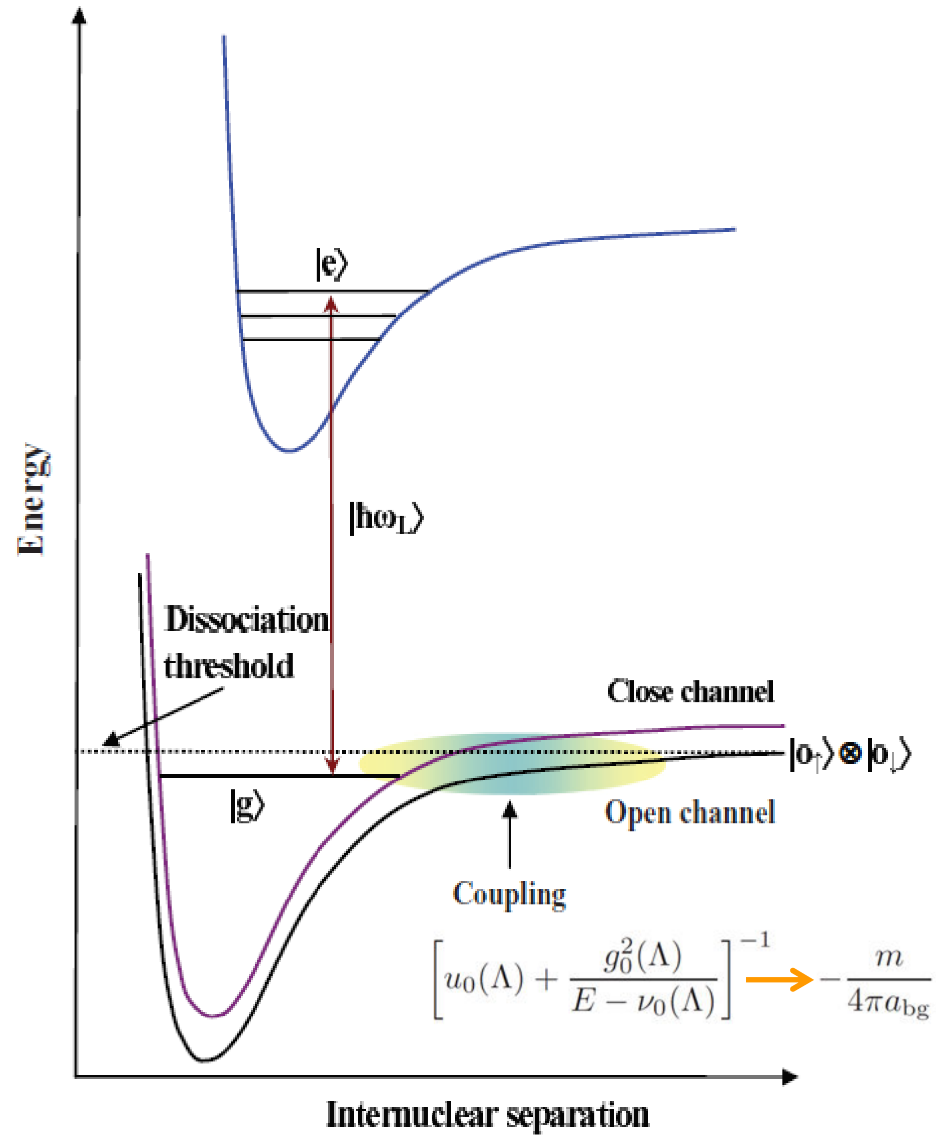
[2] *Centre for Atom Optics and Ultrafast Spectroscopy, Swinburne University of Technology, Melbourne 3122, Australia*

(Received 3 June 2013; published 9 October 2013)

We use laser light near resonant with a molecular bound-to-bound transition to control a magnetic Feshbach resonance in ultracold Fermi gases of $^{40}$K atoms. The spectrum of excited molecular states is measured by applying a laser field that couples the ground Feshbach molecular state to electronically excited molecular states. Nine strong bound-to-bound resonances are observed below the $^2P_{1/2} + {}^2S_{1/2}$ threshold. We use radio-frequency spectroscopy to characterize the laser-dressed bound state near a specific bound-to-bound resonance and show clearly the shift of the magnetic Feshbach resonance using light. The demonstrated technology could be used to modify interatomic interactions with high spatial and temporal resolutions in the crossover regime from a Bose-Einstein condensate to a Bardeen-Cooper-Schrieffer superfluid.

DOI: 10.1103/PhysRevA.88.041601 PACS number(s): 67.85.−d, 03.75.Hh, 03.75.Ss, 05.30.Fk

# Optical control of MFR: level diagrams




$$\left[u_0(\Lambda)+\frac{g_0^2(\Lambda)}{E-\nu_0(\Lambda)}\right]^{-1} \rightarrow -\frac{m}{4\pi a_{\mathrm{bg}}}\frac{E-\Delta\mu(B-B_0)}{E-\Delta\mu(B-B_0)+\Delta\mu\Delta B}$$

# Optical control of MFR: K-40 case

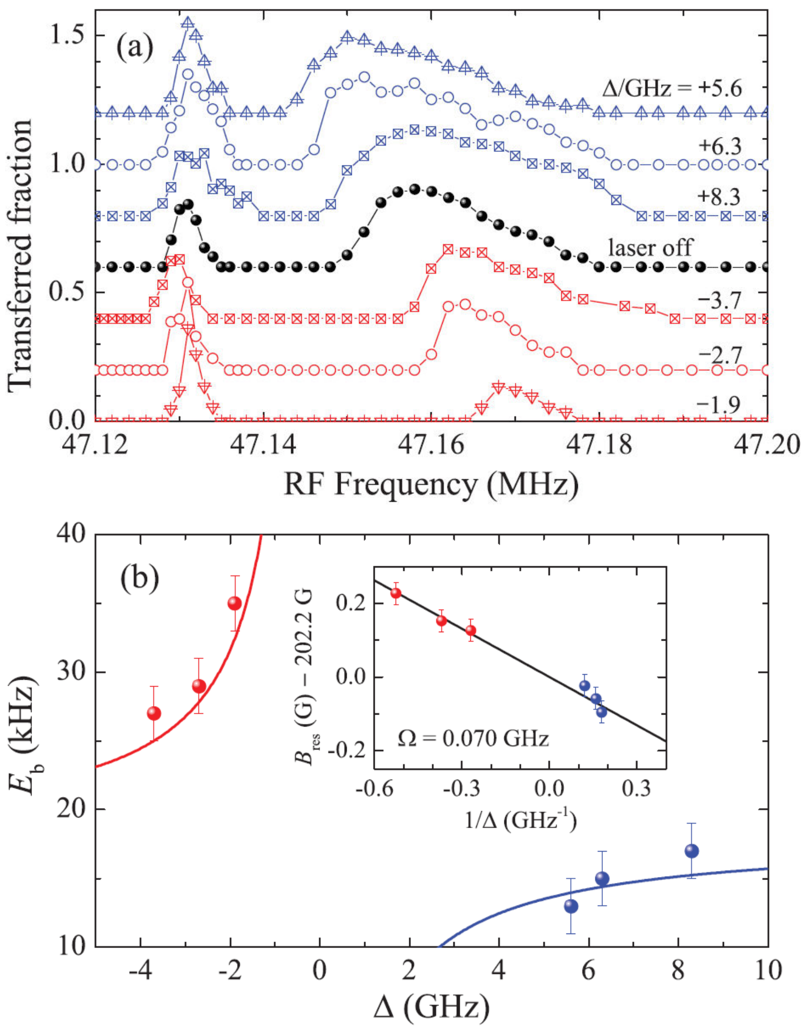



FIG. 2. (Color online) rf spectroscopy and binding energy of the laser-dressed bound state near the Feshbach resonance $B_0 = 202.20 \pm 0.02$ G. With the light off, Feshbach molecules are created below the resonance at $B = 201.60$ G, corresponding to an $s$-wave scattering length $a_s \simeq 2216a_0$, where $a_0$ is the Bohr radius. The dimensionless interaction parameter of the Fermi cloud is $1/(k_F a_s) \simeq 0.62$. (a) The rf spectroscopy at different detunings, offset vertically for clarity. (b) The binding energy as a function of the detuning. The solid lines are the theoretical predictions from a simple theory as outlined in the Supplemental Material [39]. The inset shows the resonance position of the shifted Feshbach resonance as a function of the inverse detuning.

**Stark shift** (will be explained later):

$$\delta = \frac{\Omega^2}{4(\Delta + i\gamma/2)} \simeq \frac{\Omega^2}{4\Delta} - \left(\frac{\Omega^2}{4\Delta^2}\right)\frac{i\gamma}{2}, \qquad (1)$$

where $\Omega$ is the Rabi frequency of laser beam, $\Delta = (2\pi\hbar)(\omega_L - \omega_{eg})$ is the detuning, and $\gamma \sim 2\pi \times 6$ MHz stands for the fast spontaneous radiative decay of the excited molecular state [10]. Our measurements are performed under the condition $\Omega \ll \Delta \sim (2\pi\hbar) \times 1$ GHz, so that the effective decay rate $\gamma_{\rm eff} \equiv (\gamma\Omega^2/8\Delta^2) \sim 2\pi \times 1$ kHz and therefore the atomic loss should be greatly suppressed. Such a suppression was also observed in the recent experiment for bosonic $^{87}$Rb atoms [10], where a large detuning was used.

**Problem: too short lifetime - 1ms**

# Dark state control of MFR

PRL **116,** 075301 (2016) PHYSICAL REVIEW LETTERS week ending 19 FEBRUARY 2016

**Optical Control of Magnetic Feshbach Resonances by Closed-Channel Electromagnetically Induced Transparency**

A. Jagannathan,[1,2] N. Arunkumar,[1] J. A. Joseph,[1] and J. E. Thomas[1*]
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(Received 4 November 2015; published 17 February 2016)

We control magnetic Feshbach resonances in an optically trapped mixture of the two lowest hyperfine states of a $^6$Li Fermi gas, using two optical fields to create a dark state in the closed molecular channel. In the experiments, the narrow Feshbach resonance is tuned by up to 3 G. For the broad resonance, the spontaneous lifetime is increased to 0.4 s at the dark-state resonance, compared to 0.5 ms for single-field tuning. We present a new model of light-induced loss spectra, employing continuum-dressed basis states, which agrees in shape and magnitude with loss measurements for both broad and narrow resonances. Using this model, we predict the trade-off between tunability and loss for the broad resonance in $^6$Li, showing that our two-field method substantially reduces the two-body loss rate compared to single-field methods for the same tuning range.

DOI: 10.1103/PhysRevLett.116.075301

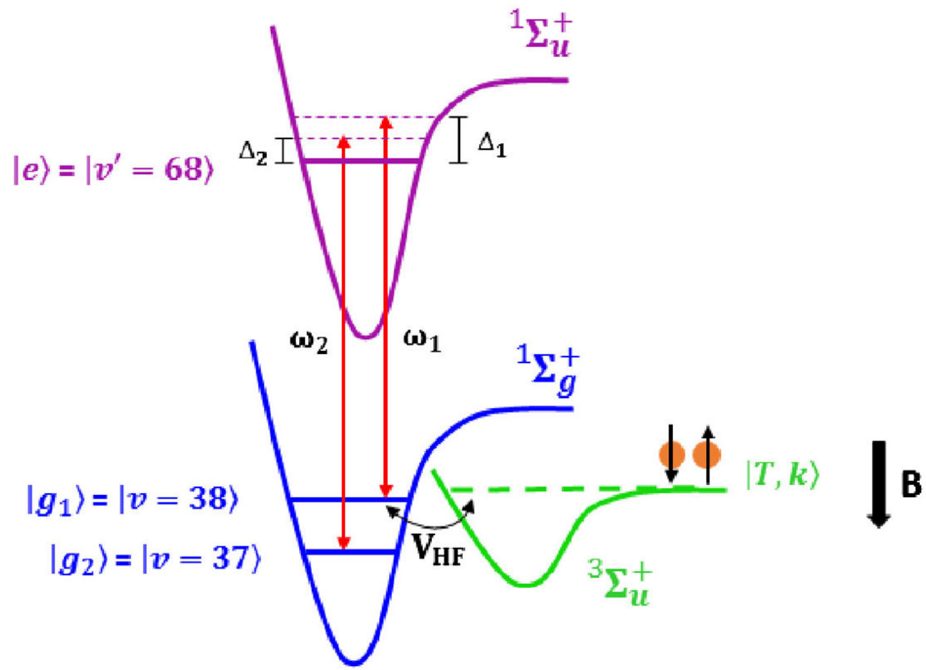



FIG. 1. Level scheme for the two-field optical technique. Optical fields of frequencies $\omega_1$ (detuning $\Delta_1$) and $\omega_2$ (detuning $\Delta_2$), respectively, couple two singlet ground molecular states $|g_1\rangle$ and $|g_2\rangle$ to the singlet excited molecular state $|e\rangle$; $V_{HF}$ is the hyperfine coupling between the incoming atomic pair state in the open triplet channel $|T, k\rangle$ and $|g_1\rangle$, and it is responsible for a magnetically controlled Feshbach resonance.

# What is the electromagnetically induced transparency (EIT)?

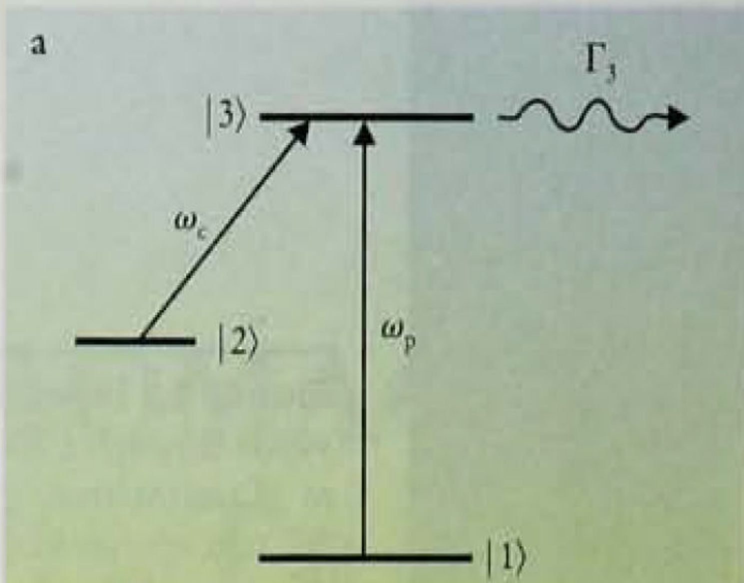



FIGURE 2. SEVERAL EIT SCHEMES. a: A three-state system in which the upper state decays with a rate $\Gamma_3$ to states outside the system. b: Transparency in the continuum. c: Use of EIT to modify the refractive index of a medium.

## Adiabatic Preparation

We use a notation system in which the Rabi frequencies $\Omega_p$ and $\Omega_c$ are constants and the temporal and spatial dependence of the probe and coupling laser pulse shapes are $f(z,t)$ and $g(z,t)$, respectively.

With both lasers tuned close to resonance, the Hamiltonian for the three-state atom of figure 2 with a decay rate $\Gamma_3$ from state $|3\rangle$ is

two-photon detuning

$$H = -\frac{1}{2}\begin{bmatrix} 0 & 0 & \Omega_p f(t) \\ 0 & 0 & \Omega_c g(t) \\ \Omega_p f^*(t) & \Omega_c g^*(t) & j(\Gamma_3/2) \end{bmatrix}$$

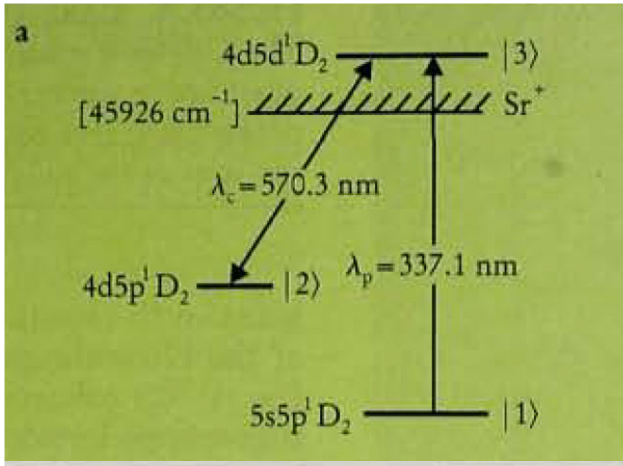
single-photon detuning

As may be verified by inspection, an eigenvector with zero eigenvalue is $[\Omega_c g^*(t), -\Omega_p f^*(t), 0]$. It is this eigenvector that represents the population-trapped state.

To use adiabatic preparation, the coupling laser pulse is applied with the probe pulse still zero. The population-trapped eigenvector is then [1,0,0] and is the same as the ground state of the atom. If both fields are then changed sufficiently slowly, the atom will remain in this eigenstate thereafter.[6]

Because it might be more intuitive to apply the lasers in a sequence that accesses the population at $t = 0$, the type of preparation described here is sometimes called counterintuitive. But from the point of view of quantum interference, it is not counterintuitive. One could ask, Would you make a tunnel, (that is, an interference) and then go through it, or would you first go through it, and then make it?

# What is EIT?



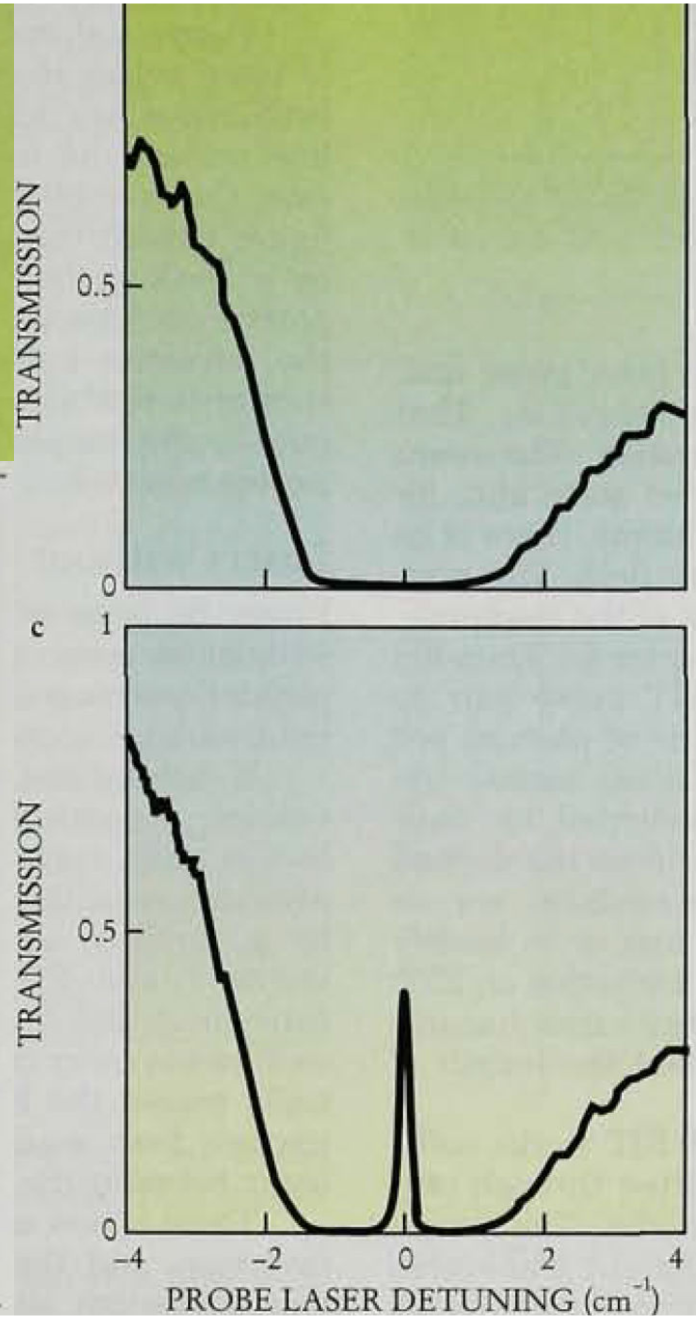




FIGURE 3. ELECTROMAGNETICALLY INDUCED TRANSPARENCY IN STRONTIUM VAPOR.[1] **a:** Partial energy-level schematic. The frequency of a laser of wavelength $\lambda_c$ is held fixed. Because this laser couples states $|2\rangle$ and $|3\rangle$, it is termed the coupling laser. The frequency of a probing laser of wavelength $\lambda_p$ is varied. **b:** Transmission versus probe laser detuning with the coupling laser absent. The minimum transmission in this figure is exp(−20). **c:** Transmission versus probe laser frequency with the coupling laser present. The ratio of the transmitted to the incident light is now about 40%. Under the conditions of this experiment, almost all of the atoms remain in the ground state $|1\rangle$.

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# Dark state control of MFR

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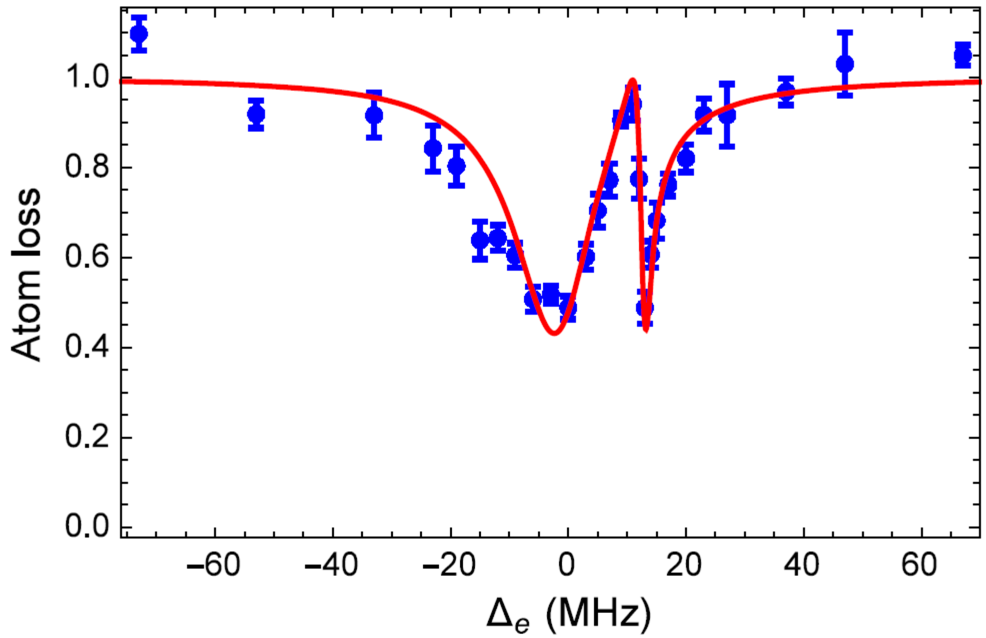


FIG. 4. Loss suppression near the broad resonance at 832.2 G, for $B = 840$ G, as a function of single photon detuning by sweeping the $\omega_1$ laser frequency. Pulse duration $\tau = 5.0$ ms; $T = 14.8\ \mu$K; $\Omega_1 = 1.36\gamma_e$; $\Omega_2 = 0.9\gamma_e$; $\Delta_2 = 10.0$ MHz. Maximum suppression occurs for $\Delta_e = \Delta_2 = 10.0$ MHz, where $\delta_e = 0$. Solid red curve, continuum-dressed-state model [16].

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**Lifetime – 0.4s**

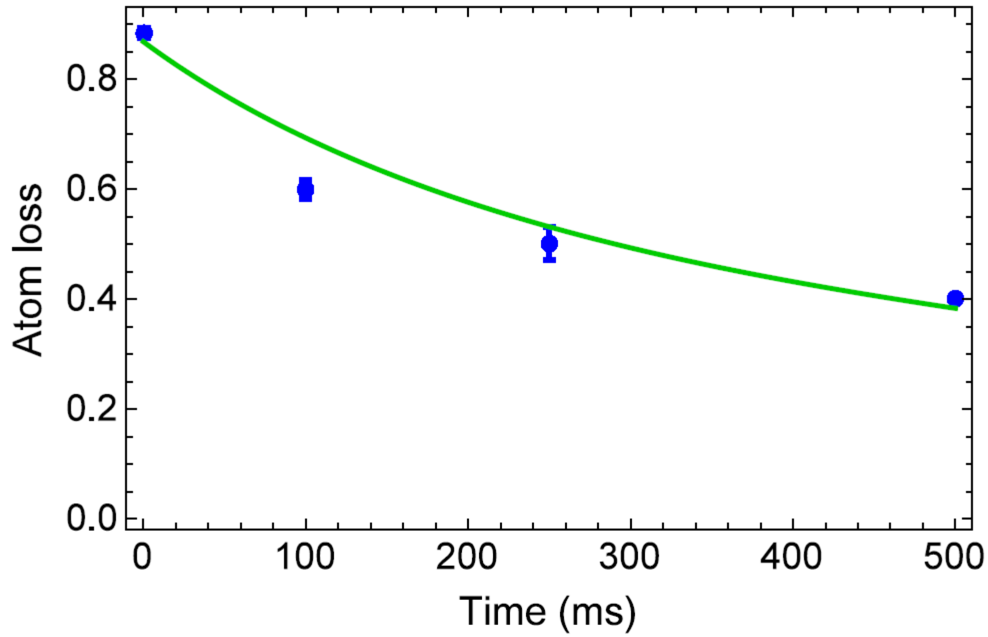
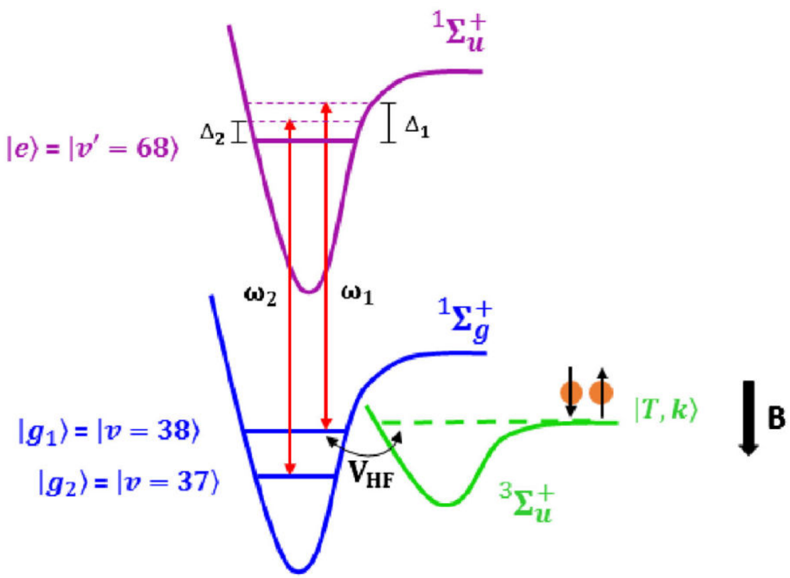



FIG. 5. Number of atoms in state $|1\rangle$ vs time with $\Omega_1 = 0.65\gamma_e$ and $\omega_1$ tuned to cause loss at 841 G. $\Omega_2 = 0.9\gamma_e$, and $\omega_2$ is tuned to suppress loss near 841 G. With $\Omega_1 = 0.65\gamma_e$ and $\Omega_2 = 0$, the corresponding decay time is $\simeq$0.5 ms. $\gamma_e = 2\pi \times 11.8$ MHz is the radiative decay rate; $T = 4.5\ \mu$K. Solid green curve, $N(t) = N(0)/(1 + \gamma t)$, where $\gamma = 2.5\text{s}^{-1}$.

# Dark state control of MFR

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**Why such a long lifetime?**

Stark shift in the case of EIT (to be shown later):

$$\Sigma_1 = \frac{\Omega_1^2/4}{I_e - \Omega_2^2/[4I_2]}$$

Recall,

$$I_e = \Delta_e + i\frac{\gamma_e}{2} \sim \mathrm{MHz}$$ single-photon detuning

$$I_2 = \delta \sim 0$$ two-photon detuning

# The wave-vector of lasers matters!

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **95**, 060701(R) (2017)

## Center-of-mass-momentum-dependent interaction between ultracold atoms

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[1]*Department of Physics, Renmin University of China, Beijing 100872, China*
[2]*Beijing Computational Science Research Center, Beijing 100084, China*
[3]*Beijing Key Laboratory of Opto-electronic Functional Materials & Micro-nano Devices, Renmin University of China, Beijing 100872, China*

(Received 21 September 2016; published 12 June 2017)

We show that a type of two-body interaction, which depends on the momentum of the center of mass (COM) of these two particles, can be realized in ultracold atom gases with a laser-modulated magnetic Feshbach resonance (MFR). Here the MFR is modulated by two laser beams propagating along different directions, which can induce Raman transition between two-body bound states. The Doppler effect causes the two-atom scattering length to be strongly dependent on the COM momentum of these two atoms. As a result, the effective two-atom interaction is COM-momentum dependent, while the one-atom free Hamiltonian is still the simple kinetic energy $\mathbf{p}^2/(2m)$.

**The Doppler effect can be significant in the dark-state regime!**

# The wave-vector of lasers matters!

Center of Mass Momentum Dependent Interaction Between Ultracold Atoms

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PACS numbers: 34.50.Cx, 34.50.Rk, 67.85.-d

## Stark shift in the case of EIT:

$$\Sigma_1(\mathbf{q}) = \frac{\Omega_1^2/4}{I_e(\mathbf{q}) - \Omega_2^2/[4I_2(\mathbf{q})]}$$

where,

$$I_e(\mathbf{q}) = \Delta_e + i\frac{\gamma_e}{2} - \frac{\hbar^2(\mathbf{q}+\mathbf{k}_1)^2}{4m}$$

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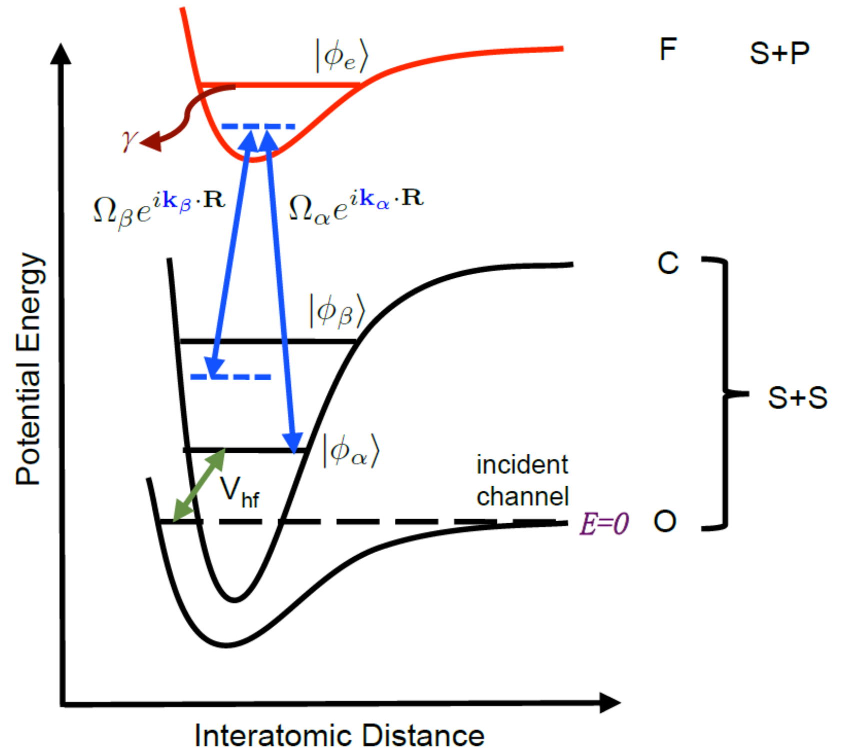


FIG. 1: (color online) Schematic diagram for the MFR modulated by Raman laser beams propagating along different directions (i.e., $\mathbf{k}_\alpha \neq \mathbf{k}_\beta$).

## The wave-vector of lasers matters!

$$\Sigma_1(\mathbf{q}) = \frac{\Omega_1^2/4}{I_e(\mathbf{q}) - \Omega_2^2/[4I_2(\mathbf{q})]}$$

Although $$\frac{\hbar^2(\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)^2}{4m} \sim 10\text{kHz} << I_e, \Omega_1, \Omega_2 \sim 1\text{MHz}$$

But, $$\frac{\hbar^2(\mathbf{q}+\mathbf{k}_1-\mathbf{k}_2)^2}{4m}\left(\frac{\Omega_1}{\Omega_2}\right)^2 \sim 1\text{MHz} \sim 0.1\Delta B\Delta\mu$$

leading to a **center-of-mass** dependent interatomic interaction, which breaks

**Galilean invariance**!

# The wave-vector of lasers matters!

Center of Mass Momentum Dependent Interaction Between Ultracold Atoms

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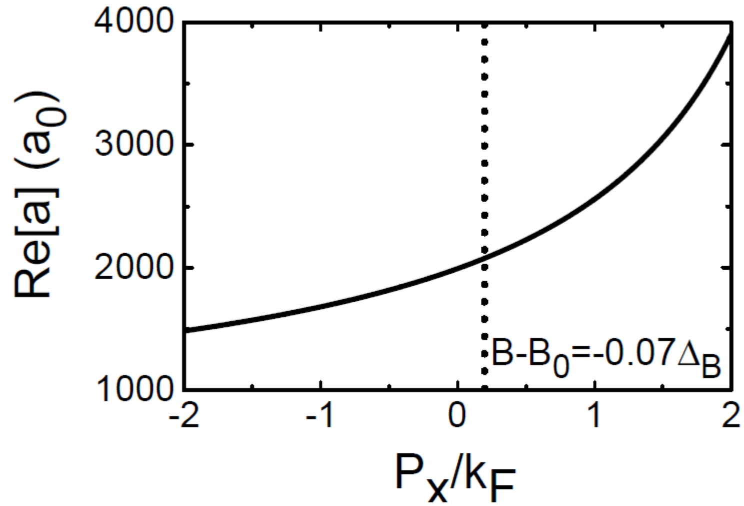


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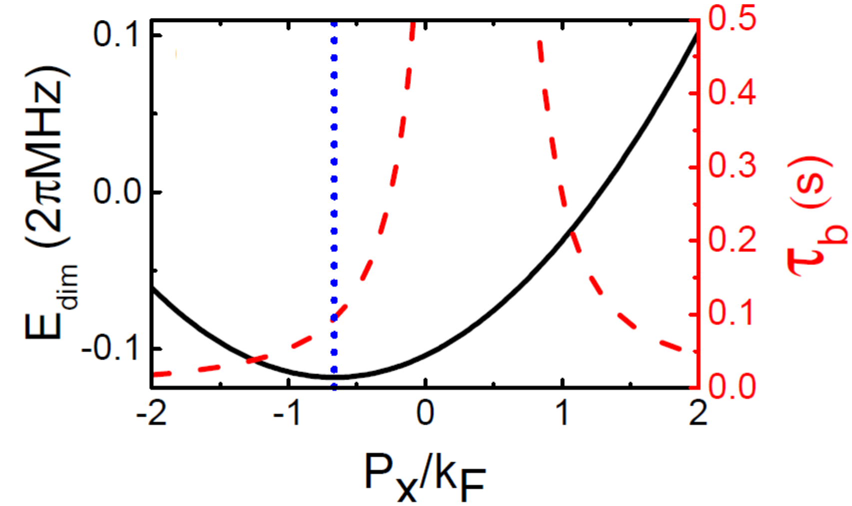



By using a parameter set for K-40 atoms, we may show the Stark shift is **tunable** and **significant**!

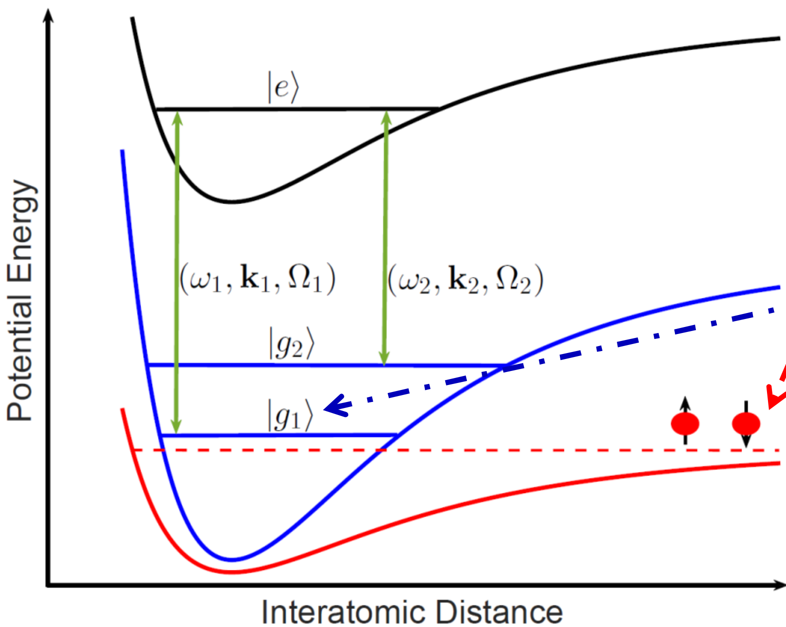
## **Part III: A new routine towards FF superfluids**

We now have a **long-lived** Fermi system with **Galilean invariance violation**, which is an ideal starting point to create **FF superfluids** ☺!

Lianyi He, HH & Xia-Ji Liu, *Phys. Rev. Lett.* **120**, 045302 (2018)

# Model Hamiltonian

$$-\eta(\Lambda) - \left[u_0(\Lambda) + \frac{g_0^2(\Lambda)}{E - \nu_0(\Lambda)}\right]^{-1} = -\frac{m}{4\pi a_{\rm bg}} \frac{E - \Delta\mu(B - B_0)}{E - \Delta\mu(B - B_0) + \Delta\mu \Delta B}$$




$$\mathcal{L}_{\rm A} = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger \hat{K}_{\rm F} \psi_\sigma - u_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow,$$

$$\mathcal{L}_{\rm M} = \varphi_1^\dagger \left(\hat{K}_{\rm B} - \nu_0\right) \varphi_1,$$

$$\mathcal{L}_{\rm AM} = -g_0 \left(\varphi_1^\dagger \psi_\downarrow \psi_\uparrow + \varphi_1 \psi_\uparrow^\dagger \psi_\downarrow^\dagger\right).$$

where,

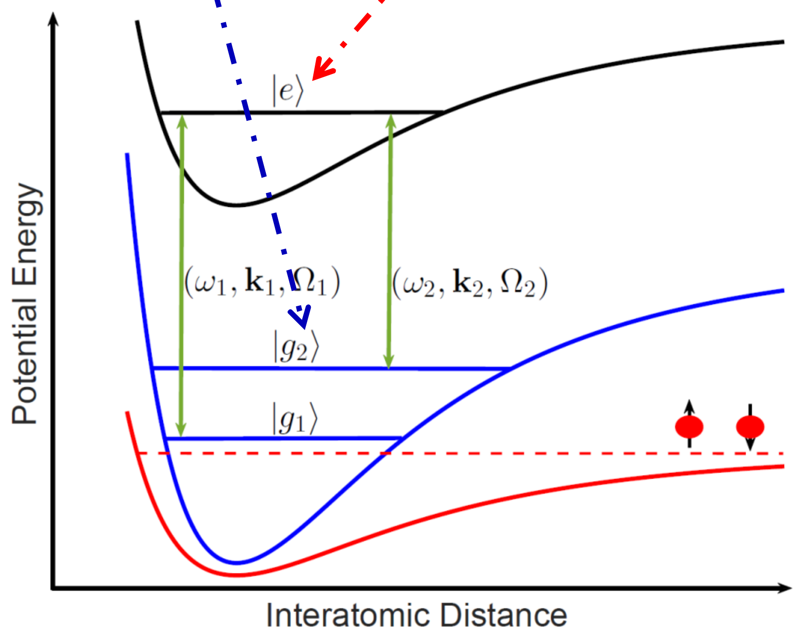
$$\hat{K}_{\rm F} = i\partial_t + \hbar^2 \nabla^2 / (2m)$$

$$\hat{K}_{\rm B} = i\partial_t + \hbar^2 \nabla^2 / (4m)$$

# Model Hamiltonian

$$\mathcal{L}'_{\rm M} = \varphi_2^\dagger\left(\hat{K}_{\rm B} - E_2\right)\varphi_2 + \varphi_{\rm e}^\dagger\left(\hat{K}_{\rm B} - E_{\rm e} + i\frac{\gamma_{\rm e}}{2}\right)\varphi_{\rm e} - \sum_{l=1,2}\left[\frac{\Omega_l}{2}\varphi_l\varphi_{\rm e}^\dagger e^{i\theta_l(\mathbf{r},t)} + \frac{\Omega_l^*}{2}\varphi_l^\dagger\varphi_{\rm e}e^{-i\theta_l(\mathbf{r},t)}\right],$$

where $\theta_l(\mathbf{r},t) = \mathbf{k}_l\cdot\mathbf{r} - \omega_l t$

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$$\hat{K}_{\rm F} = i\partial_t + \hbar^2\nabla^2/(2m)$$

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# Model Hamiltonian

By introducing new molecular fields $\phi_e = \varphi_e e^{-i\theta_1}$ and $\phi_2 = \varphi_2 e^{-i(\theta_1-\theta_2)}$ we may rewrite the molecular part (in momentum space) as,

$$\Im_M + \Im_M' = \left(\phi_1^+, \quad \phi_2^+, \quad \phi_3^+\right)\begin{bmatrix} I_1(q_0,\mathbf{q}) & 0 & -\Omega_1^*/2 \\ 0 & I_2(q_0,\mathbf{q}) & -\Omega_1^*/2 \\ -\Omega_1/2 & -\Omega_2/2 & I_e(q_0,\mathbf{q}) \end{bmatrix}\begin{pmatrix}\phi_1 \\ \phi_2 \\ \phi_3\end{pmatrix}$$

where,

$$I_1(q_0,\mathbf{q}) = \left(q_0 - \frac{\hbar^2\mathbf{q}^2}{4m}\right) - \nu_0$$

**violate Galilean invariance**

$$I_2(q_0,\mathbf{q}) = \left(q_0 - \frac{\hbar^2\mathbf{q}^2}{4m}\right) - \frac{\hbar^2\mathbf{q}\cdot(\mathbf{k}_1-\mathbf{k}_2)}{2m} - \frac{\hbar^2(\mathbf{k}_1-\mathbf{k}_2)^2}{4m} + \delta$$

**two-photon detuning**

$$I_e(q_0,\mathbf{q}) = \left(q_0 - \frac{\hbar^2\mathbf{q}^2}{4m}\right) - \frac{\hbar^2\mathbf{q}\cdot\mathbf{k}_1}{2m} - \frac{\hbar^2\mathbf{k}_1^2}{4m} + \Delta_e + i\frac{\gamma_e}{2}$$

**Galilean invariant combination**

**single-photon detuning**

## Model Hamiltonian

We may adiabatically eliminate the molecular states |*e*> and |2> and obtain the **self-energy** or the so-called **Stark shift** for the state |1>. Mathematically, we diagonalize ,

$$M(q_0,\mathbf{q}) = \begin{bmatrix} I_1(q_0,\mathbf{q}) & 0 & -\Omega_1^*/2 \\ 0 & I_2(q_0,\mathbf{q}) & -\Omega_1^*/2 \\ -\Omega_1/2 & -\Omega_2/2 & I_e(q_0,\mathbf{q}) \end{bmatrix}$$

The shift to its **11-component** is (i.e., $I_1(q_0,\mathbf{q}) \to I_1(q_0,\mathbf{q}) - \Sigma_1(q_0,\mathbf{q})$ ),

$$\Sigma_1(q_0,\mathbf{q}) = \frac{\Omega_1^2/4}{I_e(q_0,\mathbf{q}) - \Omega_2^2/[4I_2(q_0,\mathbf{q})]}$$

This leads to **an effective interaction strength**,

$$U_R(q_0,\mathbf{q}) = u + \frac{g^2}{[q_0 - \hbar^2\mathbf{q}^2/(4m) - \nu] - \Sigma_1(q_0,\mathbf{q})}$$

# Mean-field theory

For a theoretical description of this new many-body Fermi system ($\mathbf{k}_1 = -\mathbf{k}_2 \,//\, \mathbf{e}_z$), we consider the partition function and introduce explicitly a pairing order parameter,

$$\Delta(z) = \Delta e^{i\mathbf{Q}\cdot z}$$

At the mean-field level, we obtain the thermodynamic potential,

$$\Omega = \sum_{s=\pm} \sum_{\mathbf{k}} E_{\mathbf{k}}^{s} \Theta(-E_{\mathbf{k}}^{s}) + \sum_{\mathbf{k}} \left( \xi_{\mathbf{k}} - E_{\mathbf{k}} + \frac{|\Delta|^2}{2\varepsilon_{\mathbf{k}}} \right) - \frac{|\Delta|^2}{U_{\mathrm{R}}(2\mu, \mathbf{Q})}$$

where,

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} + \mathbf{Q}^2/(8m) - \mu$$

$$E_{\mathbf{k}} = (\xi_{\mathbf{k}}^2 + |\Delta|^2)^{1/2}$$

$$E_{\mathbf{k}}^{\pm} = E_{\mathbf{k}} \pm \mathbf{k} \cdot \mathbf{Q}/(2m)$$

and the effective interaction:

$$U_{\mathrm{R}}(2\mu, \mathbf{Q}) = u + \frac{g^2}{[2\mu - \hbar^2\mathbf{Q}^2/(4m)] - \nu - \Sigma_1(2\mu, \mathbf{Q})}$$

## Mean-field theory

We consider K-40 atoms near 202.02 G Feshbach resonance and take the following parameters:

$$\Omega_1 = 2\pi \times 120\text{MHz},$$

$$\Omega_2 = 2\pi \times 20\text{MHz},$$

$$\Delta_e = -2\pi \times 500\text{MHz}, \gamma_e = 2\pi \times 6\text{MHz}$$

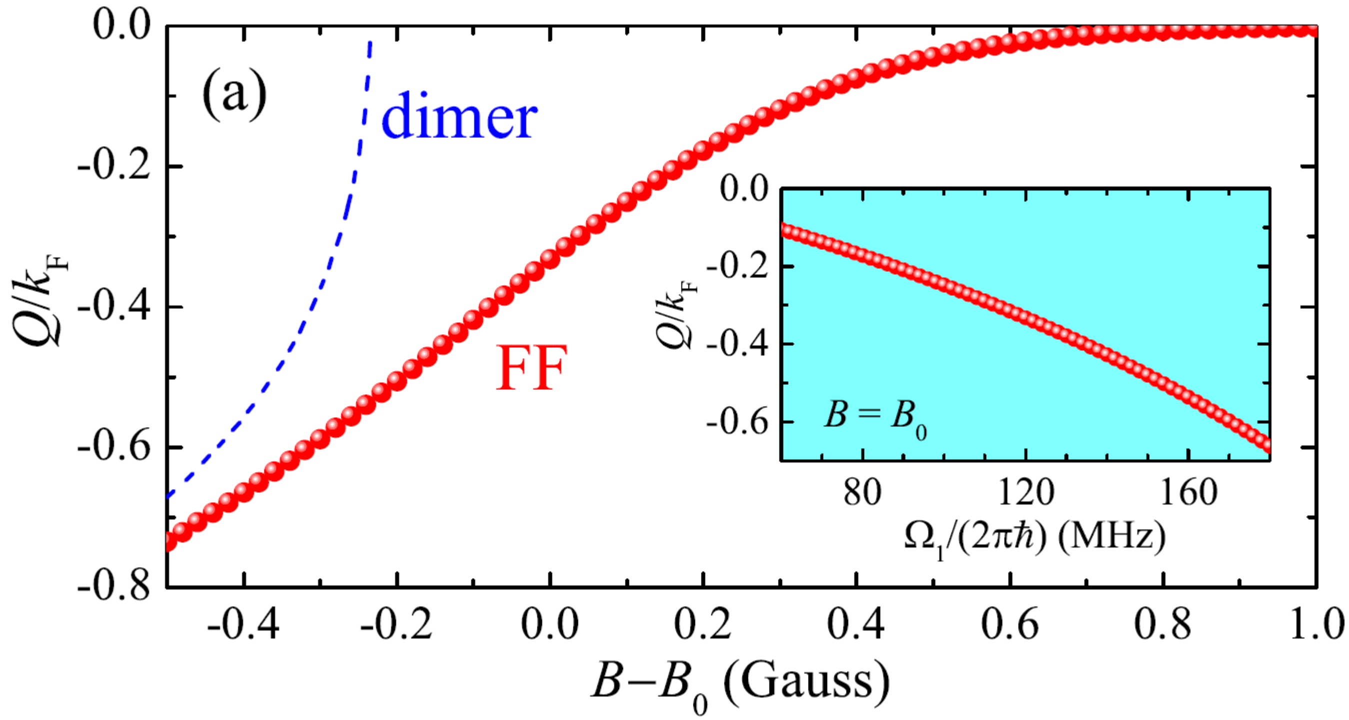
$$\delta = 0$$

$$k_R = k_F = 8.136 \times 10^6 \text{m}^{-1}$$

We solve the pairing gap and pairing momentum by,

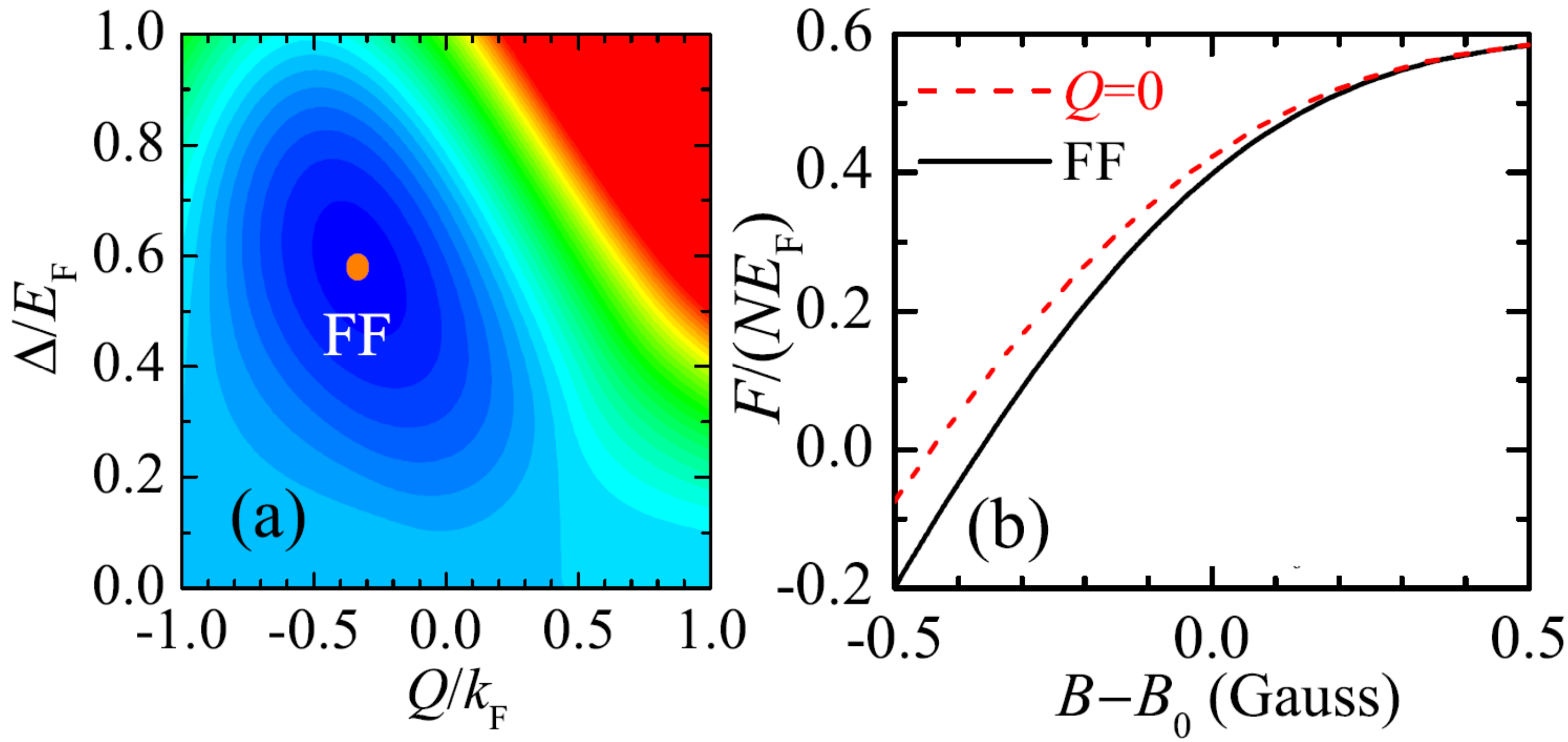
$$\frac{\partial \Omega}{\partial \Delta} = 0 \text{ and } \frac{\partial \Omega}{\partial Q} = 0$$

# Results: the emergence of a FF superfluid



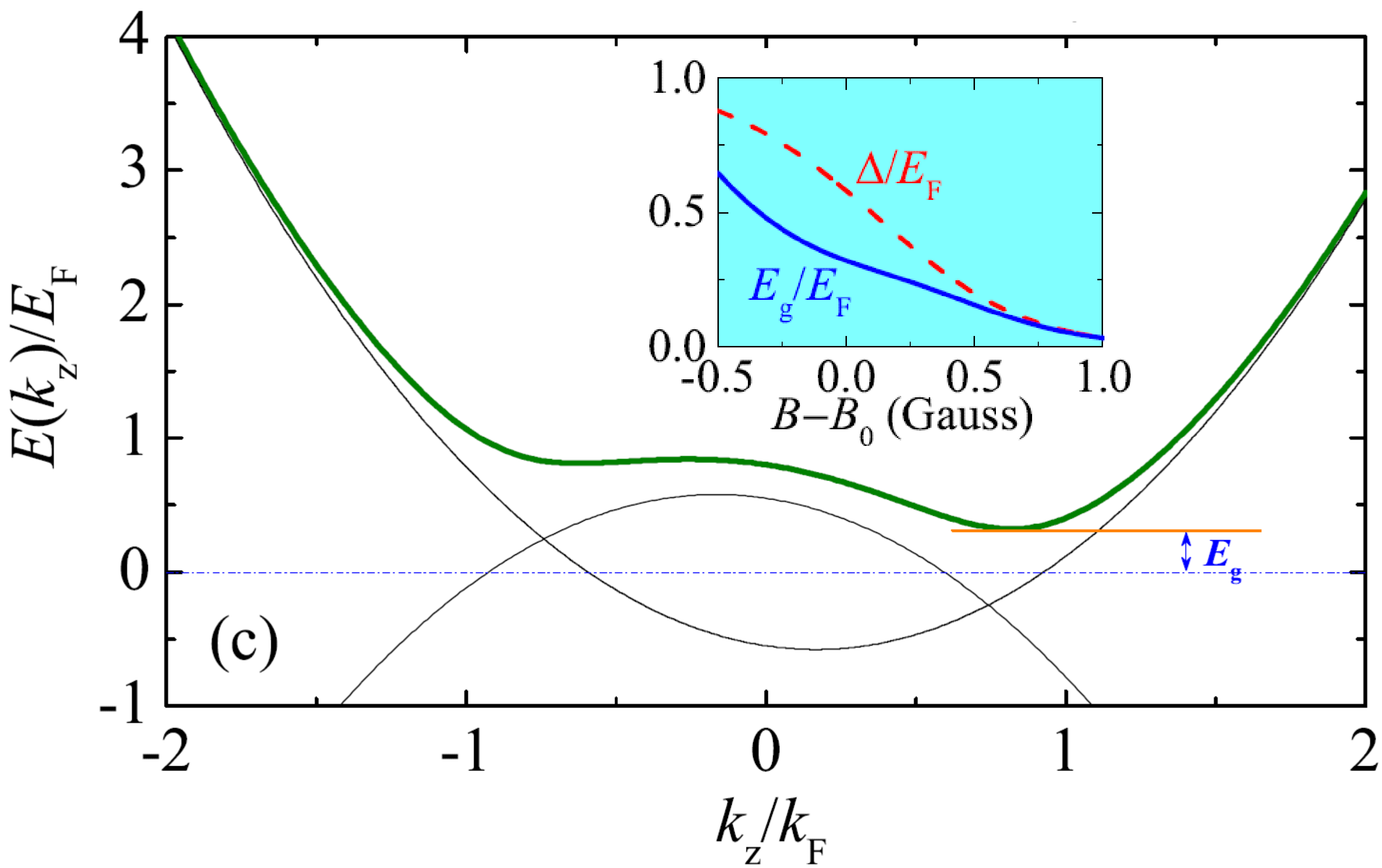

- We find **a FF superfluid** on the BCS side (excellent☺!);
- On the BEC side, we find a dimer with finite $Q$ (as anticipated);
- The pairing momentum $Q$ increases with increasing $\Omega_1$.

## Results: the emergence of a FF superfluid



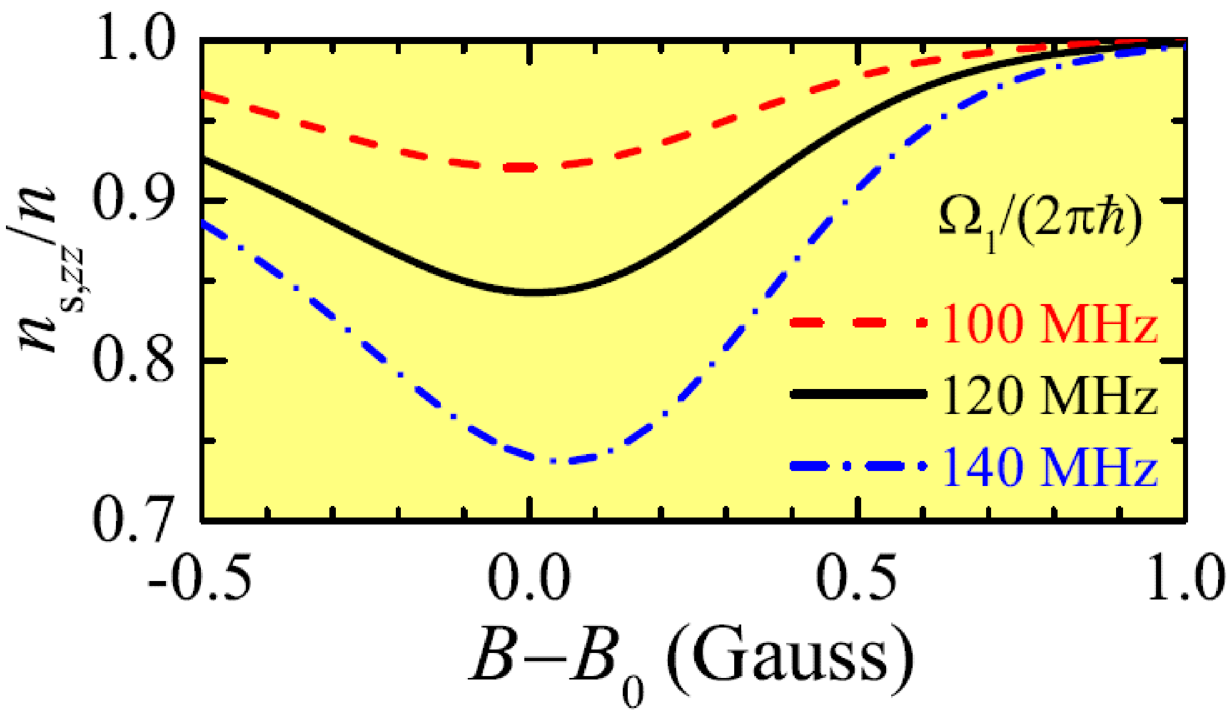

- **A FF superfluid** is energetically stable (excellent☺!).
- **Why $Q\neq0$?** Because spontaneously generated current due to violation of Galilean invariance: $\boldsymbol{j} = 2m\left(\frac{\partial\Omega}{\partial Q}\right)_{Q=0} \propto \boldsymbol{k}_1 - \boldsymbol{k}_2$

# Results: the manifest of the Galilean invariance violation



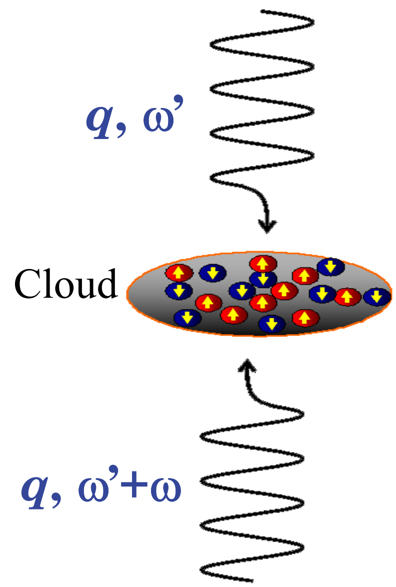

- The fermionic excitation spectrum is not symmetric;
- Due to the violation in Galilean invariance, the **energy gap** $E_g$ is not equal to the **pairing gap** $\Delta$.

## Results: the manifest of the Galilean invariance violation




- The **suppressed** **superfluid density** even at **zero** temperature!
- The consequence to the **two-fluid hydrodynamics** is unknown☹;
- The *same* suppression was predicted for a SOC Fermi gas.

# Dynamic structure factor $S(k, \omega)$!




The change in the first moment of density profile (*i.e.*, COM displacement) is proportional to the dynamic structure factor (DSF): $S(k=2q,\omega)$.

**Bragg spectroscopy** @ Swinburne: *PRL* **101**, 250403 (2008).

# Low-lying collective modes of a FF superfluid

The collective modes can be investigated by computing the effective action from the Gaussian fluctuations around the mean field [3]. The detailed derivation of the effective action will be presented in a long sequent paper. The effective action for the collective phonon mode, or the so-called Anderson-Bogoliubov mode of Fermi superfluidity, is given by

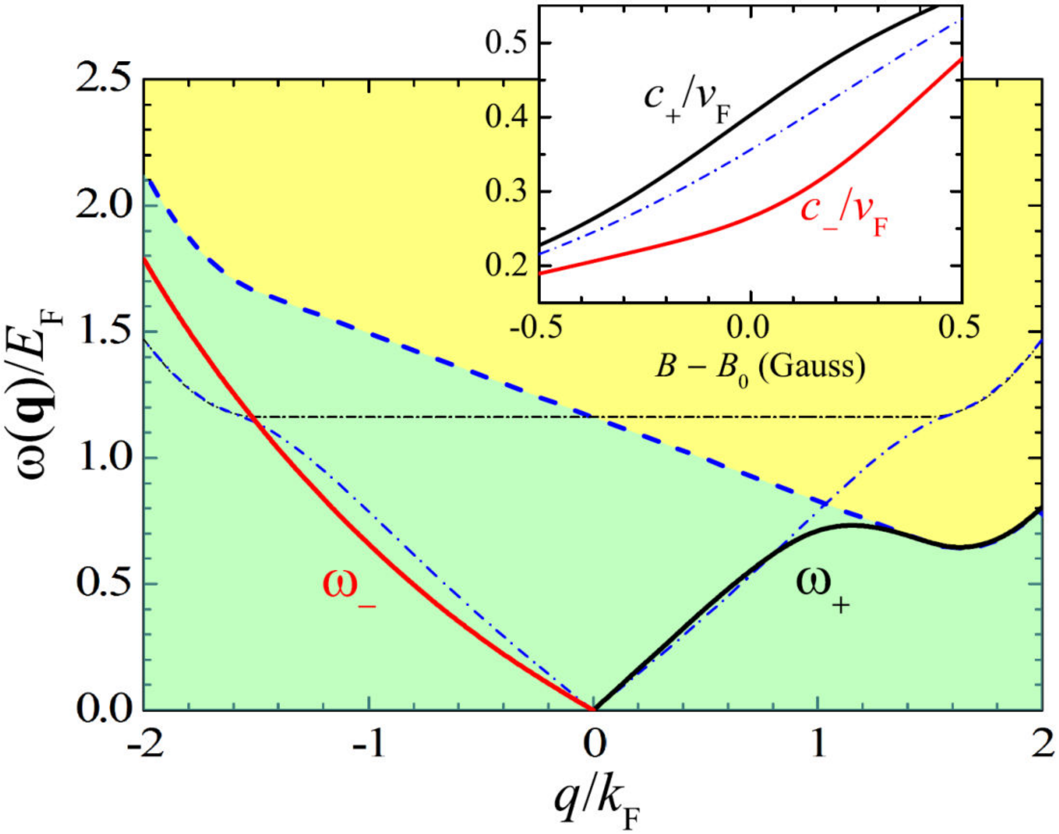
$$\mathcal{S}_{\rm coll} = \frac{1}{2}\sum_{q}\left(\delta\Delta_q^* \quad \delta\Delta_{-q}\right)\left[\begin{array}{cc} M_{11}(q) & M_{12}(q) \\ M_{21}(q) & M_{22}(q) \end{array}\right]\left(\begin{array}{c} \delta\Delta_q \\ \delta\Delta_{-q}^* \end{array}\right),$$

where we write $\Delta(x) = \Delta + \delta\Delta(x)$ with $\delta\Delta(x)$ being the quantum fluctuation around the mean field $\Delta$, and $\delta\Delta_q$ is the Fourier component of $\delta\Delta(x)$. Here $q = (i\nu_n, \mathbf{q})$ with $\nu_n = 2\pi nT$ being the boson Matsubara frequency. The inverse propagator matrix $M(q)$ determines the properties the collective modes. Its elements satisfies $M_{22}(q) = M_{11}(-q)$ and $M_{21}(q) = M_{12}(-q)$. The explicit form of $M_{11}(q)$ can be evaluated as

$$\begin{aligned} M_{11}(i\nu_n, \mathbf{q}) = & -\frac{1}{U_{\rm R}\left(i\nu_n + 2\mu, \mathbf{q}+\mathbf{Q}\right)} + \sum_{\mathbf{k}}\left[\frac{1}{2\varepsilon_{\mathbf{k}}} + u^2_{\mathbf{k}+\mathbf{q}/2}u^2_{\mathbf{k}-\mathbf{q}/2}\frac{1 - f^{(+)}_{\mathbf{k}+\mathbf{q}/2} - f^{(-)}_{\mathbf{k}-\mathbf{q}/2}}{i\nu_n - \mathbf{q}\cdot\mathbf{Q}/(2m) - E_{\mathbf{k}+\mathbf{q}/2} - E_{\mathbf{k}-\mathbf{q}/2}}\right. \\ & - u^2_{\mathbf{k}+\mathbf{q}/2}v^2_{\mathbf{k}-\mathbf{q}/2}\frac{f^{(+)}_{\mathbf{k}+\mathbf{q}/2} - f^{(+)}_{\mathbf{k}-\mathbf{q}/2}}{i\nu_n - \mathbf{q}\cdot\mathbf{Q}/(2m) - E_{\mathbf{k}+\mathbf{q}/2} + E_{\mathbf{k}-\mathbf{q}/2}} \\ & + v^2_{\mathbf{k}+\mathbf{q}/2}u^2_{\mathbf{k}-\mathbf{q}/2}\frac{f^{(-)}_{\mathbf{k}+\mathbf{q}/2} - f^{(-)}_{\mathbf{k}-\mathbf{q}/2}}{i\nu_n - \mathbf{q}\cdot\mathbf{Q}/(2m) + E_{\mathbf{k}+\mathbf{q}/2} - E_{\mathbf{k}-\mathbf{q}/2}} \\ & \left. - v^2_{\mathbf{k}+\mathbf{q}/2}v^2_{\mathbf{k}-\mathbf{q}/2}\frac{1 - f^{(-)}_{\mathbf{k}+\mathbf{q}/2} - f^{(+)}_{\mathbf{k}-\mathbf{q}/2}}{i\nu_n - \mathbf{q}\cdot\mathbf{Q}/(2m) + E_{\mathbf{k}+\mathbf{q}/2} + E_{\mathbf{k}-\mathbf{q}/2}}\right] \end{aligned}$$

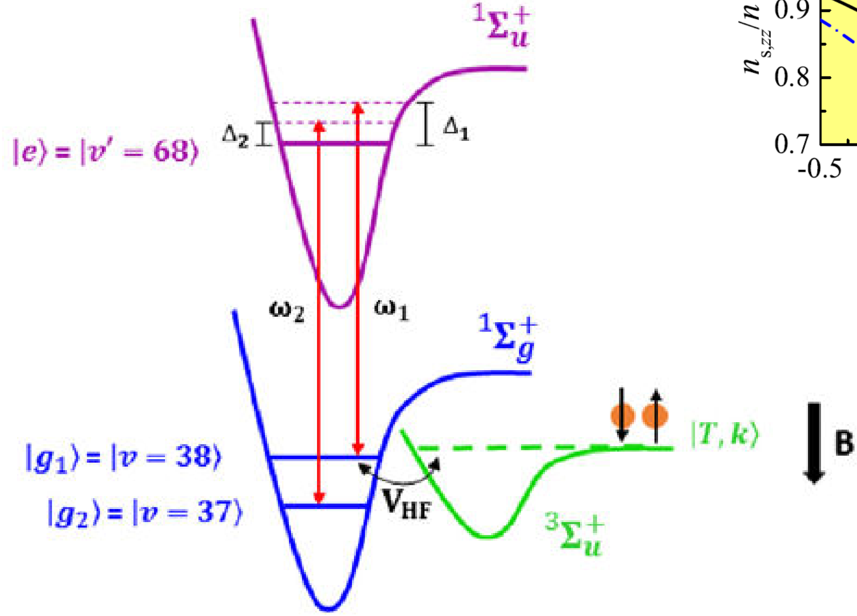
where $f^{(\pm)}_{\mathbf{k}} = f\left(E^{\pm}_{\mathbf{k}}\right)$ with $f(x) = 1/(e^{x/T}+1)$ being the Fermi-Dirac distribution. Here the BCS distributions are defined as $u^2_{\mathbf{k}} = (1 + \xi_{\mathbf{k}}/E_{\mathbf{k}})/2$ and $v^2_{\mathbf{k}} = 1 - u^2_{\mathbf{k}}$. The zero-temperature result is obtained by taking the limit $T \to 0$.

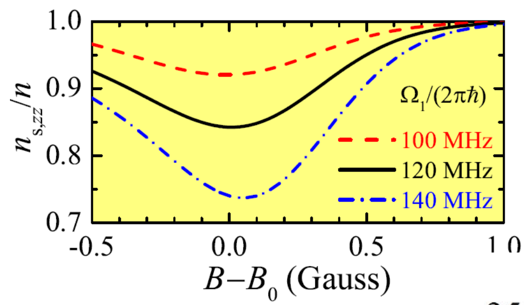
# Results: asymmetric phonon dispersion and roton-maxon?

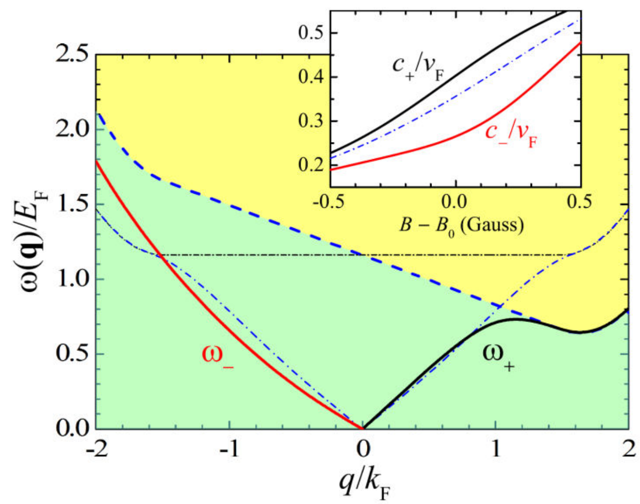



- **Asymmetric phonon dispersions** due to the FF momentum;
- **Two** sound velocities parallel (or opposite) to the FF momentum;
- The emergent **roton-maxon** structure at the 2p continuum.

# Taking home messages




The recently demonstrated **dark-state optical control** of magnetic Feshbach resonances may lead to a number of new interesting and **stable** many-body systems. Here, we only consider the center-of-mass dependent interatomic interactions.

In this case, the violation of Galilean invariance may lead to the **long-sought FF superfluids**, which features (i) anisotropic single-particle dispersion relation, (ii) suppressed superfluid density at zero temperature, (iii) anisotropic sound velocities, and (iv) rotonic collective modes. An atomic Fermi gas of K-40 could be a good candidate for observing FF superfluids.