Diagrammatic perturbation theories of a strongly interacting atomic Fermi gas - I

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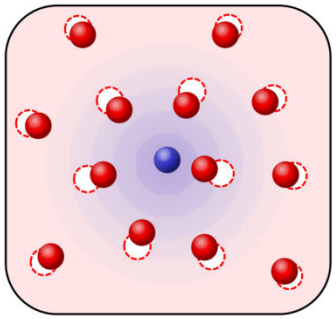
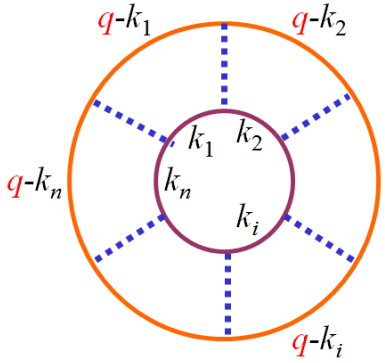
WIPM, CAS

Outline

- The standard procedure to establish **Feynman rules/diagrams**
- Application 1: Polaron problem – the simplest many-body systems



- Application 2: Nozières & Schmitt-Rink theory (pairing instability)
- Application 3: The BCS theory and GPF theory
- Application 4: Beyond-GPF (ε-expansion theory)



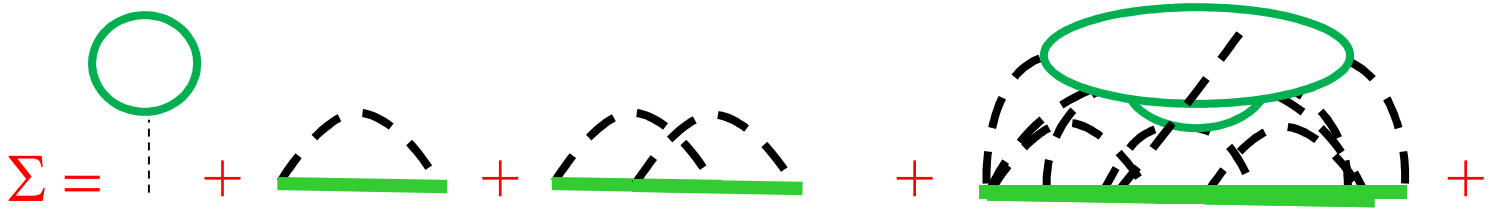
- **Any unsolved problems/challenges (FFLO)?**

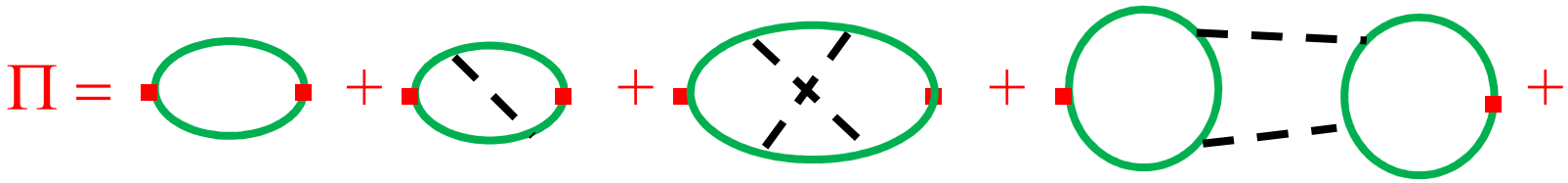
Feynman diagrams

For every physicist: Brain washed by **Feynman diagram** and **Feynman path-integral** techniques:

- Elegant and powerful to understand the physics: a simple diagram will give the essential picture!
- We may sum up all the diagrams one day…and hence solve the many-body problem completely!

Feynman diagrams





LETTERS
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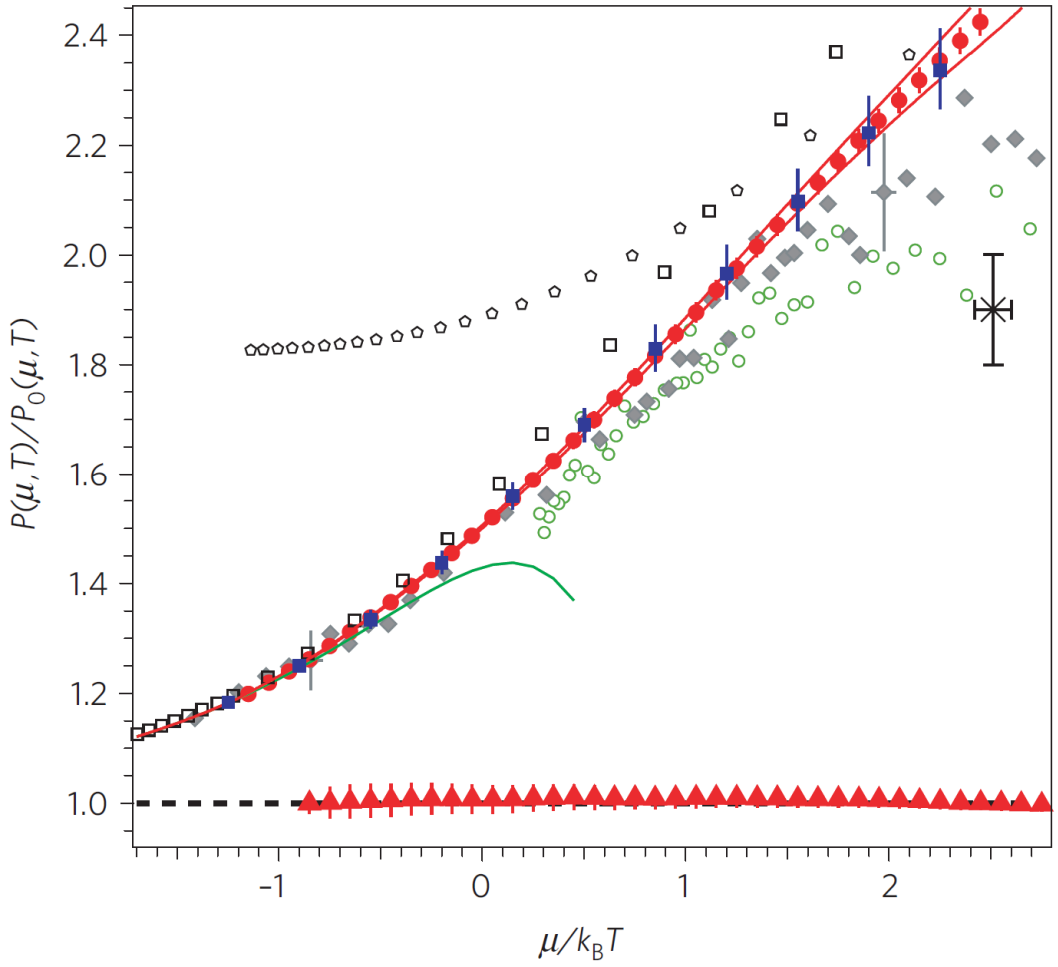
nature physics

Feynman diagrams versus Fermi-gas Feynman emulator

K. Van Houcke[1,2]*, F. Werner[1,3], E. Kozik[4,5], N. Prokof'ev[1,6], B. Svistunov[1,6], M. J. H. Ku[7], A. T. Sommer[7], L. W. Cheuk[7], A. Schirotzek[8] and M. W. Zwierlein[7]

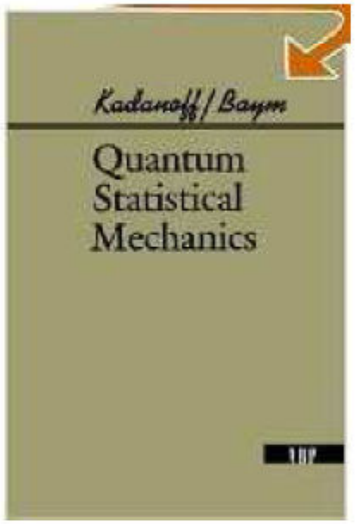
Precise understanding of strongly interacting fermions, from electrons in modern materials to nuclear matter, presents with zero-range interactions at infinite scattering length[4–6]. This system offers the unique possibility to stringently test our theory

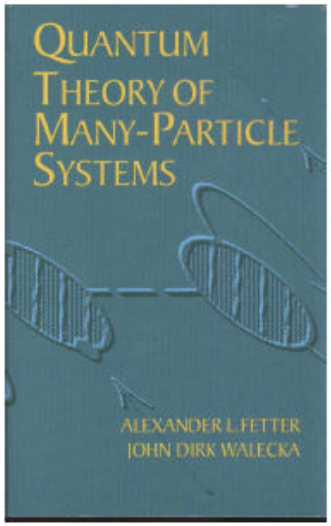
Feynman diagrams

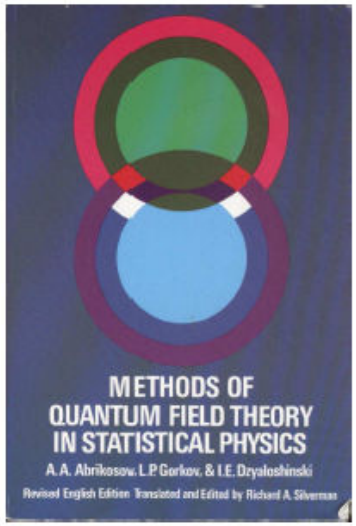


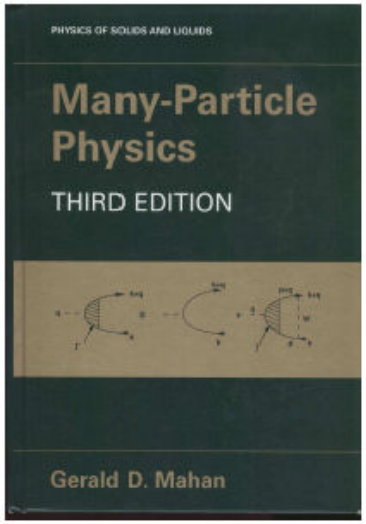
A few per-cent accuracy for the equation of state of a unitary Fermi gas:
Feynman diagrams vs. Fermi gas quantum emulator (Science 2012)

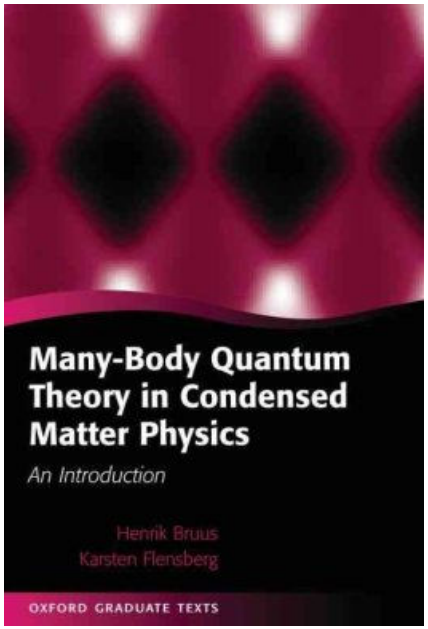
Standard textbooks





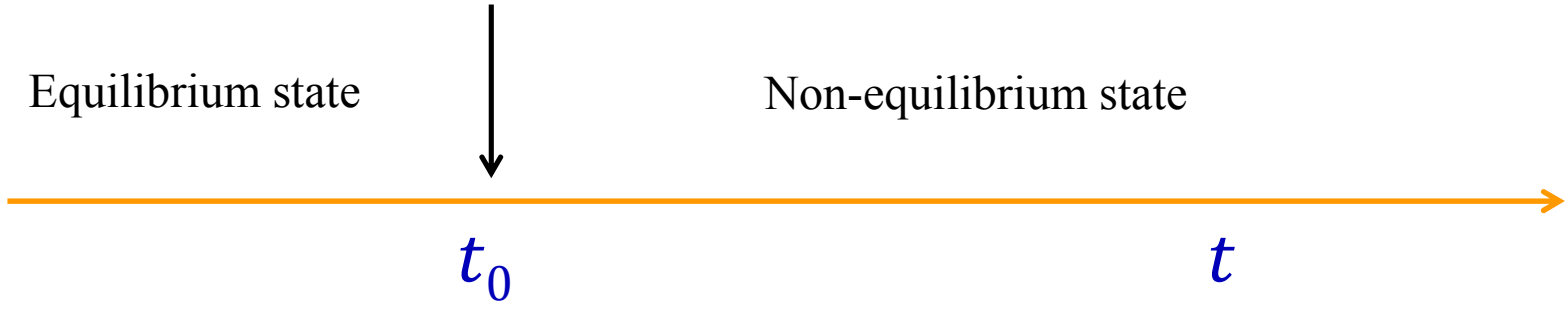






Kubo linear response at a glance

Add an external perturbation V to the system described by H_0:



$$\langle A \rangle = A_0 \qquad \delta\langle A \rangle \sim -i\theta(t-t')\langle [A(t), V(t')] \rangle$$

retarded response functions $C^R(t-t')$

where

$$\langle A \rangle = \frac{1}{Z_0} \mathrm{Tr}\left[\rho_0 A\right] = \frac{1}{Z_0} \sum_n \langle n|A|n\rangle e^{-\beta E_n}$$

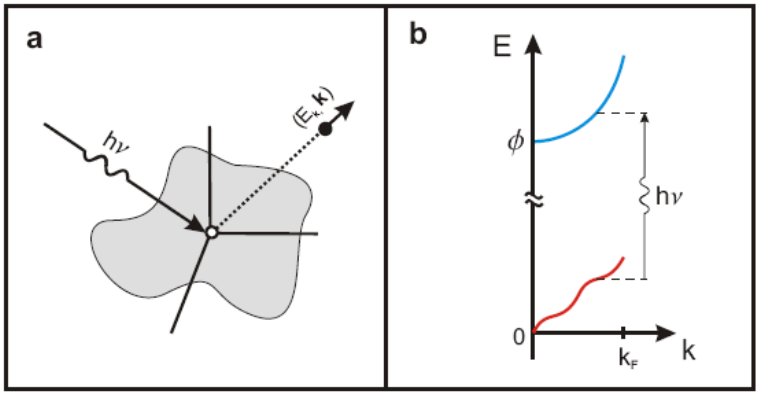
$$\rho_0 = e^{-\beta H_0} = \sum_n |n\rangle\langle n| e^{-\beta E_n}$$

Fluctuation-dissipation theorem

In the frequency domain, the response function $-\frac{1}{\pi}\mathrm{Im}\boldsymbol{C^R(\omega)}$ is directly measurable. Two very important examples:

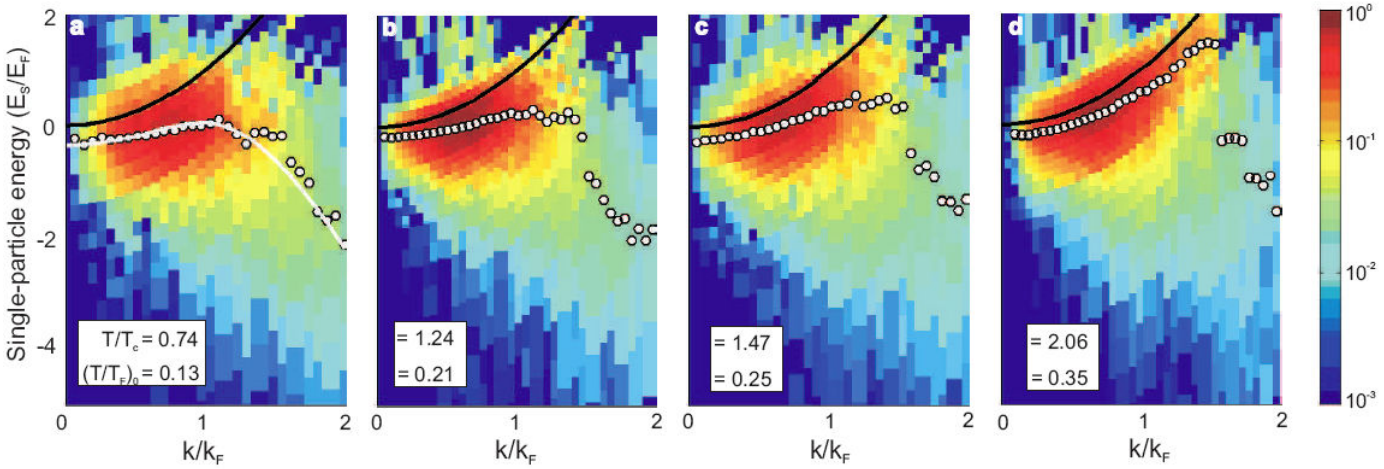
- ARPES/momentum-resolved rf (single-particle)
- Neutron/Bragg scattering (collective modes)

Single-particle spectral function $A(k,\omega)$



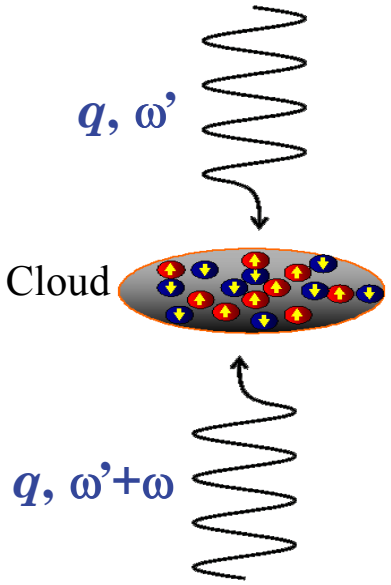
$\propto f(\omega)A(k,\omega)$

ARPES versus momentum resolved rf-spectroscopy; JILA, ***Nature*** (2008)



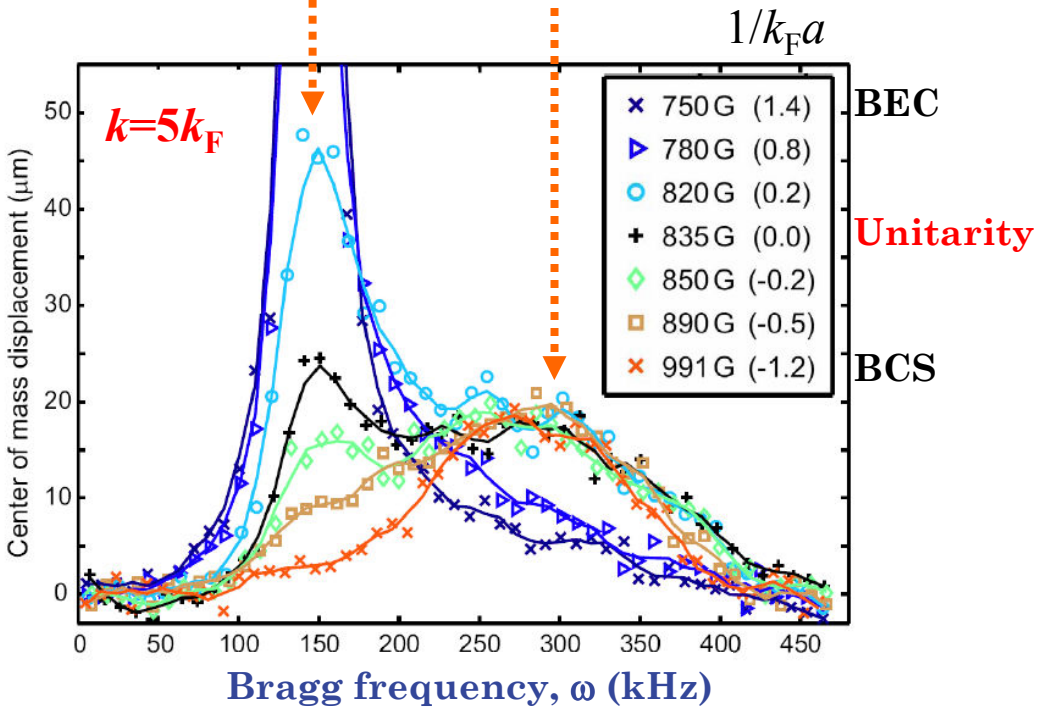
JILA, ***Nature Physics* 6**, 569 (2010): Pseudo-gap!

Dynamic structure factor $S(k, \omega)$



The change in the first moment of density profile (*i.e.*, COM displacement) is proportional to the dynamic structure factor (DSF): $S(k=2q,\omega)$.

molecule resonance freq. $\frac{\hbar^2 k^2}{2(2m)}$ $\omega_r = \frac{\hbar^2 k^2}{2m}$ atom resonance freq.



Swinburne group (Chris Vale), *PRL* **101**, 250403 (2008).

Response functions

We are interested in the **retarded** response function:

$$C^R(t,t') = -i\theta(t-t')\left\langle [A(t), B(t')]_{\pm} \right\rangle \qquad \begin{cases} +: & bosons \\ -: & fermions \end{cases}$$

But, let us consider first the response function:

$$C_{AB}(t,t') = -\langle A(t)B(t')\rangle$$

from which we find the retarded function as $C^R = i\theta(t-t')(C_{AB} \mp C_{BA})$. By definition, we have

$$C_{AB}(t,t') = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}A(t)B(t')\right]$$

Now, let us specify the **time-dependence** of the operators, if the Hamiltonian is $\boldsymbol{H = H_0 + V}$, where $\boldsymbol{V}$ is the interaction Hamiltonian.

5.1 The Schrödinger picture

The Schrödinger picture is useful when dealing with a time-independent Hamiltonian H, i.e. $\partial_t H = 0$. Any other operator A may or may not depend on time. The state vectors $|\psi(t)\rangle$ does depend on time, and their time evolution is governed by Schrödinger's equation. The time-independence of H leads to a simple formal solution:

$$i\hbar\partial_t |\psi(t)\rangle = H |\psi(t)\rangle \quad \Rightarrow \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht} |\psi_0\rangle. \tag{5.1}$$

In the following we will measure the energy in units of frequency, such that $\hbar$ drops out of the time-evolution equations: $\varepsilon/\hbar \to \varepsilon$ and $H/\hbar \to H$. At the end of the calculations one can easily convert frequencies back to energies. With this notation we can summarize the Schrödinger picture with its states $|\psi(t)\rangle$ and operators A as:

$$\text{The Schrödinger picture} \begin{cases} \text{states :} & |\psi(t)\rangle = e^{-iHt} |\psi_0\rangle, \\ \text{operators :} & A, \text{ may or may not depend on time.} \\ & H, \text{ does not depend on time.} \end{cases} \tag{5.2}$$

To interpret the operator e^{-iHt} we recall that a function $f(B)$ of any operator B is defined by the Taylor expansion of f,

$$f(B) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} B^n. \tag{5.3}$$

5.2 The Heisenberg picture

The central idea behind the Heisenberg picture is to obtain a representation where all the time dependence is transferred to the operators, $A(t)$, leaving the state vectors $|\psi_0\rangle$ time independent. The Hamiltonian H remains time-independent in the Heisenberg picture. If the matrix elements of any operator between any two states are identical in the two representations, then the two representations are fully equivalent. By using Eq. (5.2) we obtain the identity

$$\langle\psi'(t)|A|\psi(t)\rangle \quad = \quad \langle\psi'_0|e^{iHt}Ae^{-iHt}|\psi_0\rangle \quad \equiv \quad \langle\psi'_0|A(t)|\psi_0\rangle. \tag{5.4}$$

Thus we see that the correspondence between the Heisenberg picture with time-independent state vectors $|\psi_0\rangle$, but time-dependent operators $A(t)$, and the Schrödinger picture is given by the unitary transformation operator $\exp(iHt)$,

$$\text{The Heisenberg picture} \begin{cases} \text{states :} & |\psi_0\rangle \equiv e^{iHt}\,|\psi(t)\rangle, \\ \text{operators :} & A(t) \equiv e^{iHt}A\,e^{-iHt}. \\ & H \quad \text{does not depend on time.} \end{cases} \tag{5.5}$$

In (retarded) response functions, we always assume the **Heisenberg picture** for the **time-dependence** of the operators.

5.3 The interaction picture

The third and last representation, the interaction picture, is introduced to deal with the situation where a system described by a time-independent Hamiltonian H_0, with known energy eigenstates $|n_0\rangle$, is perturbed by some, possibly time-dependent, interaction $V(t)$,

$$H = H_0 + V(t), \qquad \text{with} \quad H_0|n_0\rangle = \varepsilon_{n_0}|n_0\rangle. \tag{5.7}$$

The key idea behind the interaction picture is to separate the trivial time evolution due to H_0 from the intricate one due to $V(t)$. This is obtained by using only H_0, not the full H, in the unitary transformation Eq. (5.5). As a result, in the interaction picture both the state vectors $|\hat{\psi}(t)\rangle$ and the operators $\hat{A}(t)$ depend on time. The defining equations for the interaction picture are

$$\text{The interaction picture} \begin{cases} \text{states :} & |\hat{\psi}(t)\rangle \equiv e^{iH_0t}\,|\psi(t)\rangle, \\ \text{operators :} & \hat{A}(t) \equiv e^{iH_0t}\,A\,e^{-iH_0t}. \\ & H_0 \quad \text{does not depend on time.} \end{cases} \tag{5.8}$$

The interaction picture and the Heisenberg picture coincide when $V = 0$; i.e., in the non-perturbed case. If $V(t)$ is a weak perturbation, then one can think of Eq. (5.8) as a way to pull out the fast, but trivial, time dependence due to H_0, leaving states that vary only slowly in time due to $V(t)$.

Note that: from now on the **hat** above the operator $\hat{}$ means "the interaction picture"

The interaction picture

The first hint of the usefulness of the interaction picture comes from calculating the time derivative of $|\hat{\psi}(t)\rangle$ using the definition Eq. (5.8):

$$i\partial_t|\hat{\psi}(t)\rangle \quad = \quad \left(i\partial_t e^{iH_0t}\right)|\psi(t)\rangle + e^{iH_0t}\left(i\partial_t|\psi(t)\rangle\right) \quad = \quad e^{iH_0t}(-H_0+H)|\psi(t)\rangle, \quad (5.9)$$

which by Eq. (5.8) is reduced to

$$i\partial_t|\hat{\psi}(t)\rangle = \hat{V}(t)\,|\hat{\psi}(t)\rangle. \quad (5.10)$$

The resulting Schrödinger equation for $|\hat{\psi}(t)\rangle$ thus contains explicit reference only to the interaction part $\hat{V}(t)$ of the full Hamiltonian H. This means that in the interaction picture the time evolution of a state $|\hat{\psi}(t_0)\rangle$ from time t_0 to t must be given in terms of a unitary operator $\hat{U}(t,t_0)$ which also only depends on $\hat{V}(t)$. $\hat{U}(t,t_0)$ is completely determined by

$$\boxed{|\hat{\psi}(t)\rangle = \hat{U}(t,t_0)\,|\hat{\psi}(t_0)\rangle.} \quad (5.11)$$

which leads to,

$$i\partial_t\,\hat{U}(t,t_0) = \hat{V}(t)\,\hat{U}(t,t_0), \qquad \hat{U}(t_0,t_0) = 1. \quad (5.13)$$

By integration of this differential equation we get the integral equation

$$\hat{U}(t,t_0) = 1 + \frac{1}{i}\int_{t_0}^{t} dt'\,\hat{V}(t')\,\hat{U}(t',t_0), \quad (5.14)$$

The interaction picture

which we can solve iteratively for $\hat{U}(t,t_0)$ starting from $\hat{U}(t',t_0)=1$. The solution is

$$\hat{U}(t,t_0)=1+\frac{1}{i}\int_{t_0}^{t}dt_1\,\hat{V}(t_1)+\frac{1}{i^2}\int_{t_0}^{t}dt_1\,\hat{V}(t_1)\int_{t_0}^{t_1}dt_2\,\hat{V}(t_2)+\ldots \tag{5.15}$$

Note that in the iteration the ordering of all operators is carefully kept. A more compact form is obtained by the following rewriting. Consider for example the second order term, paying special attention to the dummy variables t_1 and t_2:

$$\begin{aligned}
&\int_{t_0}^{t}dt_1\,\hat{V}(t_1)\int_{t_0}^{t_1}dt_2\,\hat{V}(t_2)\\
&=\frac{1}{2}\int_{t_0}^{t}dt_1\,\hat{V}(t_1)\int_{t_0}^{t_1}dt_2\,\hat{V}(t_2)+\frac{1}{2}\int_{t_0}^{t}dt_2\,\hat{V}(t_2)\int_{t_0}^{t_2}dt_1\,\hat{V}(t_1)\\
&=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\,\hat{V}(t_1)\hat{V}(t_2)\theta(t_1-t_2)+\frac{1}{2}\int_{t_0}^{t}dt_2\int_{t_0}^{t}dt_1\,\hat{V}(t_2)\hat{V}(t_1)\theta(t_2-t_1)\\
&=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\left[\hat{V}(t_1)\hat{V}(t_2)\theta(t_1-t_2)+\hat{V}(t_2)\hat{V}(t_1)\theta(t_2-t_1)\right]\\
&\equiv\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\,T_t[\hat{V}(t_1)\hat{V}(t_2)],
\end{aligned} \tag{5.16}$$

where we have introduced the time ordering operator T_t. Time ordering is easily generalized to higher order terms.

The interaction picture

Using the time ordering operator, we obtain the final compact form:

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{i}\right)^n \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n \, T_t\Big(\hat{V}(t_1)\ldots\hat{V}(t_n)\Big) = T_t\Big(e^{-i\int_{t_0}^{t} dt' \, \hat{V}(t')}\Big). \quad (5.18)$$

A graphical sketch of the contents of the formula is given in Fig. 5.1.

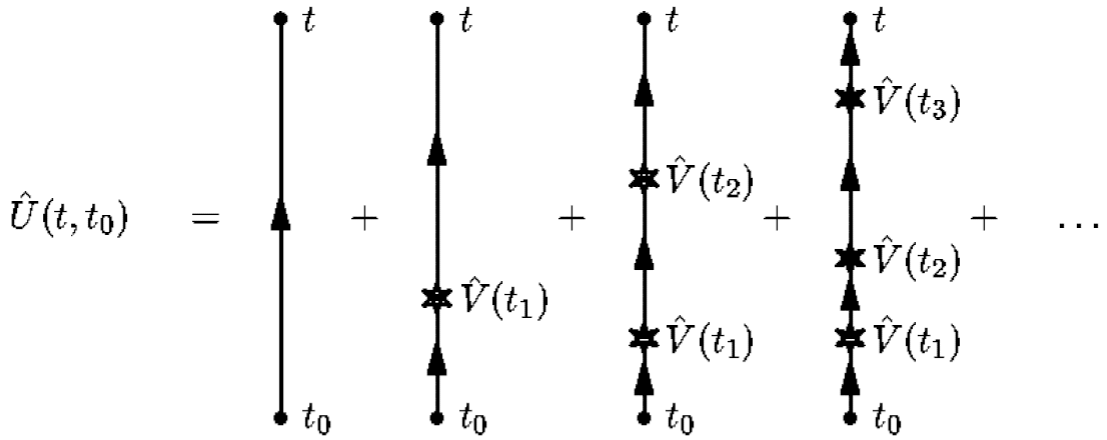


Figure 5.1: The time evolution operator $\hat{U}(t,t_0)$ can be viewed as the sum of additional phase factors due to $\hat{V}$ on top of the trivial phase factors arising from H_0. The sum contains contributions from processes with $0, 1, 2, 3, \ldots$ scattering events $\hat{V}$, which happen during the evolution from time t_0 to time t.

The interaction picture

In a brief summary, we have a unitary operator,

$$\hat{U}(t,t_0) \equiv e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} = T_t \left[e^{-i\int_{t_0}^{t} dt' \hat{V}(t')} \right]$$

For the states,

$$\left[\hat{\psi}(t)\right]_I \equiv e^{iH_0 t} [\psi(t)]_S = e^{iH_0 t} e^{-iHt} [\psi(0)]_H$$

For the operators,

$$\begin{aligned} \left[\hat{A}(t)\right]_I &\equiv e^{iH_0 t} [A]_S e^{-iH_0 t} \\ &= e^{iH_0 t} e^{-iHt} [A(t)]_H e^{iHt} e^{-iH_0 t} \\ &= \hat{U}(t,0) [A(t)]_H \hat{U}(0,t) \end{aligned}$$

Or,

$$[A(t)]_H = \hat{U}(0,t) \left[\hat{A}(t)\right]_I \hat{U}(t,0)$$

Response functions

Recall that in the Heisenberg picture, the response function is

$$C_{AB}(t,t') = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}A(t)B(t')\right]$$

In the interaction picture, it takes the form,

$$C_{AB}(t,t') = -\frac{1}{Z}\mathrm{Tr}\left[\boldsymbol{e^{-\beta H}}\hat{U}(0,t)\hat{A}(t)\hat{U}(t,t')\hat{B}(t')\hat{U}(t',0)\right]$$

In Eq. (5.18) we saw also how a single $\hat{U}$ operator could be expanded as a time-ordered exponential. This would in Eq. (10.4) result in three time-ordered exponentials, which could be collected into a single time-ordered exponential. But the trouble arises for the density matrix $e^{-\beta H}$, which should also be expanded in powers of the interaction. To make a long story short: this is a mess and a new idea is therefore needed. The solution to this problem is to use imaginary times instead of real times, but bare in mind that this is purely a mathematical trick without physical contents.

$$t \to -i\tau$$

Response functions

Thus, similar to the interaction picture defined for real times, we can define the interaction picture for **imaginary times** as,

$$\left[\hat{A}(\tau)\right]_I \equiv e^{H_0\tau}[A]_S e^{-H_0\tau}$$

And, the time-evolution operator $\hat{U}$ in the interaction picture is,

$$\hat{U}(\tau,\tau') = e^{\tau H_0} e^{-(\tau-\tau')H} e^{-\tau' H_0}. \tag{10.9}$$

From this it follows directly that

$$\hat{U}(\tau,\tau'')\hat{U}(\tau'',\tau') = \hat{U}(\tau,\tau'). \tag{10.10}$$

An explicit expression for $U(\tau,\tau')$ is found in analogy with the derivation of Eq. (5.18). First we differentiate Eq. (10.9) with respect to τ and find

$$\partial_\tau \hat{U}(\tau,\tau') = e^{\tau H_0}(H_0 - H)e^{-(\tau-\tau')H} e^{-\tau' H_0} = -\hat{V}(\tau)\hat{U}(\tau,\tau'). \tag{10.11}$$

This is analogous to Eq. (5.13) and the boundary condition, $\hat{U}(\tau,\tau) = 1$, is of course the same. Now the same iterative procedure is applied and we end with

$$\begin{aligned}\hat{U}(\tau,\tau') &= \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \int_{\tau'}^{\tau} d\tau_1 \cdots \int_{\tau'}^{\tau} d\tau_n \, T_\tau \left(\hat{V}(\tau_1)\cdots\hat{V}(\tau_n)\right) \\ &= T_\tau \exp\left(-\int_{\tau'}^{\tau} d\tau_1 \hat{V}(\tau_1)\right).\end{aligned} \tag{10.12}$$

The time ordering is again the same as defined in Sec. 5.3, i.e. the operators are ordered such that $T_\tau(A(\tau)B(\tau'))$ is equal to $A(\tau)B(\tau')$ for $\tau > \tau'$ and $B(\tau')A(\tau)$ when $\tau' > \tau$.

Temperature response functions/Green functions

On the other hand, we may write the density matrix operator (**check by yourself**),

$$\boldsymbol{e^{-\beta H}} = e^{-\beta H_0}\,\hat{U}(\beta,0) = e^{-\beta H_0} T_\tau\left[e^{-\int_0^\beta d\tau' \hat{V}(\tau')}\right]$$

Let us now define **temperature/thermal/Matsubara** response function,

$$C_{AB}(\tau,\tau') \equiv -\langle T_\tau[A(\tau)B(\tau')]\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H} T_\tau A(\tau)B(\tau')\right]$$

Thus, we find that,

$$C_{AB}(\tau,\tau') = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0) T_\tau\left\{\hat{U}(0,\tau)\hat{A}(\tau)\hat{U}(\tau,\tau')\hat{B}(\tau')\hat{U}(\tau',0)\right\}\right]$$

$$= -\frac{\mathrm{Tr}\left[e^{-\beta H_0} T_\tau\left\{\hat{U}(\beta,0)\hat{A}(\tau)\hat{B}(\tau')\right\}\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0)\right]}$$

$$= -\frac{\left\langle T_\tau\left\{\hat{U}(\beta,0)\hat{A}(\tau)\hat{B}(\tau')\right\}\right\rangle_0}{\left\langle \hat{U}(\beta,0)\right\rangle_0}$$

where we have used $Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0)\right]$, and where the averages $\langle\cdots\rangle_0$ depending on $e^{-\beta H_0}$ appear after normalizing with $1/Z_0 = 1/\mathrm{Tr}\left[e^{-\beta H_0}\right]$.

Temperature response functions/Green functions

Temperature/thermal/Matsubara Green functions,

$$C_{AB}(\tau,\tau') \equiv -\langle T_\tau[A(\tau)B(\tau')]\rangle$$
$$= -\theta(\tau-\tau')\langle A(\tau)B(\tau')\rangle \mp \theta(\tau'-\tau)\langle B(\tau')A(\tau)\rangle$$

What values can the imaginary time $\boldsymbol{\tau}$ have?

First, $\boldsymbol{C_{AB}(\tau,\tau') = C_{AB}(\tau-\tau')}$.

Proof: Let us consider the case $\boldsymbol{\tau > \tau'}$

$$\mathcal{C}_{AB}(\tau,\tau') = \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{\tau H}Ae^{-\tau H}e^{\tau' H}Be^{-\tau' H}\right]$$
$$= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{-\tau' H}e^{\tau H}Ae^{-\tau H}e^{\tau' H}B\right]$$
$$= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{(\tau-\tau')H}Ae^{-(\tau-\tau')H}B\right]$$
$$= \mathcal{C}_{AB}(\tau-\tau'),$$

and of course, likewise for the case $\boldsymbol{\tau < \tau'}$ (**check!**). Therefore, we may set $\boldsymbol{\tau' = 0}$.

Temperature response functions/Green functions

Temperature/thermal/Matsubara Green functions,

$$C_{AB}(\tau,\tau') \equiv -\langle T_\tau[A(\tau)B(\tau')]\rangle$$
$$= -\theta(\tau-\tau')\langle A(\tau)B(\tau')\rangle \mp \theta(\tau'-\tau)\langle B(\tau')A(\tau)\rangle$$

What values can the imaginary time $\boldsymbol{\tau}$ have?

Second, $\boldsymbol{C_{AB}(\tau) = \pm C_{AB}(\tau+\beta)}$. **"+" for bosons and "–" for fermions**

Proof: Let us consider the case $\boldsymbol{\tau < 0}$ (similar proof for $\boldsymbol{\tau > 0}$ (**check by yourself**)):

$$\begin{aligned}
\mathcal{C}_{AB}(\tau+\beta) &= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{(\tau+\beta)H}Ae^{-(\tau+\beta)H}B\right] \\
&= \frac{-1}{Z}\mathrm{Tr}\left[e^{\tau H}Ae^{-\tau H}e^{-\beta H}B\right] \\
&= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}Be^{\tau H}Ae^{-\tau H}\right] \\
&= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}BA(\tau)\right] \\
&= \pm\frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}T_\tau\left(A(\tau)B\right)\right] \\
&= \pm\mathcal{C}_{AB}(\tau),
\end{aligned}$$

Temperature response functions/Green functions

Let us consider the **Fourier transform** of Matsubara Green functions,

Recall that, $C_{AB}(\tau) = \pm C_{AB}(\tau + \beta)$. **"+" for bosons and "–" for fermions**

Note that the **length scale** along the imaginary time is $\beta = 1/(k_B T)$, we have,

$$C_{AB}(\tau - \tau') = k_B T \sum_{n=-\infty}^{+\infty} e^{-i\omega_n(\tau-\tau')} C_{AB}(i\omega_n)$$

and

$$C_{AB}(i\omega_n) = \int_0^{\beta} d(\tau - \tau') e^{i\omega_n(\tau-\tau')} C_{AB}(\tau - \tau')$$

The **periodic**/**anti-periodic** boundary condition for **bosons** and **fermions** leads to,

$$e^{i\omega_n\beta} = \begin{cases} +1 \\ -1 \end{cases} \quad \text{and} \quad \omega_n = \begin{cases} 2n\pi k_B T & \text{for bosons} \\ (2n+1)\pi k_B T & \text{for fermions} \end{cases}$$

Analytic continuation in the frequency domain

Let us consider now the relation between the **retarded** response functions (real time) and the **temperature** response functions (imaginary time), i.e.,

$$C^R(t,t') = -i\theta(t-t')\langle A(t)B(t') \mp B(t')A(t)\rangle$$

versus

$$C_{AB}(\tau,\tau') = -\theta(\tau-\tau')\langle A(\tau)B(\tau')\rangle \mp \theta(\tau'-\tau)\langle B(\tau')A(\tau)\rangle$$

First, consider the **temperature** response functions at $\tau > 0$,

$$\begin{aligned} \mathcal{C}_{AB}(\tau) &= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{\tau H}Ae^{-\tau H}B\right] \\ &= \frac{-1}{Z}\sum_{nn'} e^{-\beta E_n}\left\langle n\left|A\right|n'\right\rangle\left\langle n'\left|B\right|n\right\rangle e^{\tau(E_n-E_{n'})}, \end{aligned}$$

Analytic continuation in the frequency domain

And hence

$$\begin{aligned}
\mathcal{C}_{AB}(i\omega_n) &= \int_0^\beta d\tau\, e^{i\omega_n\tau}\frac{-1}{Z}\sum_{nn'} e^{-\beta E_n}\left\langle n\left|A\right|n'\right\rangle\left\langle n'\left|B\right|n\right\rangle e^{\tau(E_n-E_{n'})}, \\
&= \frac{-1}{Z}\sum_{nn'} e^{-\beta E_n}\frac{\left\langle n\left|A\right|n'\right\rangle\left\langle n'\left|B\right|n\right\rangle}{i\omega_n+E_n-E_{n'}}\left(e^{i\omega_n\beta}e^{\beta(E_n-E_{n'})}-1\right), \\
&= \frac{-1}{Z}\sum_{nn'} e^{-\beta E_n}\frac{\left\langle n\left|A\right|n'\right\rangle\left\langle n'\left|B\right|n\right\rangle}{i\omega_n+E_n-E_{n'}}\left(\pm e^{\beta(E_n-E_{n'})}-1\right) \\
&= \frac{1}{Z}\sum_{nn'}\frac{\left\langle n\left|A\right|n'\right\rangle\left\langle n'\left|B\right|n\right\rangle}{i\omega_n+E_n-E_{n'}}\left(e^{-\beta E_n}-(\pm)e^{-\beta E_{n'}}\right),
\end{aligned}$$

For the retarded functions, in the frequency domain, we obtain (**check by yourself**),

$$C^R_{AB}(\omega) = \frac{1}{Z}\sum_{nn'}\frac{\left\langle n\left|A\right|n'\right\rangle\left\langle n'\left|B\right|n\right\rangle}{\omega+E_n-E_{n'}+i\eta}\left(e^{-\beta E_n}-(\pm)e^{-\beta E_{n'}}\right)$$

This means once we find $C_{AB}(i\omega_n)$, we obtain $C^R(\omega)$ by **analytic continuation**:

$$C^R(\omega) = C_{AB}(i\omega_n \to \omega + i0^+),$$

Analytic continuation in the frequency domain

If we define, $$C_{AB}(z) = \frac{1}{Z}\sum_{nn'} \frac{\langle n|A|n'\rangle\langle n'|B|n\rangle}{z + E_n - E_{n'}}\left(e^{-\beta E_n} - (\pm)\, e^{-\beta E_{n'}}\right)$$

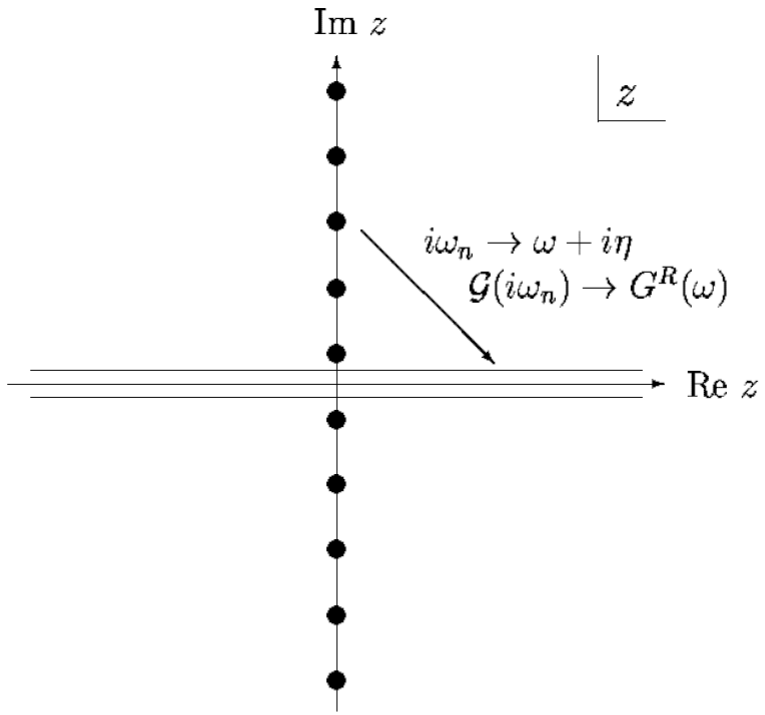


Figure 10.1: The analytic continuation procedure in the complex z-plane where the Matsubara function defined for $z = i\omega_n$ goes to the retarded Green's functions defined infinitesimally close to real axis.

$$C^R(\omega) = C_{AB}(i\omega_n \to \omega + i0^+),$$

A brief summary so far

To find the **retarded** response functions, we may calculate first the **temperature** response function in the interaction picture and then take the **analytic continuation**, i.e.,

Step 1:

$$C_{AB}(\tau,\tau') = -\frac{\left\langle T_\tau\left\{\hat{U}(\beta,0)\hat{A}(\tau)\hat{B}(\tau')\right\}\right\rangle_0}{\left\langle \hat{U}(\beta,0)\right\rangle_0}$$

where,

$$\hat{U}(\beta,0) = T_\tau\left[e^{-\int_0^\beta d\tau'\hat{V}(\tau')}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n T_\tau\left[\hat{V}(\tau_1)\cdots\hat{V}(\tau_n)\right]$$

Step 2:

$$C^R(\omega) = C_{AB}(i\omega_n \to \omega + i0^+),$$

Non-interacting temperature Green functions

Let us now consider the temperature **Green function**:

$$G(\boldsymbol{x}\sigma\tau, \boldsymbol{x}'\sigma'\tau') = -\left\langle T_\tau \left[\Psi_\sigma(\boldsymbol{x},\tau)\Psi_{\sigma'}{}^\dagger(\boldsymbol{x}',\tau')\right]\right\rangle$$

$$G(\boldsymbol{k}\sigma\tau, \boldsymbol{k}'\sigma'\tau') = -\left\langle T_\tau \left[\Psi_\sigma(\boldsymbol{k},\tau)\Psi_{\sigma'}{}^\dagger(\boldsymbol{k}',\tau')\right]\right\rangle$$

or in a general $\{\boldsymbol{\nu}\}$ representation:

$$G(\nu\tau, \nu'\tau') = -\left\langle T_\tau \left[c_\nu(\tau)c_{\nu'}{}^\dagger(\tau')\right]\right\rangle$$

Non-interacting temperature Green functions

The non-interacting temperature Green function in $\{\boldsymbol{\nu} = \boldsymbol{k\sigma}\}$ is to easy to calculate.
The non-interacting Hamiltonian is diagonal in the $\{\boldsymbol{\nu}\}$ quantum numbers.

$$H_0 = \sum_\nu \xi_\nu c_\nu^\dagger c_\nu,$$

so that

check by yourself!

$$c_\nu(\tau) = e^{\tau H_0} c_\nu e^{-\tau H_0} = e^{-\xi_\nu \tau} c_\nu, \qquad c_\nu^\dagger(\tau) = e^{\tau H_0} c_\nu^\dagger e^{-\tau H_0} = e^{\xi_\nu \tau} c_\nu^\dagger,$$

which gives

$$\begin{aligned}
\mathcal{G}_0(\nu, \tau - \tau') &= -\left\langle T_\tau \left(c_\nu(\tau) c_\nu^\dagger(\tau') \right) \right\rangle, \\
&= -\theta(\tau - \tau') \langle c_\nu(\tau) c_\nu^\dagger(\tau') \rangle - (\pm)\, \theta(\tau' - \tau) \langle c_\nu^\dagger(\tau') c_\nu(\tau) \rangle \\
&= -\left[\theta(\tau - \tau') \langle c_\nu c_\nu^\dagger \rangle (\pm) \theta(\tau' - \tau) \langle c_\nu^\dagger c_\nu \rangle \right] e^{-\xi_\nu(\tau - \tau')},
\end{aligned}$$

For fermions this is

$$\mathcal{G}_{0,F}(\nu, \tau - \tau') = -\left[\theta(\tau - \tau')(1 - n_F(\xi_\nu)) - \theta(\tau' - \tau) n_F(\xi_\nu) \right] e^{-\xi_\nu(\tau - \tau')}$$

while the bosonic free particle Green's function reads

$$\mathcal{G}_{0,B}(\nu, \tau - \tau') = -\left[\theta(\tau - \tau')\,(1 + n_B(\xi_\nu)) + \theta(\tau' - \tau) n_B(\xi_\nu) \right] e^{-\xi_\nu(\tau - \tau')}.$$

Non-interacting temperature Green functions

In the frequency representation, the fermionic Green's function is

$$\begin{aligned}
\mathcal{G}_{0,F}(\nu, ik_n) &= \int_0^\beta d\tau\, e^{ik_n\tau}\mathcal{G}_{0,F}(\nu,\tau), \quad k_n = (2n+1)\,\pi/\beta \\
&= -\left(1 - n_F(\xi_\nu)\right)\int_0^\beta d\tau\, e^{ik_n\tau}e^{-\xi_\nu\tau}, \\
&= -\left(1 - n_F(\xi_\nu)\right)\frac{1}{ik_n - \xi_\nu}\left(e^{ik_n\beta}e^{-\xi_\nu\beta} - 1\right), \\
&= \frac{1}{ik_n - \xi_\nu},
\end{aligned}$$

because $e^{ik_n\beta} = -1$ and $1 - n_F(\varepsilon) = \left(e^{-\beta\varepsilon} + 1\right)^{-1}$, while the bosonic one becomes

$$\begin{aligned}
\mathcal{G}_{0,B}(\nu, iq_n) &= \int_0^\beta d\tau\, e^{iq_n\tau}\mathcal{G}_{0,B}(\nu,\tau), \quad q_n = 2n\pi/\beta \\
&= -\left(1 + n_B(\xi_\nu)\right)\int_0^\beta d\tau\, e^{iq_n\tau}e^{-\xi_\nu\tau}, \\
&= -\left(1 + n_B(\xi_\nu)\right)\frac{1}{iq_n - \xi_\nu}\left(e^{iq_n\beta}e^{-\xi_\nu\beta} - 1\right), \\
&= \frac{1}{iq_n - \xi_\nu},
\end{aligned}$$

because $e^{iq_n\beta} = 1$ and $1 + n_B(\varepsilon) = -\left(e^{-\beta\varepsilon} - 1\right)^{-1}$.

Evaluation of Matsubara sums

To calculate Feynman diagrams, we often encounter different summation over the bosonic or fermionic Matsubara frequency. Let us define,

$$S_F = -k_B T \sum_{i\omega_m} g(i\omega_m) e^{+i\omega_m 0^+}$$

$$S_B = +k_B T \sum_{i\nu_n} h(i\nu_n) e^{+i\nu_n 0^+}$$

To evaluate these, the trick is to rewrite them as integrals over a complex variable and to use residue theory. For this we need two functions, $n(z)$, which have poles at $z = ik_n$ and $z = i\omega_n$, respectively. These functions turn out to be the well known Fermi and Bose distribution functions

$$n_F(z) = \frac{1}{e^{\beta z} + 1}, \quad \text{poles for } z = i(2n+1)\pi/\beta,$$

$$n_B(z) = \frac{1}{e^{\beta z} - 1}, \quad \text{poles for } z = i(2n)\pi/\beta.$$

The residues at these values are

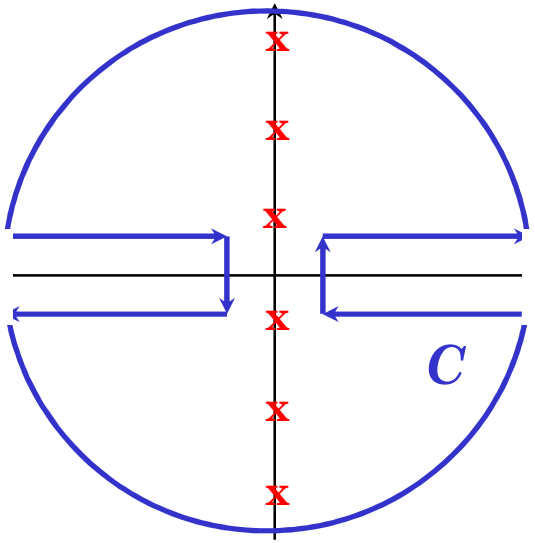
$$\operatorname*{Res}_{z=ik_n} [n_F(z)] = \lim_{z\to ik_n} \frac{(z - ik_n)}{e^{\beta z} + 1} = \lim_{\delta\to 0} \frac{\delta}{e^{\beta i k_n} e^{\beta\delta} + 1} = -\frac{1}{\beta},$$

$$\operatorname*{Res}_{z=i\omega_n} [n_B(z)] = \lim_{z\to i\omega_n} \frac{(z - i\omega_n)}{e^{\beta z} - 1} = \lim_{\delta\to 0} \frac{\delta}{e^{\beta i \omega_n} e^{\beta\delta} - 1} = +\frac{1}{\beta}.$$

Evaluation of Matsubara sums

To calculate Feynman diagrams, we often encounter different summation over the bosonic or fermionic Matsubara frequency. Let us consider first the **fermionic** case. For any function $g(x)$,

$$-k_B T\sum_{i\omega_m} g(i\omega_m)e^{+i\omega_m 0^+} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{e^{\beta\omega}+1}\mathrm{Im}\, g(i\omega_m \to \omega + i0^+). \quad \textbf{(fermionic)}$$



This is because the left side of equation can be written as a contour integral over ***C*** (see the left graph):

$$-k_B T\sum_{i\omega_m} g(i\omega_m)e^{+i\omega_m 0^+} = \frac{1}{2\pi i}\oint_C \frac{e^{\omega 0^+}}{e^{\beta\omega}+1}g(\omega)$$

Due to the *convergence* factor, the integral at two half circles vanishes. The contribution near the real axis gives the right hand side of the equation (**fermionic**).

Evaluation of Matsubara sums

Consider the fermionic Green function,

$$G(\boldsymbol{k}\sigma; 0^-) = -\langle T_\tau[\Psi_\sigma(\boldsymbol{k}, 0^-)\Psi_\sigma{}^\dagger(\boldsymbol{k}, 0)]\rangle = n_{\boldsymbol{k}\sigma}$$

$$n_{\boldsymbol{k}\sigma} = k_B T \sum_{m=-\infty}^{+\infty} G(\boldsymbol{k}\sigma; i\omega_m) e^{-i\omega_m 0^-}$$

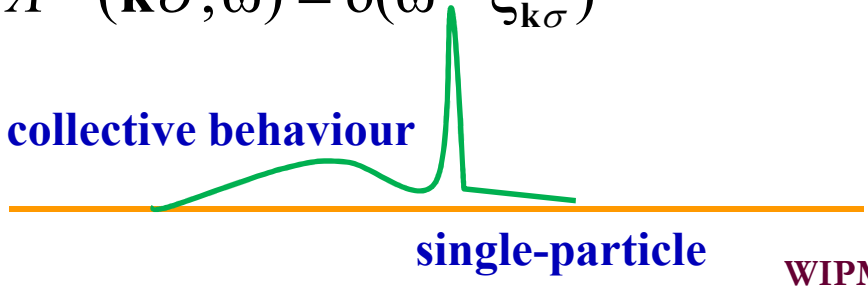
By taking the summation, we have,

$$n_{\mathbf{k}\sigma} = \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega}+1}\left(-\frac{1}{\pi}\right)\mathrm{Im}\,G(\mathbf{k}\sigma;\omega+i0^+) = \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega}+1} A(\mathbf{k}\sigma;\omega).$$

This **single-particle spectral function** $A(\boldsymbol{k}, \omega)$ is experimentally measurable!

For a non-interacting Fermi system: $A^{(0)}(\mathbf{k}\sigma;\omega) = \delta(\omega - \xi_{\mathbf{k}\sigma})$

For an interacting Fermi system:



Tutorial on Matsubara sums:

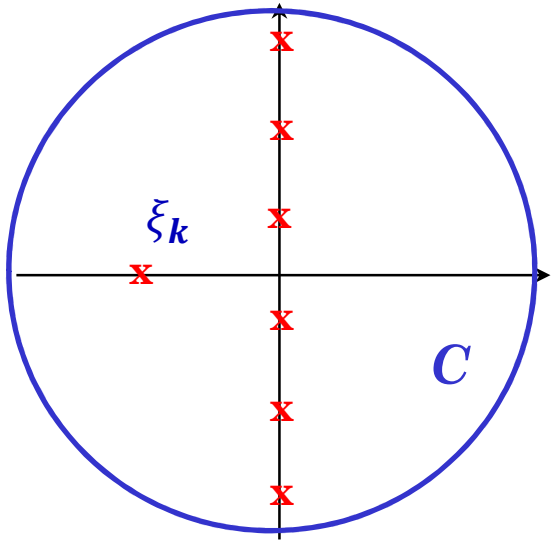
Question 1: Consider the Matsubara summation,

$$n_{\boldsymbol{k}}{}^{(0)} = k_B T \sum_{m=-\infty}^{+\infty} \frac{e^{-i\omega_m 0^-}}{i\omega_m - \xi_{\boldsymbol{k}}} = n_F(\xi_{\boldsymbol{k}})$$

Answer: Let us consider ,

$$\oint \frac{1}{e^{\beta\omega}+1} \frac{e^{\omega 0^+}}{\omega - \xi_{\boldsymbol{k}}}$$

and the contour C in the right plot



Question 2: What is the result of,

$$A = k_B T \sum_{m=-\infty}^{+\infty} \frac{e^{+i\omega_m 0^-}}{i\omega_m - \xi_{\boldsymbol{k}}}$$

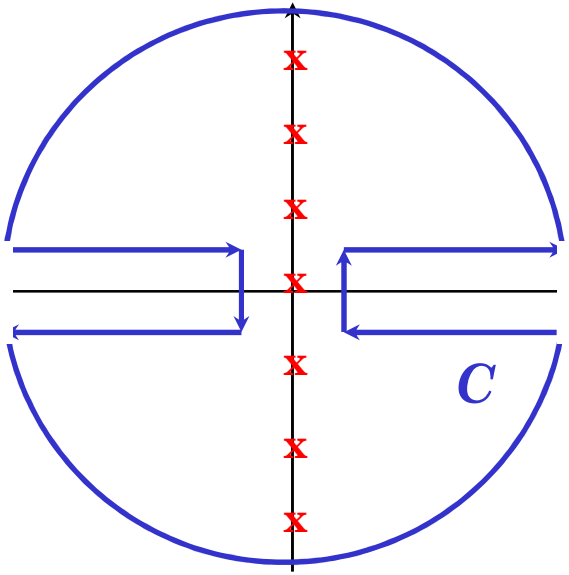
Remark: We don't need the convergence factor if

$$B = k_B T \sum_{m=-\infty}^{+\infty} \frac{1}{i\omega_m - \xi_{\boldsymbol{k}}} \frac{1}{i\omega_m - \xi_{\boldsymbol{p}}}$$

Evaluation of Matsubara sums

Let us consider next the **bosonic** case. For any function $h(x)$,

$$k_B T \sum_{i\nu_n} h(i\nu_n) e^{+i\nu_n 0^+} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{e^{\beta\omega} - 1} \mathrm{Im}\, h(i\nu_n \to \omega + i0^+). \qquad \textbf{(bosonic)}$$



This is because the left side of equation can be written as a contour integral over ***C*** (see again the left graph):

$$k_B T \sum_{i\nu_n} h(i\nu_n) e^{+i\nu_n 0^+} = \frac{1}{2\pi i} \oint_C \frac{e^{\omega 0^+}}{e^{\beta\omega} - 1} h(\omega)$$

Due to the *convergence* factor, the integral at two half circles vanishes. The contribution near the real axis gives the right hand side of equation (**bosonic**).

Evaluation of Matsubara sums

No worries if you don't remember the details, simply search wiki, https://en.wikipedia.org/wiki/Matsubara_frequency:

Distribution Function

Bosons

$$G_B(\tau=0^-)=\frac{1}{\beta}\sum_{i\omega_n}\frac{e^{i\omega_n 0^+}}{i\omega_n-\xi}=-n_B(\xi),$$

$$G_B(\tau=0^+)=\frac{1}{\beta}\sum_{i\omega_n}\frac{e^{-i\omega_n 0^+}}{i\omega_n-\xi}=-(n_B(\xi)+1).$$

Fermions

$$G_F(\tau=0^-)=\frac{1}{\beta}\sum_{i\omega_m}\frac{e^{i\omega_m 0^+}}{i\omega_m-\xi}=n_F(\xi),$$

$$G_F(\tau=0^+)=\frac{1}{\beta}\sum_{i\omega_m}\frac{e^{-i\omega_m 0^+}}{i\omega_m-\xi}=-(1-n_F(\xi)).$$

Free Energy

Bosons

$$\frac{1}{\beta}\sum_{i\omega_n}\ln(\beta(-i\omega_n+\xi))=\frac{1}{\beta}\ln(1-e^{-\beta\xi}),$$

Fermions

$$-\frac{1}{\beta}\sum_{i\omega_m}\ln(\beta(-i\omega_m+\xi))=-\frac{1}{\beta}\ln(1+e^{-\beta\xi}).$$

Diagrams Evaluation [edit]

Frequently encountered diagrams are evaluated here with the single mode setting.

Fermion Self Energy [edit]

$$\Sigma(i\omega_m)=-\frac{1}{\beta}\sum_{i\omega_n}\frac{1}{i\omega_m+i\omega_n-\epsilon}\frac{1}{i\omega_n-\Omega}=\frac{n_F(\epsilon)-n_F(\Omega)}{i\omega_m-\epsilon+\Omega}.$$

Particle-Hole Bubble [edit]

$$\Pi(i\omega_n)=\frac{1}{\beta}\sum_{i\omega_m}\frac{1}{i\omega_m+i\omega_n-\epsilon}\frac{1}{i\omega_m-\epsilon'}=-\frac{n_F(\epsilon)-n_F(\epsilon')}{i\omega_n-\epsilon+\epsilon'}.$$

Particle-Particle Bubble [edit]

$$\Pi(i\omega_n)=-\frac{1}{\beta}\sum_{i\omega_m}\frac{1}{i\omega_m+i\omega_n-\epsilon}\frac{1}{-i\omega_m-\epsilon'}=\frac{1-n_F(\epsilon)-n_F(\epsilon')}{i\omega_n-\epsilon-\epsilon'}.$$

Wick theorem

We now consider the final technical issue, the calculation of

$$\left\langle T_\tau\left(A_1A_2A_3A_4\cdots A_{2n-1}A_{2n}\right)\right\rangle_0$$

where all the field operators are defined in the interaction picture.

The **Wick theorem** states that in the interaction picture with $\boldsymbol{H_0}$,

$$\left\langle T_\tau\left(A_1A_2A_3A_4\cdots A_{2n-1}A_{2n}\right)\right\rangle_0 = \sum_{\substack{\text{all}\\ \text{possible}\\ \text{permutation } P}} (\pm)^P \left\langle T_\tau\left(A_{P_1}A_{P_2}\right)\right\rangle_0 \left\langle T_\tau\left(A_{P_3}A_{P_4}\right)\right\rangle_0 \cdots \left\langle T_\tau\left(A_{P_{2n-1}}A_{P_{2n}}\right)\right\rangle_0$$

For example (for a normal Fermi gas):

$$\left\langle T_\tau\left[c^+_{\mathbf{k}\sigma}(\tau)c_{\mathbf{k}+\mathbf{q}\sigma}(\tau)c^+_{\mathbf{k}'\sigma'}(0)c_{\mathbf{k}'-\mathbf{q}\sigma'}(0)\right]\right\rangle_0 =$$

$$+\left\langle T_\tau\left[c^+_{\mathbf{k}\sigma}(\tau)c_{\mathbf{k}+\mathbf{q}\sigma}(\tau)\right]\right\rangle_0\left\langle T_\tau\left[c^+_{\mathbf{k}'\sigma'}(0)c_{\mathbf{k}'-\mathbf{q}\sigma'}(0)\right]\right\rangle_0 - \left\langle T_\tau\left[c_{\mathbf{k}+\mathbf{q}\sigma}(\tau)c^+_{\mathbf{k}'\sigma'}(0)\right]\right\rangle_0\left\langle T_\tau\left[c_{\mathbf{k}'-\mathbf{q}\sigma'}(0)c^+_{\mathbf{k}\sigma}(\tau)\right]\right\rangle_0$$

Wick theorem

Why this happens?

(1) Recall, for example, $c_\nu(\tau) = e^{\tau H_0} c_\nu e^{-\tau H_0} = e^{-\xi_\nu \tau} c_\nu$

$$\left\langle T_\tau \left(c_1(\tau_1) c_2(\tau_2) c_3^+(\tau_3) c_4^+(\tau_4) \right) \right\rangle_0 = \left\langle T_\tau \left(e^{-\xi_1 \tau_1} c_1 e^{-\xi_2 \tau_2} c_2 e^{\xi_3 \tau_3} c_3^+ e^{\xi_4 \tau_4} c_4^+ \right) \right\rangle_0$$

We may pull out all the time-dependent factors and consider only the average such as,

$$\left\langle c_1 c_2 c_3^+ c_4^+ \right\rangle_0$$

(2) The Wick theorem is clearly valid for these averages.

(3) We then send back the time-dependent factors to the factorised product.

<u>Reminder</u>: Wick theorem holds only for the product under thermal average $\langle \quad \rangle_0$

We are ready to draw Feynman diagrams!

We aim to calculate the Green functions,

$$G(\boldsymbol{x}\tau, \boldsymbol{x}\tau') = -\frac{\left\langle T_\tau\left\{\hat{U}(\beta,0)\hat{\Psi}(\boldsymbol{x},\tau)\hat{\Psi}^\dagger(\boldsymbol{x}',\tau')\right\}\right\rangle_0}{\left\langle \hat{U}(\beta,0)\right\rangle_0}$$

where,

$$\hat{U}(\beta,0) = T_\tau\left[e^{-\int_0^\beta d\tau'\hat{V}(\tau')}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n T_\tau\left[\hat{V}(\tau_1)\cdots\hat{V}(\tau_n)\right]$$

Step 1: We represent each term in the perturbative expansion using a **Feynman diagram**.

Step 2: We then use **Wick theorem** to decouple each term as a product of Green functions. By performing the Fourier transform, we obtain the expression in terms of,

$$G_0(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \xi_{\boldsymbol{k}}}$$

Step 3: We may sum some sorts of geometric diagrams as **a series** and take the Matsubara frequency summation, either *numerically* or *analytically*.

Step 4: We finally take the **analytic continuation** to obtain the desired **retarded response function** or **spectral function**.

Hamiltonian

Our starting Hamiltonian is (for a two-component Fermi gas)

$$H_0 = \int d\mathbf{x} \sum_\sigma \psi_\sigma^+(\mathbf{x}) \left[-\frac{\hbar^2 \nabla^2}{2m} - \mu \right] \psi_\sigma(\mathbf{x})$$

$$H_{\text{int}} = U_0 \int d\mathbf{x} d\mathbf{x}' \psi_\uparrow^+(\mathbf{x}) \psi_\downarrow^+(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \psi_\downarrow(\mathbf{x}') \psi_\uparrow(\mathbf{x})$$

In momentum space, the Hamiltonian takes the form,

$$H_0 = \sum_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}^+ \left[\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu \right] c_{\mathbf{k}\sigma} = \sum_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}^+ \xi_{\mathbf{k}} c_{\mathbf{k}\sigma}$$

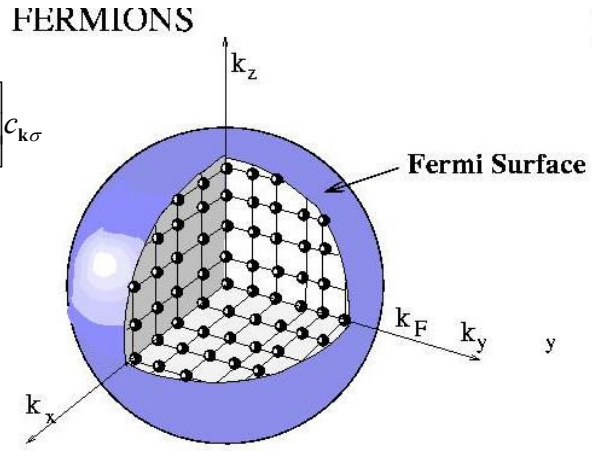
$$H_{\text{int}} = U_0 \sum_{\mathbf{qkk'}} c_{\mathbf{k}\uparrow}^+ c_{\mathbf{q}-\mathbf{k}\downarrow}^+ c_{\mathbf{q}-\mathbf{k'}\downarrow} c_{\mathbf{k'}\uparrow}$$

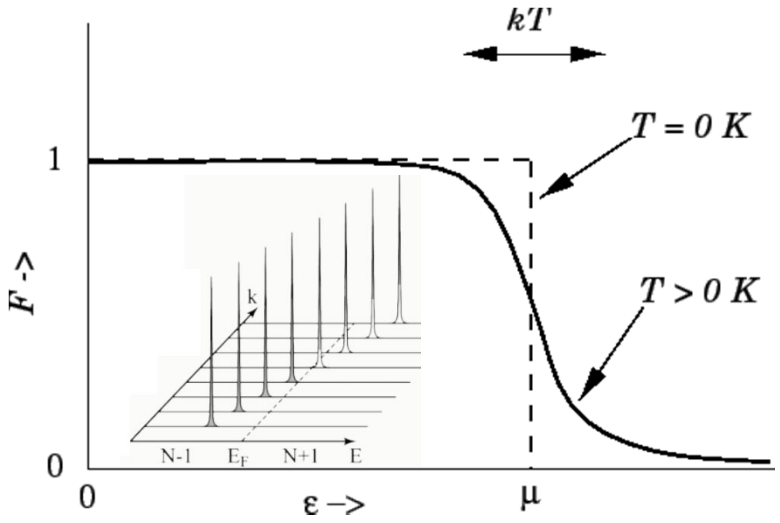
Definition: μ - chemical potential (**grand-canonical ensemble**), U_0 – interaction strength

Hamiltonian

In the absence of the inter-particle interactions, we have perfect Fermi sea at **T=0K**:

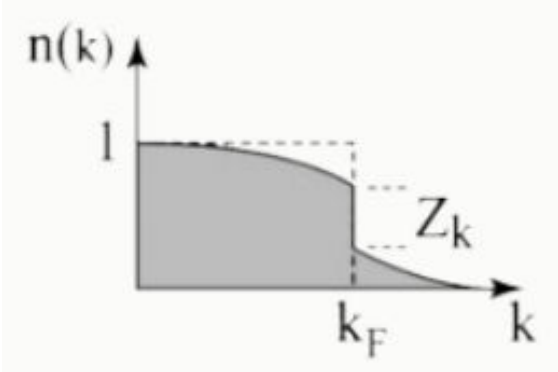
$$\sum_{k\sigma} c^{+}_{k\sigma}\left[\frac{\hbar^2 \mathbf{k}^2}{2m}-\mu\right]c_{k\sigma}$$

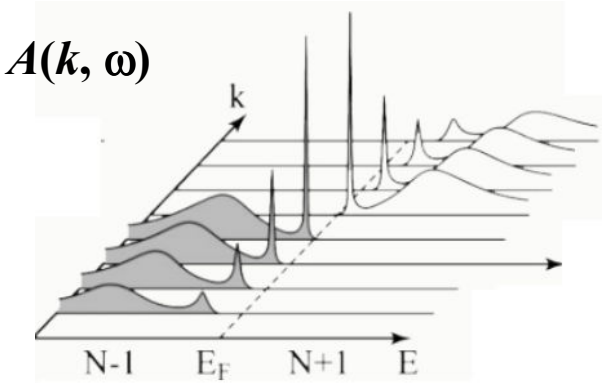




$\xi_k = 0$

What will happen if we switch on the repulsive interactions (normal state)?





SWIN BURNE — CENTRE FOR QUANTUM AND OPTICAL SCIENCE

Fermi sea is real !!!

PRL 118, 123401 (2017) PHYSICAL REVIEW LETTERS week ending 24 MARCH 2017

Homogeneous Atomic Fermi Gases

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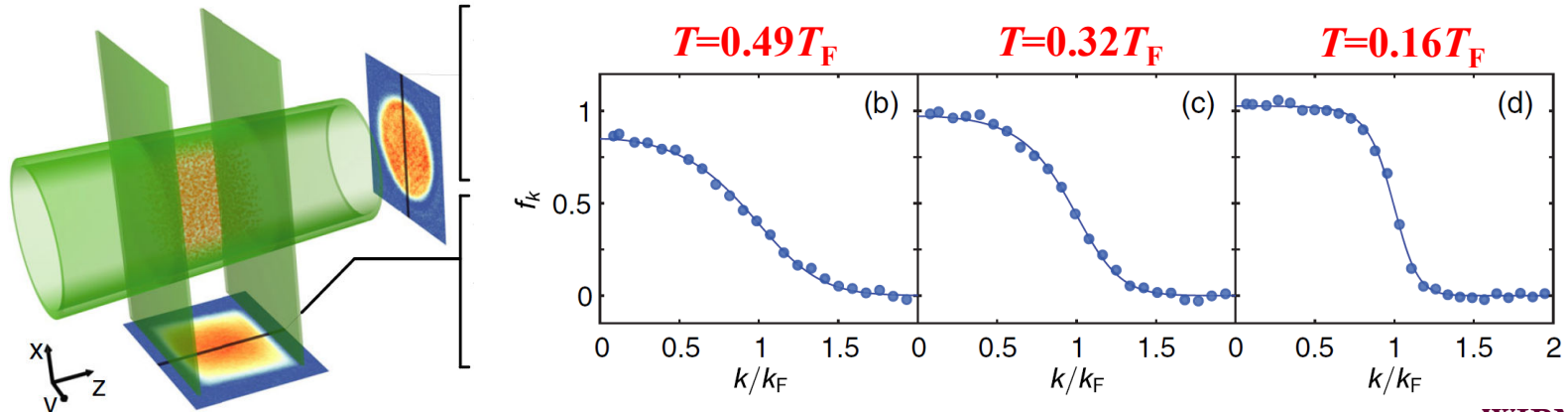
We report on the creation of homogeneous Fermi gases of ultracold atoms in a uniform potential. In the momentum distribution of a spin-polarized gas, we observe the emergence of the Fermi surface and the saturated occupation of one particle per momentum state: the striking consequence of Pauli blocking in momentum space for a degenerate gas. Cooling a spin-balanced Fermi gas at unitarity, we create homogeneous superfluids and observe spatially uniform pair condensates. For thermodynamic measure-



Fermi sea is real !!!

PRL 2018 (Editors' Suggestion)

Two-Dimensional Homogeneous Fermi Gases

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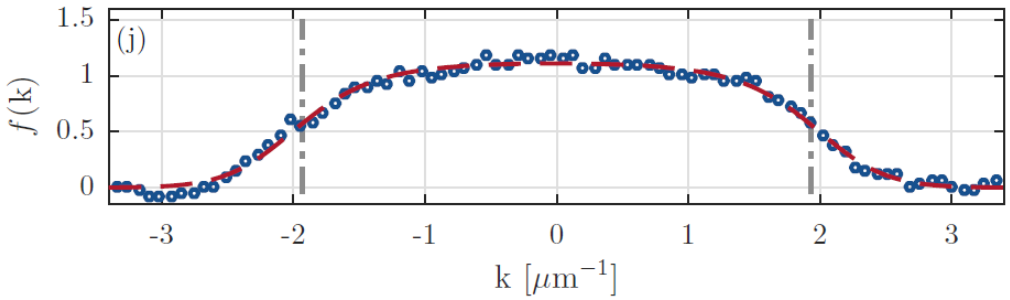
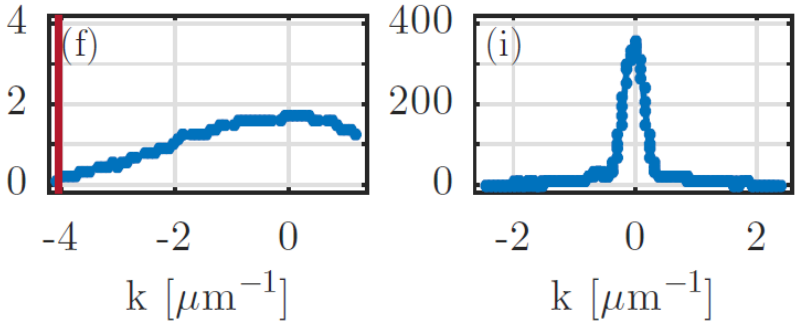
(Dated: April 24, 2017)

We report on the experimental realization of homogeneous two-dimensional (2D) Fermi gases trapped in a box potential. In contrast to harmonically trapped gases, these homogeneous 2D systems are ideally suited to probe local as well as non-local properties of strongly interacting many-body systems. As a first measurement, we use a local probe to extract the equation of state (EOS) of a non-interacting Fermi gas. We then perform matter wave focusing to extract its momentum distribution and directly observe Pauli blocking in a near unity occupation of momentum states. Finally, we measure the momentum distribution of homogeneous 2D Fermi gases in the crossover between weakly-bound fermionic pairs and deeply-bound bosonic molecules and observe a dramatic increase in the occupation of low momentum states with increasing attractive interactions.

A non-interacting Fermi gas T=0.31T_F



A strongly attractively interacting Fermi gas



We may explain the observation by using Feynman diagrams!

Perturbative expansion of GF

Let us check the expansion of the Green function,

$$G_{\uparrow}(\boldsymbol{x}\tau,\boldsymbol{x}\tau') = -\frac{\left\langle T_\tau\left\{\hat{U}(\beta,0)\,\hat{\Psi}_{\uparrow}(\boldsymbol{x},\tau)\hat{\Psi}_{\uparrow}{}^{\dagger}(\boldsymbol{x}',\tau')\right\}\right\rangle_0}{\left\langle \hat{U}(\beta,0)\right\rangle_0}$$

with,

$$\hat{U}(\beta,0) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n T_\tau\left[\hat{H}_{int}(\tau_1)\cdots\hat{H}_{int}(\tau_n)\right]$$

Consider first the numerator. The n=0 contribution is trivial, simply gives G_0. The n=1 contribution from the numerator is,

$$-\frac{(-1)^1}{1!}\int_0^\beta d\tau_1\left\langle T_\tau\left\{\hat{\Psi}_{\uparrow}(\boldsymbol{x},\tau)\hat{H}_{int}(\tau_1)\hat{\Psi}_{\uparrow}{}^{\dagger}(\boldsymbol{x}',\tau')\right\}\right\rangle_0$$

or

$$-\frac{(-1)^1}{1!}\int_0^\beta d\tau_1\iint d\boldsymbol{x}_1 d\boldsymbol{x}_1{}'$$

$$\left\langle T_\tau\left\{\hat{\Psi}_{\uparrow}(\boldsymbol{x},\tau)\hat{\Psi}_{\uparrow}{}^{\dagger}(\boldsymbol{x}_1,\tau_1)\hat{\Psi}_{\downarrow}{}^{\dagger}(\boldsymbol{x}_1{}',\tau_1)U_0(\boldsymbol{x}_1\boldsymbol{x}_1{}')\hat{\Psi}_{\downarrow}(\boldsymbol{x}_1{}',\tau_1)\hat{\Psi}_{\uparrow}(\boldsymbol{x}_1,\tau_1)\hat{\Psi}_{\uparrow}{}^{\dagger}(\boldsymbol{x}',\tau')\right\}\right\rangle_0$$

Perturbative expansion of GF in real space

Let us introduce the short-hand notations and rewrite the interaction,

$$x=(\mathbf{x},\tau),$$
$$x'=(\mathbf{x}',\tau'),$$
$$x_1=(\mathbf{x}_1,\tau_1),$$
$$x'_1=(\mathbf{x}'_1,\tau'_1),$$

$$U_0(x_1-x'_1)=\int d\tau'_1 U_0\delta(\mathbf{x}_1-\mathbf{x}'_1)\delta(\tau_1-\tau'_1)$$

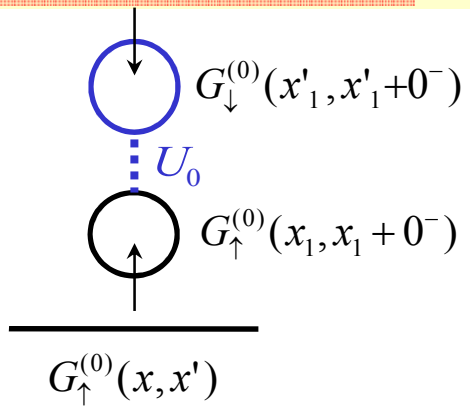
Then, the first-order contribution is,

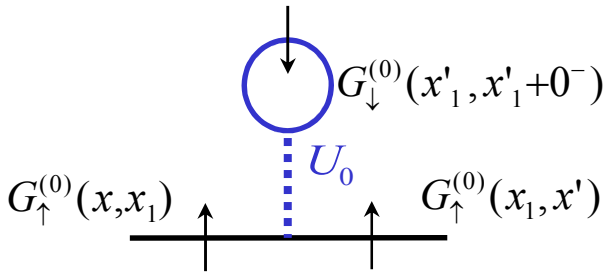
$$(-)(-)\iint dx_1dx'_1U_0(x_1-x'_1)\left\langle T_\tau\left\{\psi_\uparrow(x)\psi^+_\uparrow(x_1)\psi^+_\downarrow(x'_1)\psi_\downarrow(x'_1)\psi_\uparrow(x_1)\psi^+_\uparrow(x')\right\}\right\rangle_0$$

By using Wick theorem, we may obtain for $\langle\quad\rangle$,

(a) $(-)G^{(0)}_\uparrow(x,x')G^{(0)}_\uparrow(x_1,x_1+0^-)G^{(0)}_\downarrow(x'_1,x'_1+0^-)$

(b) $(+)G^{(0)}_\uparrow(x,x_1)G^{(0)}_\uparrow(x_1,x')G^{(0)}_\downarrow(x'_1,x'_1+0^-)$



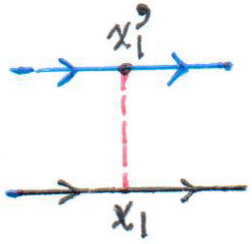


Perturbative expansion of GF in real space

A few observations:

(1) We use " 

" to represent $G^{(0)}(x, x')$



to represent $\hat{\mathcal{H}}_{int}(x_1, x_1')$

(2). The application of Wick theorem corresponds to connect the lines in different ways.

(3) There is disconnected diagram. see, (a).

(4). About the "sign" in applying Wick theorem: if there is a Fermi loop, add "(−)"

Perturbative expansion of GF in real space

On the disconnected diagrams:

$$(-)\frac{(-)^n}{n!}\int dx_1 \cdots \langle T_\tau \{ \psi_\uparrow(x)\, \hat{\mathcal{H}}_{int}(x_1,x_1')\, \hat{\mathcal{H}}_{int}(x_2,x_2') \cdots \hat{\mathcal{H}}_{int}(x_i,x_i') \cdots \hat{\mathcal{H}}_{int}(x_n,x_n')\, \psi_\uparrow^+(x') \} \rangle$$

if one $\hat{\mathcal{H}}_{int}$ or more does not connect to other parts, we create a disconnected diagram!

can be found in $\langle \hat{U}(\beta,0) \rangle_0$

This means:

$$\left(\text{All diagrams}\right) = \left(\begin{array}{c}\text{All}\\ \text{connected}\\ \text{diagrams}\end{array}\right) \times \langle \hat{U}(\beta,0) \rangle_0$$

OR:

$$-\frac{\langle T_\tau \{ \hat{U}(\beta,0)\, \psi_\uparrow(\vec{x},\tau)\, \psi_\uparrow^+(\vec{x}',\tau') \} \rangle_0}{\langle \hat{U}(\beta,0) \rangle_0} = \left(\begin{array}{c}\text{All}\\ \text{connected}\\ \text{diagrams}\end{array}\right).$$

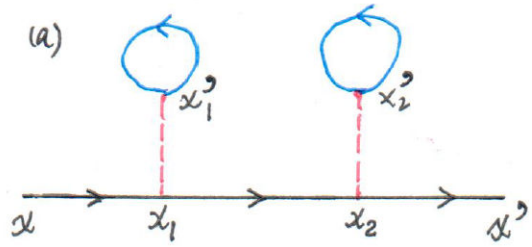
Perturbative expansion of GF in real space

Let us check the second-order contribution:

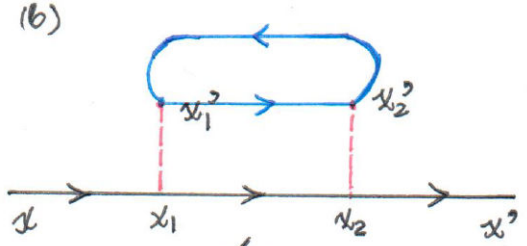
$$\left(\frac{1}{2!}\right)\langle T_c\{\psi_\uparrow(x)\psi_\uparrow^+(x_1)\psi_\downarrow^+(x_1')\psi_\downarrow(x_1')\psi_\uparrow(x_1)\,\psi_\uparrow^+(x_2)\psi_\downarrow^+(x_2')\psi_\downarrow(x_2')\psi_\uparrow(x_2)\,\psi_\uparrow^+(x')\}\rangle_0\, U_0(x_1-x_1')U_0(x_2-x_2')$$

if we consider only connected diagrams:

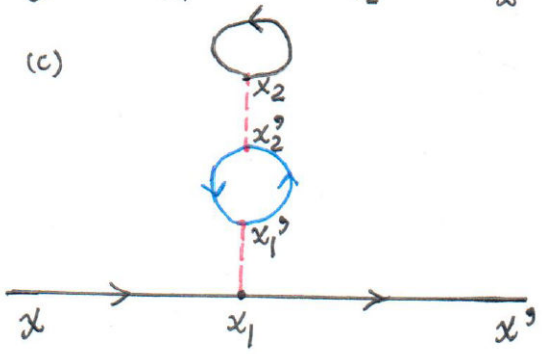
(a) $+\; G_\uparrow^{(0)}(x,x_1)\,G_\uparrow^{(0)}(x_1,x_2)\,G_\uparrow^{(0)}(x_2,x')\,G_\downarrow^{(0)}(x_1',x_1')\,G_\downarrow^{(0)}(x_2',x_2')$



(b) $-\; G_\uparrow^{(0)}(x,x_1)\,G_\uparrow^{(0)}(x_1,x_2)\,G_\uparrow^{(0)}(x_2,x')\,G_\downarrow^{(0)}(x_1',x_2')\,G_\downarrow^{(0)}(x_2',x_1')$

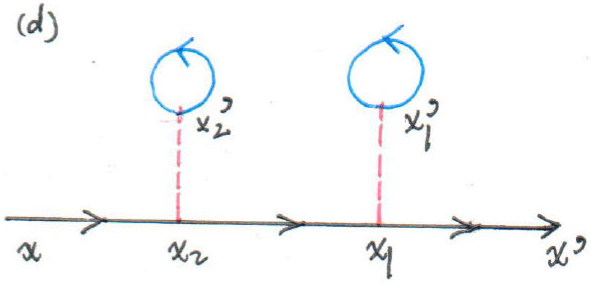


(c) $+\; G_\uparrow^{(0)}(x,x_1)\,G_\uparrow^{(0)}(x_1,x')\,G_\uparrow^{(0)}(x_2,x_2)\,G_\downarrow^{(0)}(x_1',x_2')\,G_\downarrow^{(0)}(x_2',x_1')$

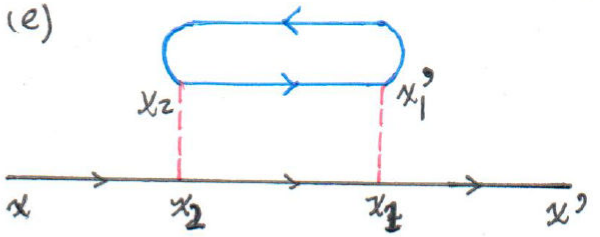


Perturbative expansion of GF in real space

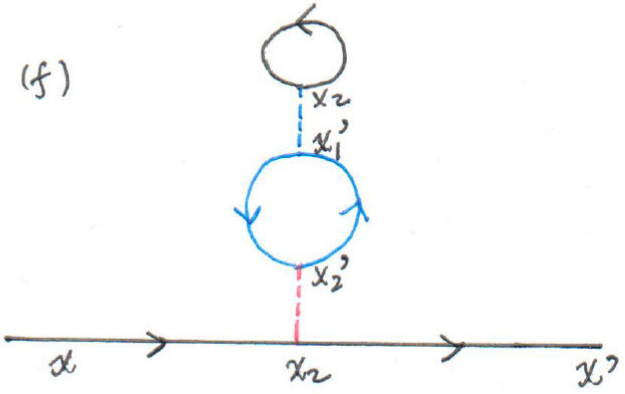
(d) $+ \; G_{\uparrow}^{(0)}(x,x_2)\, G_{\uparrow}^{(0)}(x_2,x_1)\, G_{\uparrow}^{(0)}(x_1,x')\, G_{\downarrow}^{(0)}(x_2',x_2')\, G_{\downarrow}^{(0)}(x_1',x_1')$



(e) $- \; G_{\uparrow}^{(0)}(x,x_2)\, G_{\uparrow}^{(0)}(x_2,x_1)\, G_{\uparrow}^{(0)}(x_1,x')\, G_{\downarrow}^{(0)}(x_2',x_1')\, G_{\downarrow}^{(0)}(x_1',x_2')$



(f) $+ \; G_{\uparrow}^{(0)}(x,x_2)\, G_{\uparrow}^{(0)}(x_2,x')\, G_{\uparrow}^{(0)}(x_1,x_1)\, G_{\downarrow}^{(0)}(x_2',x_1')\, G_{\downarrow}^{(0)}(x_1',x_2')$



(5) fifth notation: the prefactor $\left(\frac{1}{n!}\right)$ can be canceled !

Perturbative expansion of GF in momentum space

Now, let us consider the Fourier transform:

$$f(\vec{x},\tau) = k_B T \sum_{i\omega_m} \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-i\omega_m \tau} f(\vec{k}, i\omega_m).$$

$$\begin{cases} f(\vec{k}, i\omega_m) = \int_0^\beta d\tau \int d\vec{x}\; e^{-i\vec{k}\cdot\vec{x}} e^{i\omega_m \tau} f(\vec{x},\tau). \end{cases}$$

If we introduce four-component vector $k \equiv (\vec{k}, i\omega_m)$, $x = (\vec{x}, \tau)$, and define.

$$k\cdot x \equiv \vec{k}\cdot\vec{x} - i\omega_m \tau, \quad \left[\text{Also: } \sum_k \equiv k_B T \sum_{i\omega_m} \int \frac{d\vec{k}}{(2\pi)^3}, \quad \int dx \equiv \int_0^\beta d\tau \int d\vec{x} \right]$$

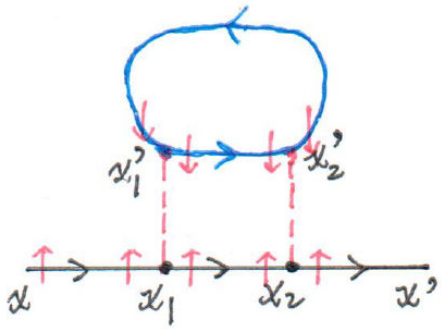
then, we may write the Fourier transform:

$$\begin{cases} f(x) = \sum_k e^{ikx} f(k) \\ f(k) = \int dx\, e^{-ikx} f(x) \end{cases}$$

Perturbative expansion of GF in momentum space

Let us consider the 2nd–order diagram:

$$\Delta G_{\uparrow}^{(2)}(x-x') =$$



$$= \underbrace{(-)^2}_{\text{second order}} \cdot \underbrace{(-)}_{\text{one Fermi loop}} \int dx_1 dx_1' dx_2 dx_2'$$

$$G_{\uparrow}^{(0)}(x,x_1)\, G_{\uparrow}^{(0)}(x_1-x_2)\, G_{\uparrow}^{(0)}(x_2,x')\, G_{\downarrow}^{(0)}(x_1',x_2')\, G_{\downarrow}^{(0)}(x_2',x_1') *$$

$$U_0(x_1-x_1')\, U_0(x_2-x_2') ;$$

and let us take Fourier transform:

$$\Delta G_{\uparrow}^{(2)}(k) = \int d(x-x')\, e^{-ik(x-x')} (-)(-)^2(-) U_0^2 \cdot \int dx_1 dx_2 \cdot G_{\uparrow}^{(0)}(x-x_1)\, G_{\uparrow}^{(0)}(x_1-x_2)\, G_{\uparrow}^{(0)}(x_2-x')\, G_{\downarrow}^{(0)}(x_1-x_2)\, G_{\downarrow}^{(0)}(x_2-x_1) ;$$

$$G_{\uparrow}^{(0)}(x-x_1) = \sum_{p_1} G_{\uparrow}^{(0)}(p_1)\, e^{ip_1(x-x_1)}$$

$$G_{\uparrow}^{(0)}(x_1-x_2) = \sum_{p_2} G_{\uparrow}^{(0)}(p_2)\, e^{ip_2(x_1-x_2)}$$

$$G_{\uparrow}^{(0)}(x_2-x') = \sum_{p_3} G_{\uparrow}^{(0)}(p_3)\, e^{ip_3(x_2-x')}$$

$$G_{\downarrow}^{(0)}(x_1-x_2) = \sum_{p_4} G_{\downarrow}^{(0)}(p_4)\, e^{ip_4(x_1-x_2)}$$

$$G_{\downarrow}^{(0)}(x_2-x_1) = \sum_{p_5} G_{\downarrow}^{(0)}(p_5)\, e^{ip_5(x_2-x_1)}$$

Perturbative expansion of GF in momentum space

To demonstrate the momentum conservation:

$$\Delta G_{\uparrow}^{(2)}(k) = \int d(x-x') e^{-ik(x-x')} (-)^2 (-) U_0^2 \cdot \int dx_1 dx_2 \cdot G_{\uparrow}^{(0)}(x-x_1) G_{\uparrow}^{(0)}(x_1-x_2) G_{\uparrow}^{(0)}(x_2-x') G_{\downarrow}^{(0)}(x_1-x_2) G_{\downarrow}^{(0)}(x_2-x_1);$$

$$G_{\uparrow}^{(0)}(x-x_1) = \sum_{p_1} G_{\uparrow}^{(0)}(p_1) e^{ip_1(x-x_1)}$$

$$G_{\uparrow}^{(0)}(x_1-x_2) = \sum_{p_2} G_{\uparrow}^{(0)}(p_2) e^{ip_2(x_1-x_2)}$$

$$G_{\uparrow}^{(0)}(x_2-x') = \sum_{p_3} G_{\uparrow}^{(0)}(p_3) e^{ip_3(x_2-x')}$$

$$G_{\downarrow}^{(0)}(x_1-x_2) = \sum_{p_4} G_{\downarrow}^{(0)}(p_4) e^{ip_4(x_1-x_2)}$$

$$G_{\downarrow}^{(0)}(x_2-x_1) = \sum_{p_5} G_{\downarrow}^{(0)}(p_5) e^{ip_5(x_2-x_1)}$$

$$= (-)^2(-) U_0^2 \sum_{p_1 \cdots p_5} G_{\uparrow}^{(0)}(p_1) G_{\uparrow}^{(0)}(p_2) G_{\uparrow}^{(0)}(p_3) G_{\downarrow}^{(0)}(p_4) G_{\downarrow}^{(0)}(p_5) \times \int d(x-x') dx_1 dx_2 \cdot$$

$$e^{-ik(x-x')} e^{ip_1(x-x_1)} e^{ip_2(x_1-x_2)} e^{ip_3(x_2-x')} e^{ip_4(x_1-x_2)} e^{ip_5(x_2-x_1)};$$

Let us collect (x_1, x_2), $\left[e^{-ip_1 x_1} e^{i(p_2+p_4-p_5)(x_1-x_2)} e^{ip_3 x_2} \right]$; $\therefore$ the $\int dx_1 dx_2$ leads to two δ-functions:

$$\begin{cases} p_2 + p_4 - p_5 - p_1 = 0 \\ p_2 + p_4 - p_5 - p_3 = 0 \end{cases} \quad \therefore p_1 = +p_3; \quad \text{then: } \left[\int dx \cdot e^{-ik(x-x')} e^{ip_1 x} e^{-ip_3 x'} \right] \Rightarrow p_1 = p_3 = k.$$

$\therefore$ there are two independent momentum, say, p_2 and p_4, then: $p_5 = p_2 + p_4 - k$;

Perturbative expansion of GF in momentum space

The final result:

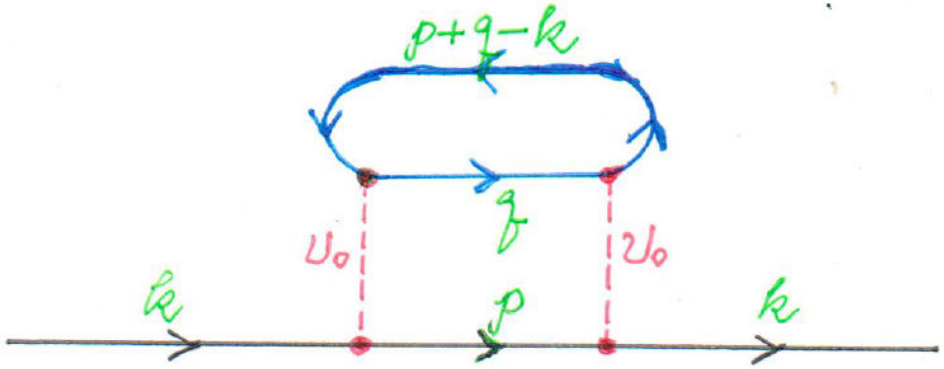
∴ there are two independent momentum: say, p_2 and p_4, then: $p_5 = p_2 + p_4 - k$;

∴ the result is:

$$\Delta G_{\uparrow}^{(2)}(k) = (-)^2(-)U_0^2 \sum_{p_2 p_4} \left[G_{\uparrow}^{(0)}(k)\right]^2 \cdot G_{\uparrow}^{(0)}(p_2)\, G_{\downarrow}^{(0)}(p_4)\, G_{\downarrow}^{(0)}(p_2+p_4-k) ;$$

p q

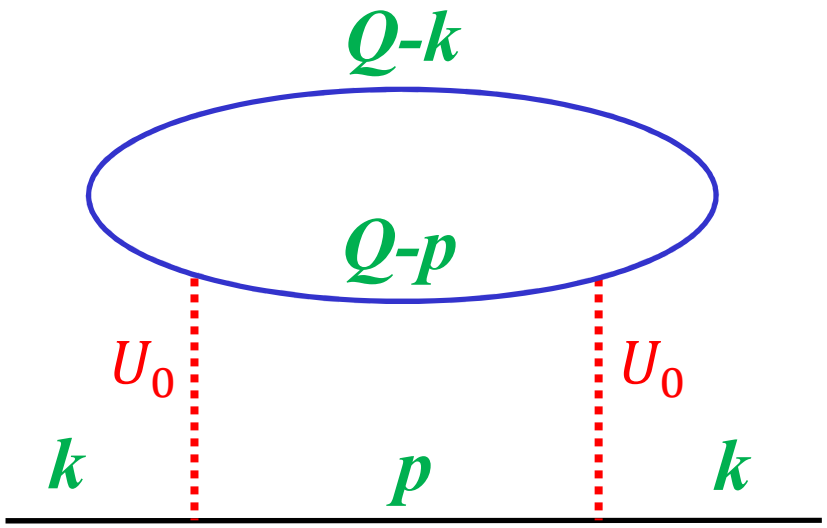
Actually, it is:



$$(-)^2(-)U_0^2(k_B T)^2 \sum_{i\omega_p, i\omega_q} \iint \frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3}\left[G_{\uparrow}^0(\mathbf{k}, i\omega_m)\right]^2 G_{\uparrow}^0(\mathbf{p}, i\omega_p) G_{\downarrow}^0(\mathbf{q}, i\omega_q) G_{\downarrow}^0(\mathbf{p}+\mathbf{q}-\mathbf{k}, i\omega_p + i\omega_q - i\omega_m)$$

Perturbative expansion of GF in momentum space

Alternatively



$$(-)^2(-)U_0^2(k_BT)^2\sum_{i\omega_p,i\nu_n}\iint\frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{Q}}{(2\pi)^3}\left[G_\uparrow^0(\mathbf{k},i\omega_m)\right]^2G_\uparrow^0(\mathbf{p},i\omega_p)G_\downarrow^0(\mathbf{Q}-\mathbf{p},i\nu_n-i\omega_p)G_\downarrow^0(\mathbf{Q}-\mathbf{k},i\nu_n-i\omega_m)$$

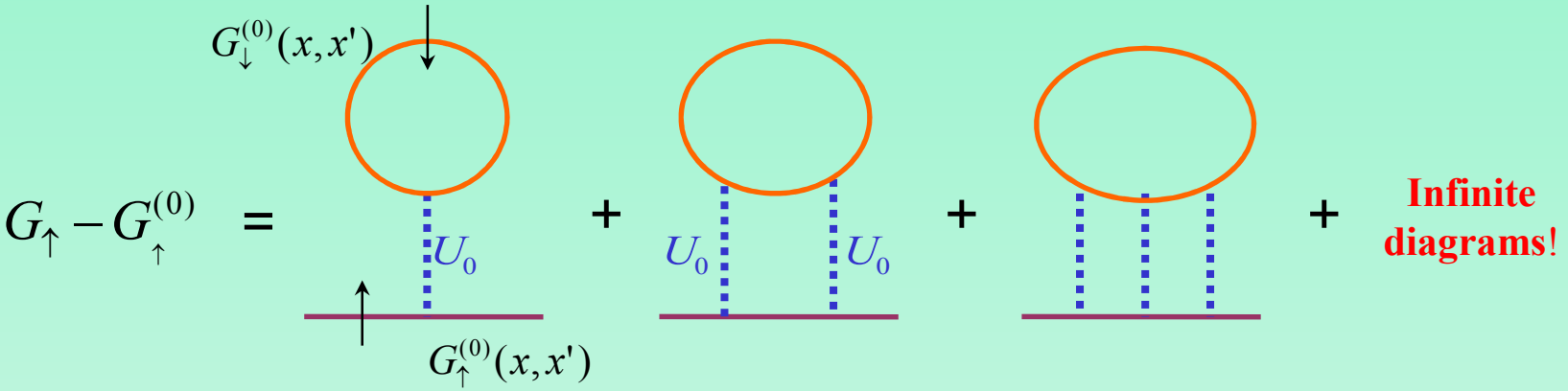
ω_m, ω_p: fermionic Matsubara frequency

ν_n: bosonic Matsubara frequency

Perturbative expansion of GF in momentum space

Let assume the interaction terms: $H_{\text{int}} = U_0 \int dx \psi_\uparrow^+(x)\psi_\downarrow^+(x')\delta(x-x')\psi_\downarrow(x')\psi_\uparrow(x)$, where $x \equiv (\mathbf{r}, \tau)$.

Green functions (GF):



FFT to *k*-space, using the following **Feynman rules** to calculate the Green function:

- With each *thin* line, associate it with an ideal GF: $G^{(1)}(k, i\omega_m) = 1/[i\omega_m - (\varepsilon_{\mathbf{k}} - \mu)]$;
- With *n*-interaction lines and *F* Fermi loops, add a prefactor $(-1)^{n+F}$;
- Integrate and sum over *independent* internal $\mathbf{k}$ and $i\omega_m$, using $\sum_k \equiv k_B T \sum_{i\omega_m} \int d\mathbf{k}/(2\pi)^3$.

Why this? All (diagrams) → connected → *different connected*!

Recall $G(x,x') = -\langle T_\tau \psi(x)\psi^+(x')U(\beta)\rangle_{con}$ where $U(\beta) = T_\tau \exp\{-\int_0^\beta H_{\text{int}}(\tau)d\tau\}$.

Perturbative expansion of Ω

Let us now consider the **thermodynamic potential** $\Omega(\mu, T)$. Once we know it, we may work out the equation of state of the system. According to its fundamental definition,

$$e^{-\beta\Omega}=\mathrm{Tr}\left[e^{-\beta H}\right]=\mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0)\right]=\mathrm{Tr}\left[e^{-\beta H_0}\right]\frac{\mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}=e^{-\beta\Omega_0}\left\langle\hat{U}(\beta,0)\right\rangle_0$$

We thus have,

$$\Omega=\Omega_0-k_BT\ln\left\langle\hat{U}(\beta,0)\right\rangle_0$$

where,

$$\hat{U}(\beta,0)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n T_\tau\left[\hat{H}_{int}(\tau_1)\cdots\hat{H}_{int}(\tau_n)\right]$$

Perturbative expansion of Ω

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We thus have,

$$\Omega=\Omega_0-k_BT\,\mathbf{ln}\left\langle\hat{U}(\beta,0)\right\rangle_0$$

where,

$$\hat{U}(\beta,0)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n T_\tau\left[\hat{H}_{int}(\tau_1)\cdots\hat{H}_{int}(\tau_n)\right]$$

But, how to handle the "**ln**"? We don't need to consider "**ln**", if we keep only **connected diagrams**!

Perturbative expansion of Ω

Let us now consider the **thermodynamic potential** $\boldsymbol{\Omega(\mu, T)}$. Once we know it, we may work out the equation of state of the system. According to its fundamental definition,

$$e^{-\beta\Omega}=\mathrm{Tr}\left[e^{-\beta H}\right]=\mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0)\right]=\mathrm{Tr}\left[e^{-\beta H_0}\right]\frac{\mathrm{Tr}\left[e^{-\beta H_0}\hat{U}(\beta,0)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}=e^{-\beta\Omega_0}\left\langle\hat{U}(\beta,0)\right\rangle_0$$

We thus have,

$$\Omega=\Omega_0-k_BT\,\mathbf{ln}\left\langle\hat{U}(\beta,0)\right\rangle_0$$

where,

$$\hat{U}(\beta,0)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n T_\tau\left[\hat{H}_{int}(\tau_1)\cdots\hat{H}_{int}(\tau_n)\right]$$

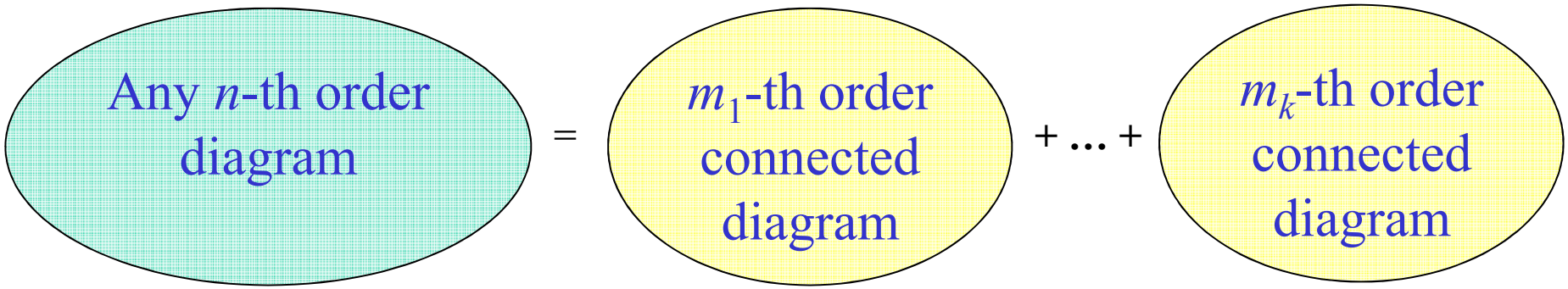
But, how to handle the "**ln**"? We don't need to consider "**ln**", if we keep only **connected diagrams**!

Perturbative expansion of Ω

The basic idea is for a n-th order diagrams,

$$\frac{(-1)^n}{n!}\int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n T_\tau\left[\hat{H}_{int}(\tau_1)\cdots\hat{H}_{int}(\tau_n)\right]$$

it can be written as ($n = m_1 + m_2 + \cdots + m_k$),



Mathematically, the right-hand side has the structure of the expansion of the function $\boldsymbol{e^{(x_1+\cdots+x_k)}}$! Here, $\left[\langle\hat{U}(\beta,0)\rangle_0\right]_{connected} = \mathbf{1} + \boldsymbol{x_1} + \cdots +.$

Thus, it is clear,

$$\Omega = \Omega_0 - k_B T\left[\langle\hat{U}(\beta,0)\rangle_0 - 1\right]_{connected}$$

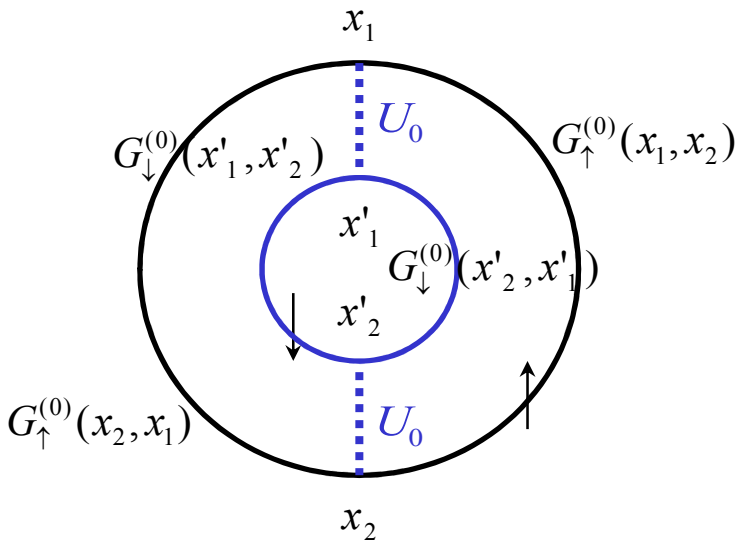
Tutorial: Perturbative expansion of Ω in real space

Let us consider the second-order contribution to **Ω**,

$$(-)k_B T \frac{1}{2}\iint dx_1 dx'_1 \iint dx_2 dx'_2 U_0(x_1 - x'_1)U_0(x_2 - x'_2)\times$$
$$\left\langle T_\tau \left\{\psi^+_\uparrow(x_1)\psi^+_\downarrow(x'_1)\psi_\downarrow(x'_1)\psi_\uparrow(x_1)\psi^+_\uparrow(x_2)\psi^+_\downarrow(x'_2)\psi_\downarrow(x'_2)\psi_\uparrow(x_2)\right\}\right\rangle_0$$

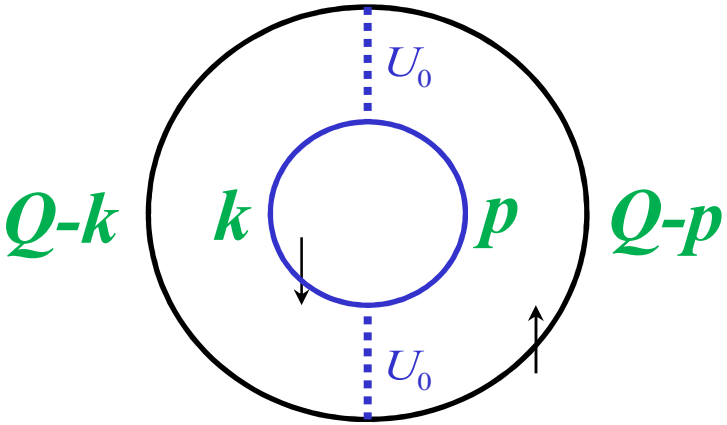
By using Wick theorem, we may obtain one connected diagram,

$$(-)k_B T \frac{U_0^2}{2}\iint dx_1 dx_2 G^{(0)}_\uparrow(x_1,x_2)G^{(0)}_\uparrow(x_2,x_1)G^{(0)}_\downarrow(x_1,x_2)G^{(0)}_\downarrow(x_2,x_1)$$



Tutorial: Perturbative expansion of Ω in momentum space

In momentum space, the result will be,



$$(-)V\frac{U_0^2}{2}\times$$

$$(k_BT)^3\sum_{i\omega_m,i\omega_p,i\nu_n}\iiint\frac{d\mathbf{Q}}{(2\pi)^3}\frac{d\mathbf{k}}{(2\pi)^3}\frac{d\mathbf{p}}{(2\pi)^3}G^0_{\uparrow}(\mathbf{Q}-\mathbf{k},i\nu_n-i\omega_m)G^0_{\uparrow}(\mathbf{Q}-\mathbf{p},i\nu_n-i\omega_p)G^0_{\downarrow}(\mathbf{k},i\omega_m)G^0_{\downarrow}(\mathbf{p},i\omega_p)$$

Perturbative expansion of Ω

It is readily to see that, the rules for **Ω** should be slightly modified:

$$\Omega = \Omega_0 - k_B T \left[\left\langle \hat{U}(\beta, 0) \right\rangle_0 - 1 \right]_{connected}$$

(1) Because the "−" before $k_B T$ in the above equation, we have an additional "−"

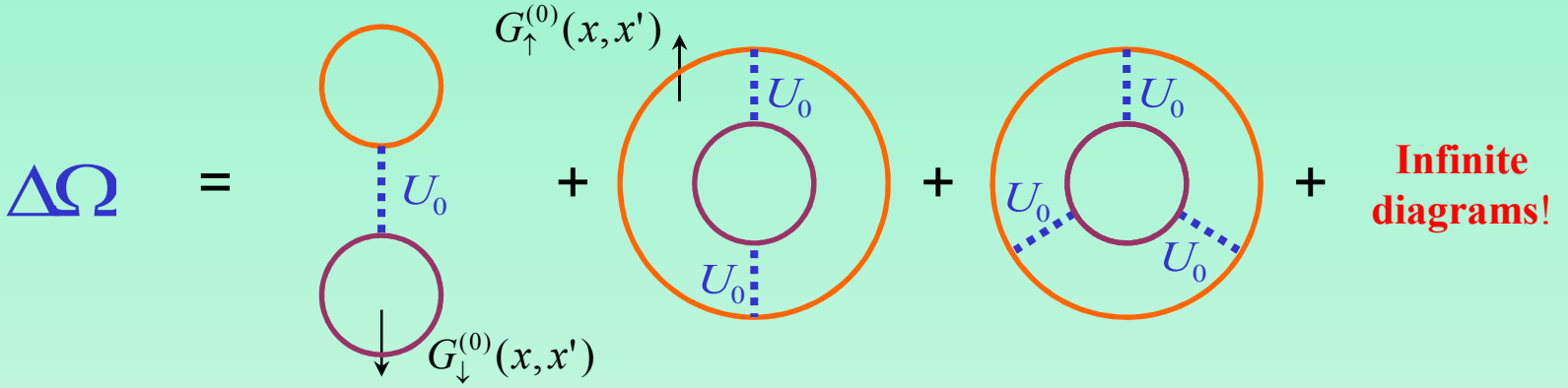
(2) In connecting the different terms in $T_\tau \left[\hat{H}_{int}(\tau_1) \cdots \hat{H}_{int}(\tau_n) \right]$, because we don't have external field operators, the prefactor $1/(n!)$ cannot be fully compensated. Each **topologically differently connected** diagram has (n-1)! possibility, leading to a remaining factor **1/*n***. Therefore, summation of different diagrams for the thermodynamic potential is more difficult than for the single-particle Green function.

(3) In performing Fourier transform to momentum space, the absence of the external field operators leads to additional integration over the center-of-mass coordinate and hence a volume factor ***V*** (this is reasonable, since the thermodynamic potential is an extensive quantity.

Perturbative expansion of Ω in momentum space

Let assume the interaction terms: $H_{\text{int}} = U_0 \int dx \psi_\uparrow^+(x)\psi_\downarrow^+(x')\delta(x-x')\psi_\downarrow(x')\psi_\uparrow(x)$, where $x \equiv (\mathbf{r}, \tau)$.

Thermodynamic potential $(\Delta\Omega = \Omega - \Omega^{(1)})$:

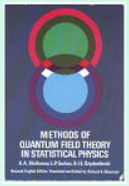


FFT to *k*-space, we have additional **Feynman rules** for the thermodynamic potential:

- With n-interaction lines and F Fermi loops, add a prefactor $(-1)^{n+F+1}$;
- With n-interaction lines, add a prefactor $1/n$;
- Additional integral over the centre-of-mass, leading to a factor of volume (V) to $\Delta\Omega$.

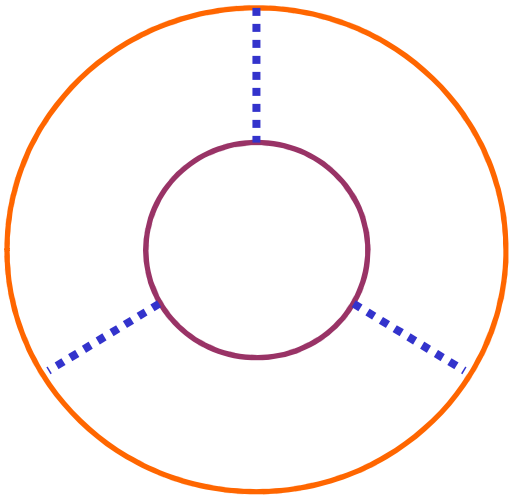
Why this? All (diagrams) → connected → *different connected* !

Recall $\Omega = \Omega^{(0)} - k_B T \ln \langle U(\beta) \rangle$ where $U(\beta) = T_\tau \exp\{-\int_0^\beta H_{\text{int}}(\tau) d\tau\}$.



Tutorial: Perturbative expansion of Ω in momentum space

In momentum space, please write down the expression for



$$(+)V\frac{U_0^3}{3}k_BT\sum_{i\nu_n}\int\frac{d\mathbf{Q}}{(2\pi)^3}\left[k_BT\sum_{i\omega_m}\int\frac{d\mathbf{k}}{(2\pi)^3}G_{\uparrow}^0(\mathbf{Q}-\mathbf{k},i\nu_n-i\omega_m)G_{\downarrow}^0(\mathbf{k},i\omega_m)\right]^3$$

Ideal gas thermodynamic potential

But, still, there is a minor problem: what is the **thermodynamic potential** $\boldsymbol{\Omega_0(\mu, T)}$ of an ideal Fermi or Bose gas?

We may use the thermodynamic relation, $-\dfrac{\partial \Omega}{\partial \mu} = n$

Recall that,

$$n_{k\sigma}{}^{(0)} = n_F(\xi_{k\sigma}) = k_B T \sum_{m=-\infty}^{+\infty} \frac{e^{i\omega_m 0^+}}{i\omega_m - \xi_{k\sigma}} \qquad n_k{}^{(0)} = n_B(\xi_k) = -k_B T \sum_{n=-\infty}^{+\infty} \frac{e^{i\nu_n 0^+}}{i\nu_n - \xi_k}$$

(fermions) (bosons)

We may have,

$$\Omega_{0,k\sigma} = -k_B T \sum_{m=-\infty}^{+\infty} \ln[-(i\omega_m - \xi_{k\sigma})]\, e^{i\omega_m 0^+} \qquad \Omega_{0,k} = +k_B T \sum_{n=-\infty}^{+\infty} \ln[-(i\nu_n - \xi_k)]\, e^{i\nu_n 0^+}$$

Or

$$\Omega_0 = +k_B T \sum_{n=-\infty}^{+\infty} \sum_{k} \ln\left[-G_0{}^{-1}(k, i\nu_n)\right] e^{i\nu_n 0^+}$$

Ideal gas thermodynamic potential

For bosons, we now have,

$$\Omega_0 = +k_B T \sum_{n=-\infty}^{+\infty} \sum_{\boldsymbol{k}} \ln\left[-{G_0}^{-1}(\boldsymbol{k}, i\nu_n)\right] e^{i\nu_n 0^+}$$

where,

$${G_0}^{-1}(\boldsymbol{k}, i\nu_n) = (i\nu_n - \xi_{\boldsymbol{k}})$$

What will happen if $\xi_{\boldsymbol{k}} < \mathbf{0}$?

Bose-Einstein condensation!

A brief summary

1. Feynman rules for the **Green function** (or **any response functions**!!!):

- With each *thin* line, associate it with an ideal GF: $G^{(1)}(k,i\omega_m)=1/[i\omega_m-(\varepsilon_{\mathbf{k}}-\mu)]$;
- With n-interaction lines and F Fermi loops, add a prefactor $(-1)^{n+F}$;
- Integrate and sum over *independent* internal $\mathbf{k}$ and $i\omega_m$, using $\sum_k \equiv k_B T \sum_{i\omega_m} \int d\mathbf{k}/(2\pi)^3$.

2. Feynman rules for the **thermodynamic potential** (additional):

- With n-interaction lines and F Fermi loops, add a prefactor $(-1)^{n+F+1}$;
- With n-interaction lines, add a prefactor $1/n$;
- Additional integral over the centre-of-mass, leading to a factor of volume (V) to $\Delta\Omega$.

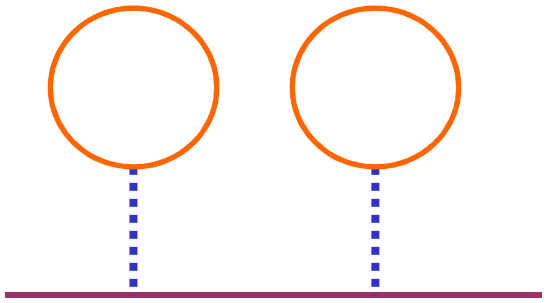
3. You may build **new Feynman rules** for a new Hamiltonian (system) in a few minutes!

Dyson equation

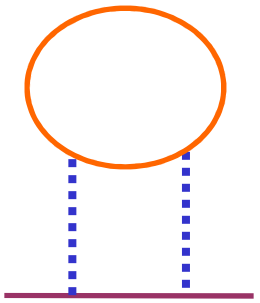
Let us think more about the diagrams: (i) How to simplify diagrams? (ii) How can we find the most important diagrams, or a series of important diagrams and sum them up?

We already consider **disconnected** diagrams, which can be taken into account if we account for the **connected** diagram only!

Any similar considerations? Yes, we have, for the one-particle **reducible** diagrams!



one-particle **reducible** diagram



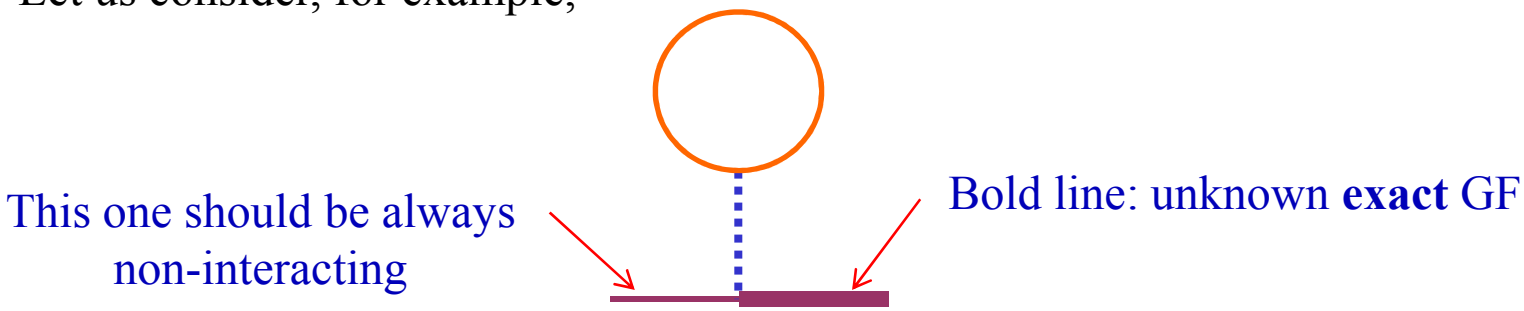
one-particle **irreducible** diagram

May we consider the one-particle irreducible diagrams only? Yes, you can!

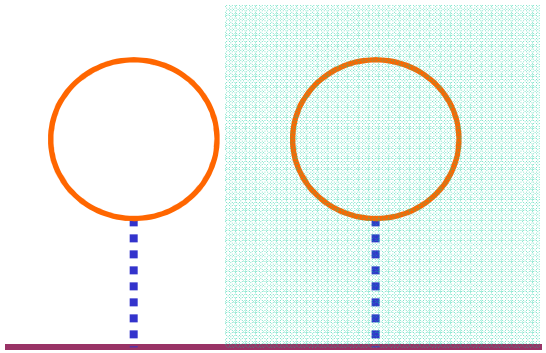
Dyson equation

In all the diagrams (i.e., for the Green function), we may replace all the non-interacting Green functions by the (*unknown*) **exact** Green function, except **one** non-interacting Green function!

Let us consider, for example,

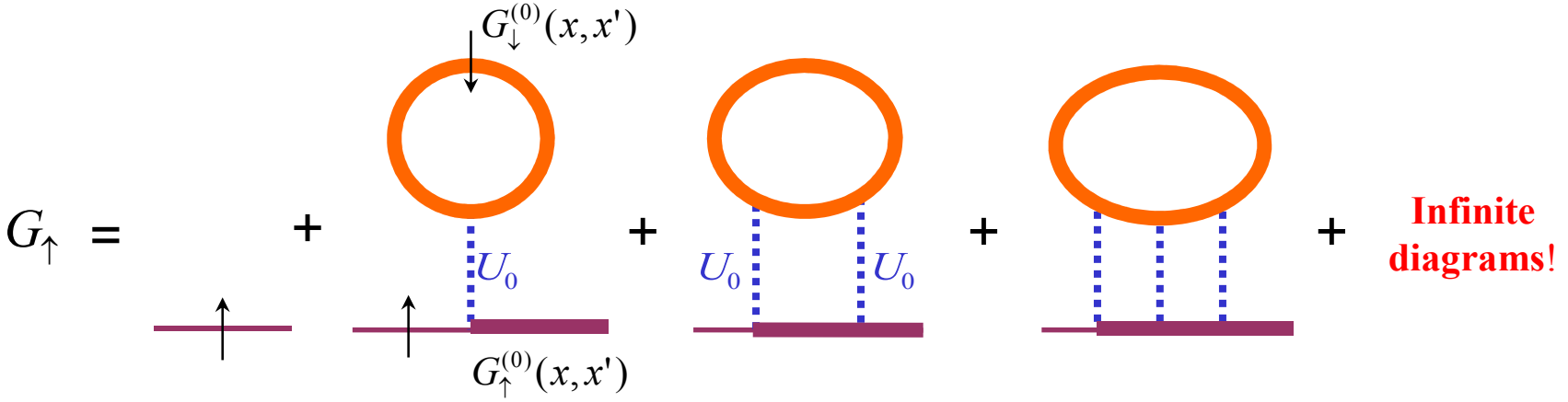


This diagram actually already includes,

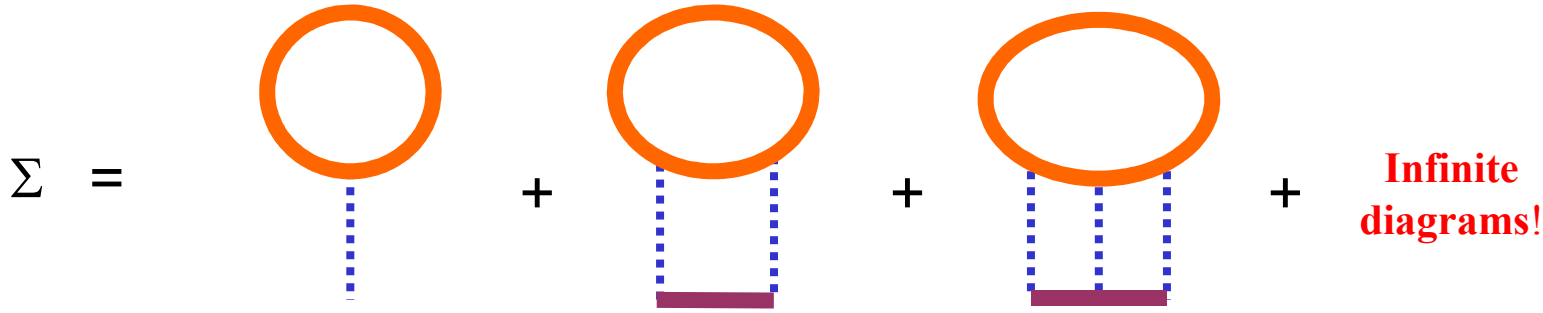


Dyson equation

Therefore the diagrams for the Green function can be represented by,



Thus, we may rewrite $G_{\uparrow} = G_{\uparrow}^{0} + G_{\uparrow}^{0}\Sigma G_{\uparrow}$, where the self-energy Σ is diagrammatically given by,

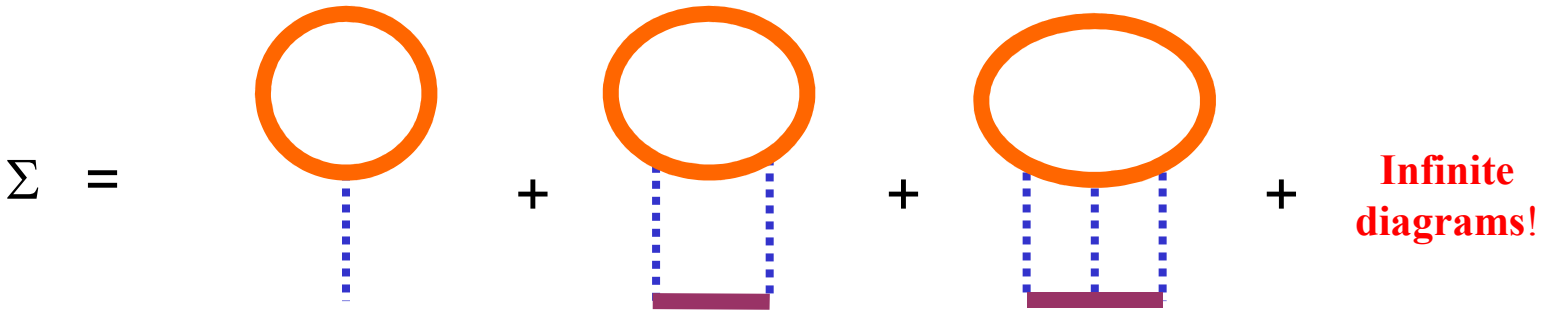


Dyson equation

We thus have the well-known **Dyson equation**:

$$G^{-1} = {G_0}^{-1} - \Sigma$$

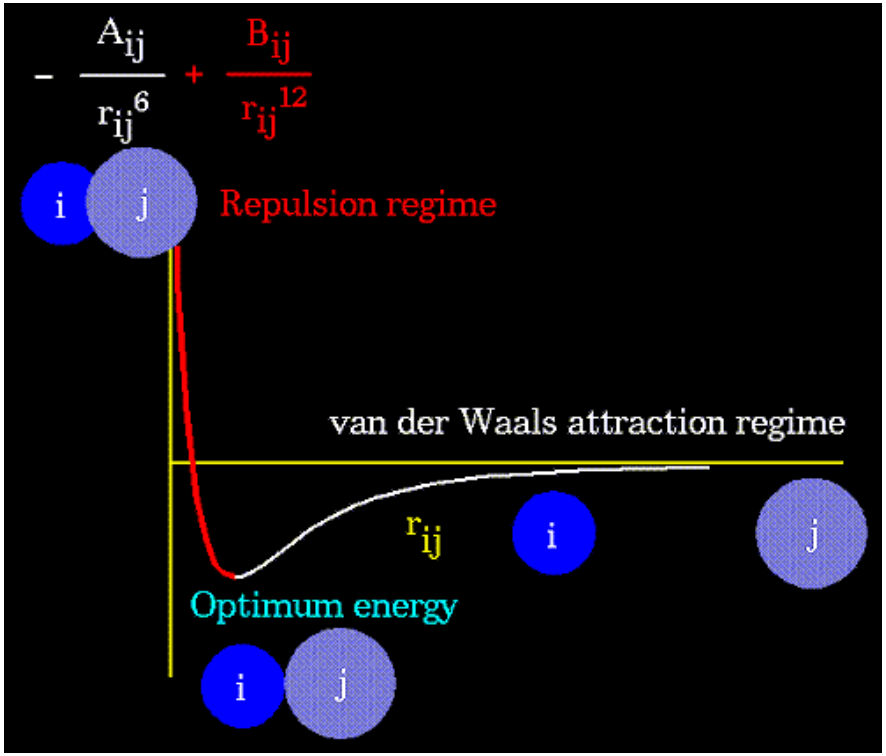
A few diagrams for the self-energy are:



Then, how about the two-particle **irreducible** diagrams?

Renormalization of the contact interaction

Why we use contact interactions? $H_{\text{int}} = U_0 \int dx \psi_\uparrow^+(x)\psi_\downarrow^+(x')\delta(x-x')\psi_\downarrow(x')\psi_\uparrow(x)$?



For *ultra-low temperature dilute* gas, we may use any interaction potentials (including contact interaction!), provided that they give the *same* s-wave scattering length. However, the contact interaction is physical only when the momentum $k < \Lambda = 1/r^*$, where r^* is the effective range of interactions. This requires re-normalization…

What we have learned about scattering

We can define an effective scattering potential,

$$V_{eff} = V + VgV + VgVgV + \cdots$$

At the low energy, the scattering amplitude is given by,

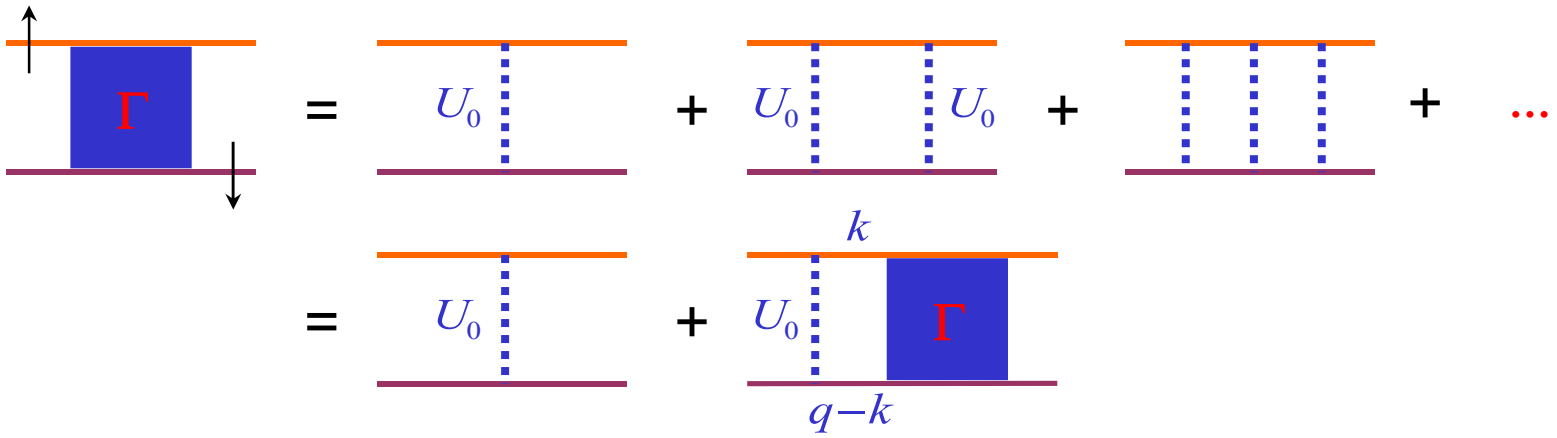
$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int V_{eff}(\mathbf{r})d\mathbf{r}$$

We then consider the different partial-wave amplitude, i.e.,

$$f = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell P_\ell(\cos\theta)$$

Renormalization of the contact interaction

Here, let us consider the vertex function (T-matrix),



We thus obtain $\Gamma(q) = U_0 + (-1)U_0 \sum_k G^{(0)}(k) G^{(0)}(q-k) \Gamma(q)$ or

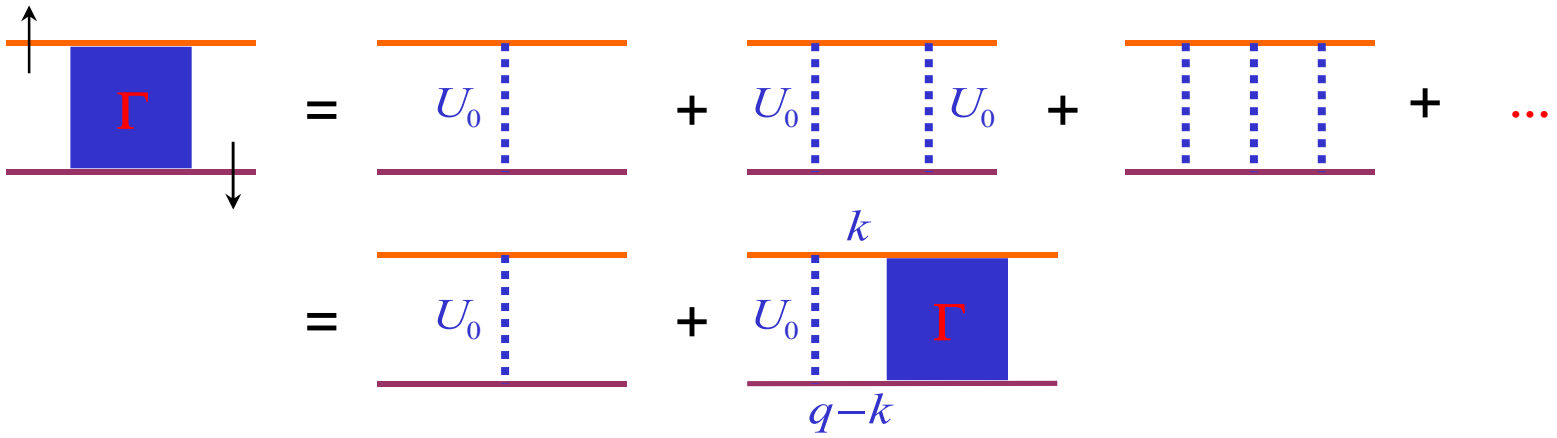
$$\Gamma^{-1}(q) = U_0^{-1} + \chi(q)$$

In more detail ($\xi_k = \varepsilon_k - \mu$ and $f(x)$ is the Fermi distribution function)

$$\frac{1}{\Gamma(q)} = \frac{1}{U_0} + \sum_{\mathbf{k}} k_B T \sum_{i\omega_m} \frac{1}{[i\omega_m - \xi_{\mathbf{k}}]} \frac{1}{[i\nu_n - i\omega_m - \xi_{\mathbf{q-k}}]} = \frac{1}{U_0} + \sum_{\mathbf{k}} \frac{f(\xi_{\mathbf{k}}) + f(\xi_{\mathbf{q-k}}) - 1}{i\nu_n - \xi_{\mathbf{k}} - \xi_{\mathbf{q-k}}}$$

Renormalization of the contact interaction

Here, let us consider the vertex function (T-matrix),



versus

$$V_{eff} = V + VgV + VgVgV + \cdots$$

If we identify $V = U_0$ and $g = -\chi(q)$

Renormalization of the contact interaction

In our case, we consider *s*-wave interaction only. What is the relation between the vertex function and the scattering amplitude?

The vertex function in vacuum (two-body) gives the scattering amplitude !!!

$$\frac{m}{\hbar^2}\Gamma(q=0)_{vac} = f_{sc} \approx 4\pi a$$

Here, the vacuum (two-body) means (there is no Fermi sea or μ=0):

$$\frac{1}{\Gamma(q)_{vac}} = \frac{1}{U_0} + \sum_{\mathbf{k}} \frac{\quad}{i\nu_n - \xi_{\mathbf{k}} - \xi_{\mathbf{q}-\mathbf{k}}} - 1$$

One may immediately find that (**re-normalization**),

$$\frac{m}{4\pi\hbar^2 a} = \frac{1}{U_0} + \sum_{\mathbf{k}} \frac{m}{\hbar^2\mathbf{k}^2}$$

Recall the momentum k should be smaller than $1/r^*$. However, this scale is very large and can be sent to infinitely large. This implies the bare interaction U_0 is infinitely small!

Renormalization of the contact interaction

More physics with the vertex function in vacuum (two-body)?

$$\frac{1}{\left[\Gamma(\mathbf{q},i\nu_n)\right]_{vac}}=\frac{1}{U_0}+\sum_{\mathbf{k}}\frac{-1}{i\nu_n-\xi_{\mathbf{k}}-\xi_{\mathbf{q}-\mathbf{k}}}$$

For a given **q**, the two-body vertex function may have poles, which correspond to bound states! And the position of the bound state determine the bound state energy, i.e.,

$$\left[\Gamma(\mathbf{q},i\nu_n\to E_{\mathbf{d}}(\mathbf{q})<0)\right]^{-1}_{vac}=0$$

Actually, we may prove that,

$$\Gamma^{-1}_{vac}(\mathbf{q},\omega+i0^+)=\frac{m}{4\pi\hbar^2 a}+\frac{im^{3/2}}{4\pi\hbar^3}\sqrt{\omega+i0^+-\frac{\varepsilon_{\mathbf{q}}}{2}};$$

What happens if $a > 0$?

Consider the two-body vertex function,

$$\Gamma_{vac}^{-1}(\mathbf{q},\omega+i0^+)=\frac{m}{4\pi\hbar^2 a}+\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\omega+i0^+-\xi_{\mathbf{q}/2+\mathbf{k}}-\xi_{\mathbf{q}/2-\mathbf{k}}}-\frac{m}{\hbar^2\mathbf{k}^2}\right]$$

Let us define,

$$A^2=\omega+i0^+-\frac{\varepsilon_q}{2}+2\mu$$

Then,

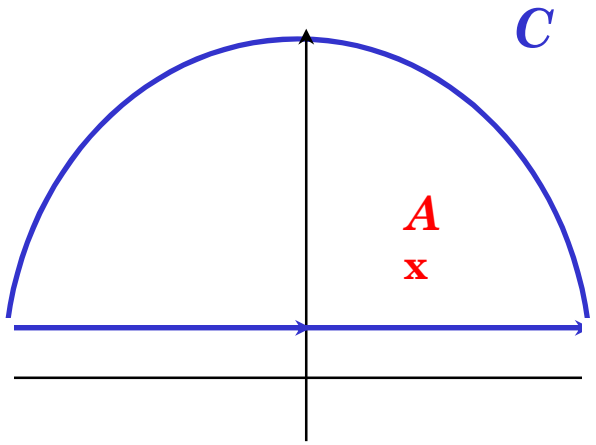
$$\Gamma_{vac}^{-1}(\mathbf{q},\omega+i0^+)$$

$$=\frac{m}{4\pi\hbar^2 a}+\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\frac{\hbar^2\mathbf{k}^2}{m}-A^2}-\frac{1}{\frac{\hbar^2\mathbf{k}^2}{m}}\right]$$

$$=\frac{m}{4\pi\hbar^2 a}+\frac{m^{3/2}}{4\pi^2\hbar^3}\int_0^\infty q^2dq\left[\frac{1}{q^2-A^2}-\frac{1}{q^2}\right]$$

$$=\frac{m}{4\pi\hbar^2 a}+\frac{m^{3/2}A}{4\pi^2\hbar^3}\int_0^\infty dq\left[\frac{1}{(q-A)(q+A)}\right]$$

$$=\frac{m}{4\pi\hbar^2 a}+\frac{im^{3/2}}{4\pi\hbar^3}\sqrt{A}$$

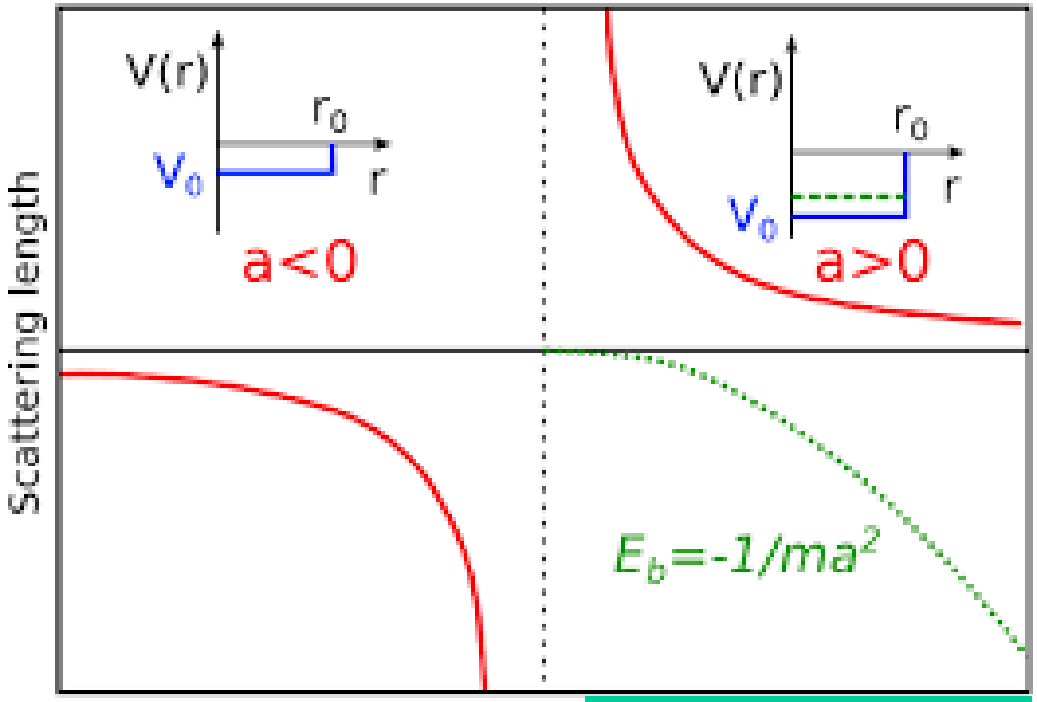


Renormalization of the contact interaction

We can see immediately that if $a > 0$,

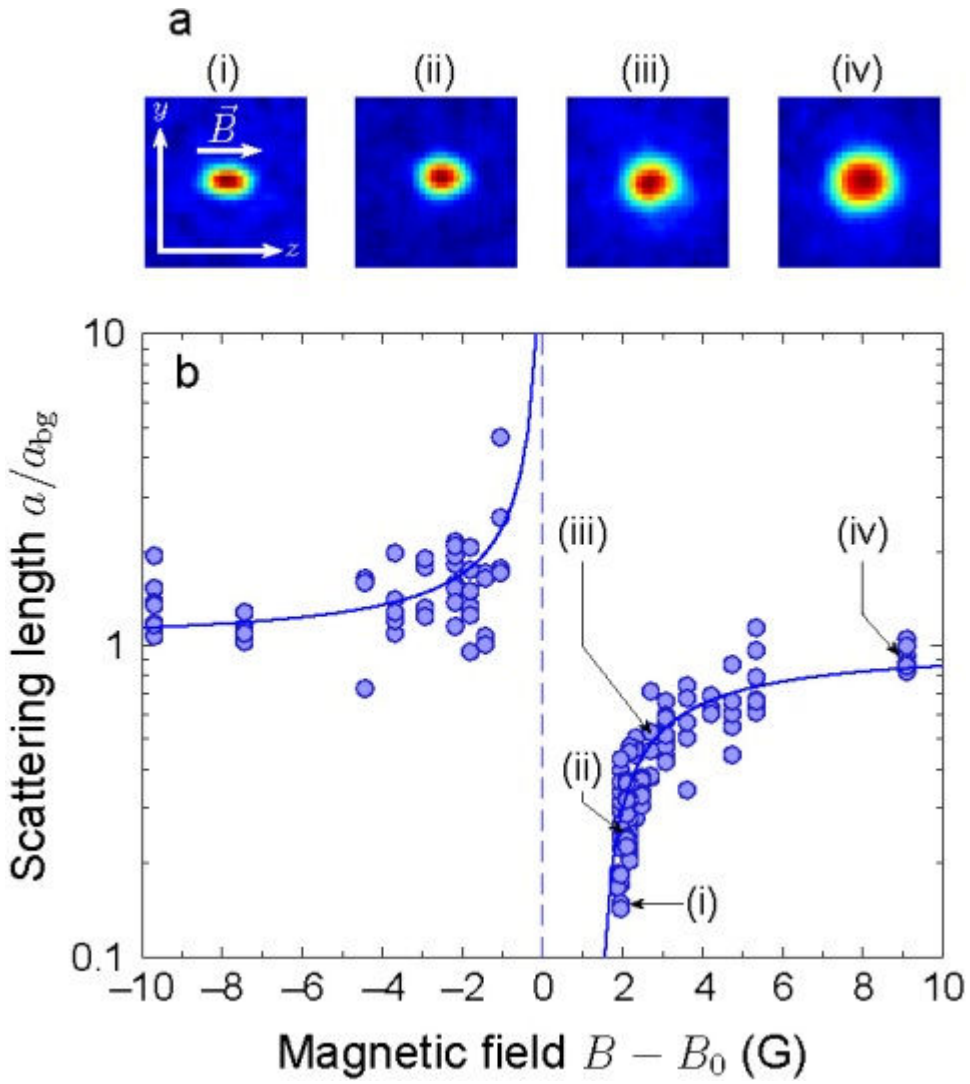
$$E_{\mathbf{d}}(\mathbf{q}) = -\frac{\hbar^2}{ma^2} + \frac{\varepsilon_{\mathbf{q}}}{2}$$

Therefore, we have the following picture:



Magnetic Feshbach resonance

Experimentally, the interaction depth can be changed by using Magnetic Feshbach resonance!!!



Potential depth U_0<0

Application 1:
Moving impurity (Fermi/Bose polaron)

Many-body problems are difficult; can you recommend the easiest one? Is the problem exactly solvable?

We are actually building a Fermi liquid from the bottom up!

Swimming in the Fermi sea

What is the fate of a single impurity in a Fermi sea?

This is a crucial question for

- **electron transport in lattices**
- **Kondo problem (single magnetic impurity)**
- **mobility of ^{3}He in ^{4}He**
- **…**
- **determines the properties of many condensed matter systems at low temperature**

credit to Martin Zwierlein **WIPM, CAS**

Impurity interacts with a Fermi sea

Example: Kondo effect

A spin impurity interacting with Fermi sea of electrons leads to increase in resistance at low temperatures

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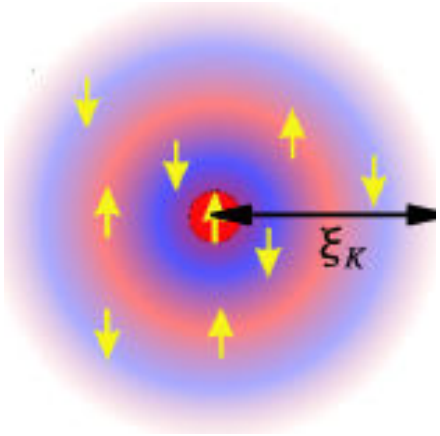

Mesoscopic Kondo Screening Effect in a Single-Electron Transistor Embedded in a Metallic Ring

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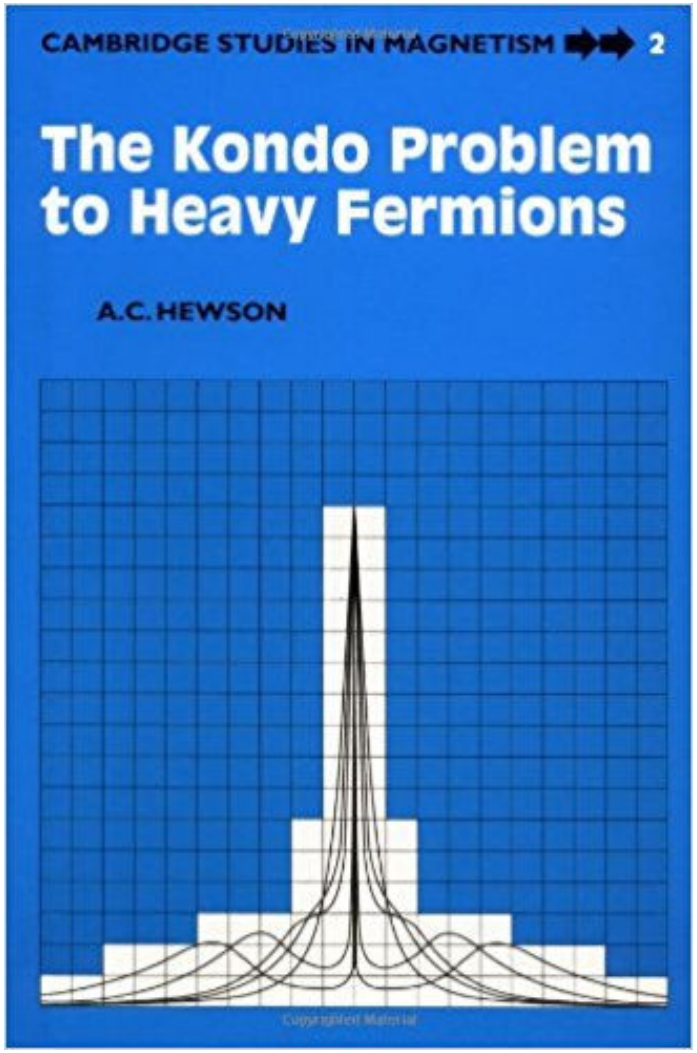
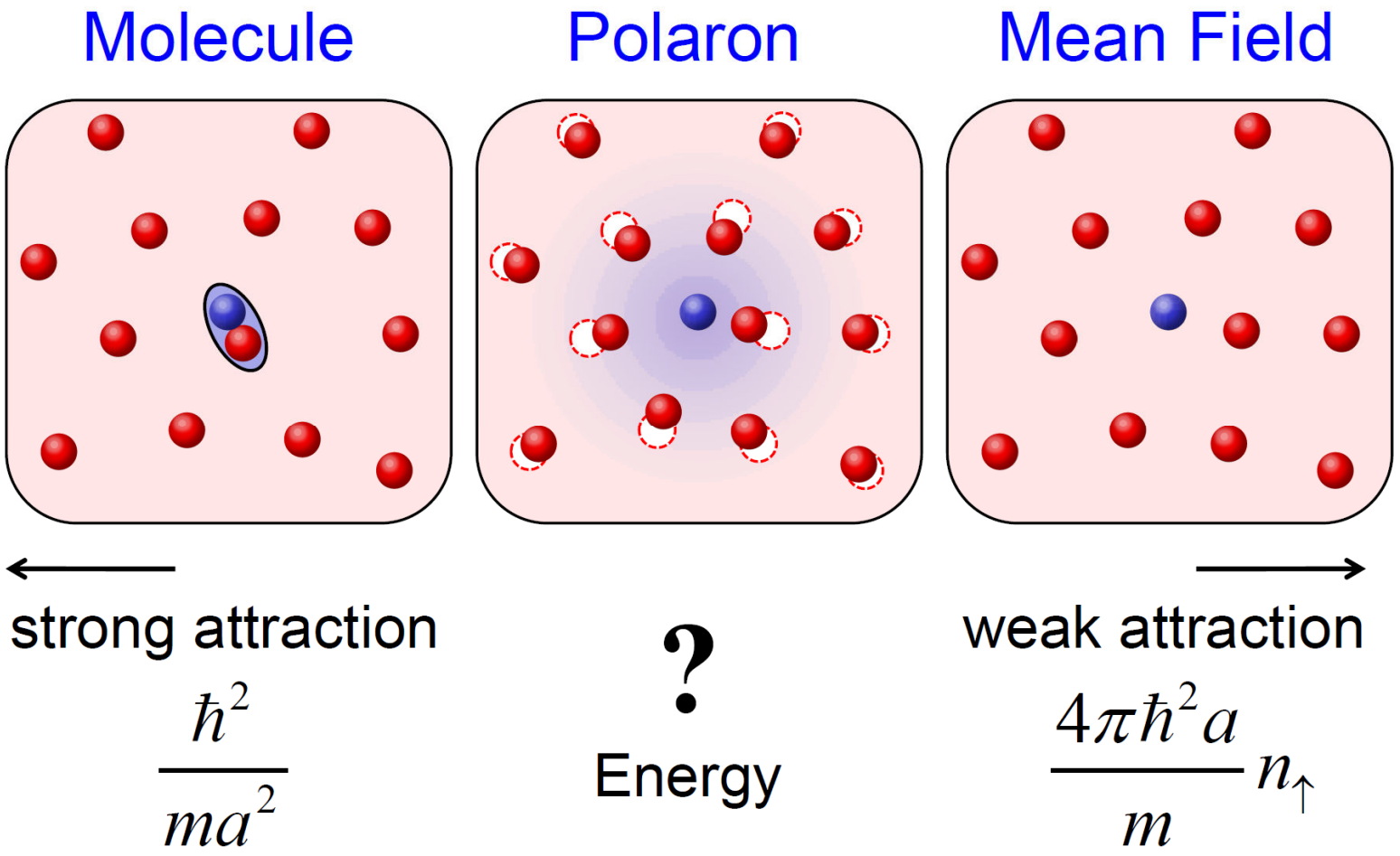
(Received 9 February 2001)

We study the Kondo screening effect generated by a single-electron transistor or quantum dot embedded in a small metallic ring. When the ring circumference L becomes comparable to the fundamental length scale $\xi_K^0 = \hbar v_F/T_K^0$ associated with the *bulk* Kondo temperature, the Kondo resonance is strongly affected, depending on the total number of electrons (mod4) and magnetic flux threading the ring. The resulting Kondo-assisted persistent currents are also calculated in both Kondo and mixed-valence regimes, and the maximum values are found in the crossover region.

Textbook for Kondo problem



Swimming in the Fermi sea



credit to Martin Zwierlein WIPM, CAS

SWINBURNE CENTRE FOR QUANTUM AND OPTICAL SCIENCE

Feynman diagrammatic theory of Fermi polaron

PRL **98,** 180402 (2007) PHYSICAL REVIEW LETTERS week ending 4 MAY 2007

Normal State of Highly Polarized Fermi Gases: Simple Many-Body Approaches

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(Received 16 February 2007; published 1 May 2007)

We consider the problem of a single ↓ atom in the presence of a Fermi sea of ↑ atoms, in the vicinity of a Feshbach resonance. We calculate the chemical potential and the effective mass of the ↓ atom using two simple approaches: a many-body variational wave function and a T-matrix approximation. These two methods lead to the same results and are in good agreement with existing quantum Monte Carlo calculations performed at unitarity and, in one dimension, with the known exact solution. Surprisingly, our results suggest that, even at unitarity, the effect of interactions is fairly weak and can be accurately described using single particle-hole excitations. We also consider the case of unequal masses.

10 years for polaron problem in cold-atoms, still a lot of surprise/fun and challenge!

First cold-atom on Fermi polaron

PRL **102,** 230402 (2009) Selected for a Viewpoint in *Physics* PHYSICAL REVIEW LETTERS week ending 12 JUNE 2009

Observation of Fermi Polarons in a Tunable Fermi Liquid of Ultracold Atoms

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We have observed Fermi polarons, dressed spin-down impurities in a spin-up Fermi sea of ultracold atoms. The polaron manifests itself as a narrow peak in the impurities' rf spectrum that emerges from a broad incoherent background. We determine the polaron energy and the quasiparticle residue for various interaction strengths around a Feshbach resonance. At a critical interaction, we observe the transition from polaronic to molecular binding. Here, the imbalanced Fermi liquid undergoes a phase transition into a Bose liquid, coexisting with a Fermi sea.

DOI: 10.1103/PhysRevLett.102.230402 PACS numbers: 05.30.Fk, 03.75.Ss, 32.30.Bv, 67.60.Fp

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Latest cold-atom on Fermi polaron

Repulsive Fermi Polarons in a Resonant Mixture of Ultracold ^{6}Li Atoms

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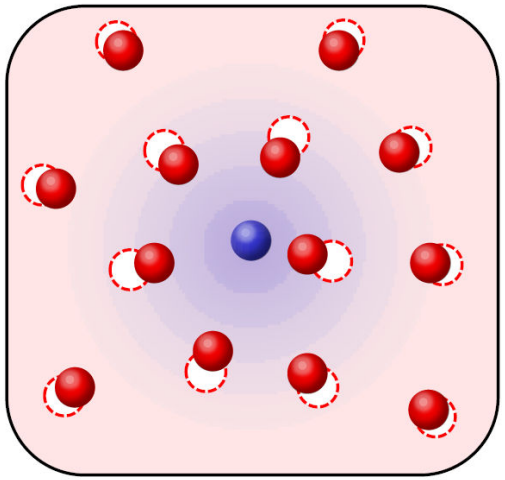
(Received 30 September 2016; published 21 February 2017)

We employ radio-frequency spectroscopy to investigate a polarized spin mixture of ultracold ^{6}Li atoms close to a broad Feshbach scattering resonance. Focusing on the regime of strong repulsive interactions, we observe well-defined coherent quasiparticles even for unitarity-limited interactions. We characterize the many-body system by extracting the key properties of repulsive Fermi polarons: the energy E_+, the effective mass m^*, the residue Z, and the decay rate Γ. Above a critical interaction, E_+ is found to exceed the Fermi energy of the bath, while m^* diverges and even turns negative, thereby indicating that the repulsive Fermi liquid state becomes energetically and thermodynamically unstable.

DOI: 10.1103/PhysRevLett.118.083602

Feynman diagrammatic theory of Fermi polaron



In momentum space, the Hamiltonian takes the form,

$$H_0 = \sum_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow}\left[\overbrace{\frac{\hbar^2\mathbf{k}^2}{2m_\uparrow}}^{\xi_{\uparrow k}} - \mu_\uparrow\right] c_{\mathbf{k}\uparrow} + \sum_{\mathbf{k}} c^{+}_{\mathbf{k}\downarrow}\left[\overbrace{\frac{\hbar^2\mathbf{k}^2}{2m_\downarrow}}^{\varepsilon_{\downarrow k}} - \mu_\downarrow\right] c_{\mathbf{k}\downarrow}$$

$$H_{\text{int}} = U_0 \sum_{\mathbf{qkk'}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{\mathbf{q}-\mathbf{k}\downarrow} c_{\mathbf{q}-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

Definition: $\mu_{\uparrow,\downarrow}$ – chemical potentials (**grand-canonical ensemble**), U_0 – interaction strength

The Green function of *N* spin-up atoms is exact!

$$G_\uparrow(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \xi_{\uparrow\boldsymbol{k}}}$$

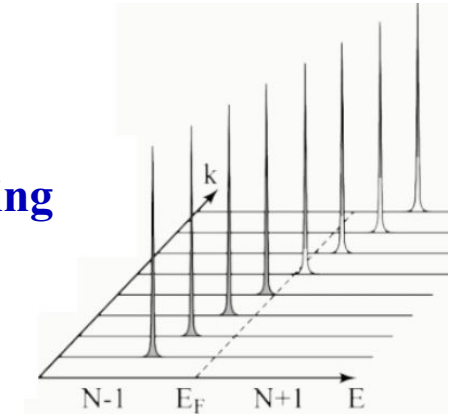
The Green function of 1 spin-down atom is to determined!

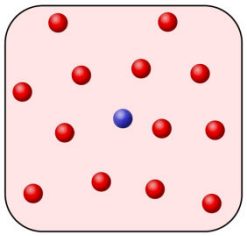
$$G_\downarrow(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \varepsilon_{\downarrow\boldsymbol{k}} + \mu_\downarrow - \Sigma(\boldsymbol{k}, i\omega_m)}$$

Feynman diagrammatic theory of Fermi polaron

A **naïve picture** of the spectral function $A_{\downarrow}(k, \omega)$

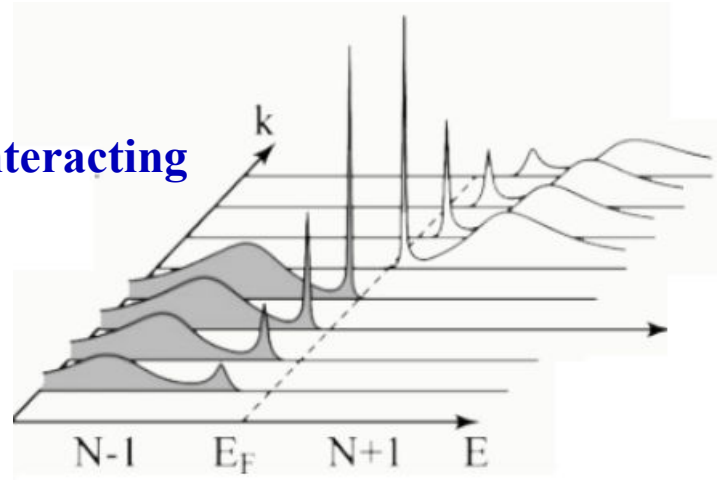
Non-interacting

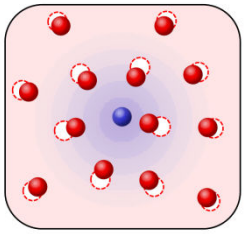




$$A_{\downarrow}^{(0)}(\mathbf{k},\omega) = \delta(\omega - \frac{\hbar^2\mathbf{k}^2}{2m_{\downarrow}})$$

Interacting





$$A_{\downarrow}(\mathbf{k},\omega) = Z\delta[\omega - \frac{\hbar^2\mathbf{k}^2}{2m^*}] + \cdots$$

Feynman diagrammatic theory of Fermi polaron

This means that we approximate the spin-down Green function:

$$G_\downarrow(\boldsymbol{k}, i\omega_m) = \frac{Z}{i\omega_m - \frac{\hbar^2\boldsymbol{k}^2}{2m^*}} + \cdots$$

Actually, we may Taylor-expand the self-energy at small $\boldsymbol{k}$ and ω

$$\Sigma(\boldsymbol{k}, i\omega_m) = \Sigma(0,0) + \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \varepsilon_{\downarrow\boldsymbol{k}}}\right]\frac{\hbar^2\boldsymbol{k}^2}{2m_\downarrow} + \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\right] i\omega_m + \cdots$$

Then, the Green function takes the form,

$$G_\downarrow(\boldsymbol{k}, i\omega_m) = \frac{1}{\underbrace{i\omega_m - \frac{\hbar^2\boldsymbol{k}^2}{2m_\downarrow} + \mu_\downarrow} - \Sigma(0,0) - \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \varepsilon_{\downarrow\boldsymbol{k}}}\right]\frac{\hbar^2\boldsymbol{k}^2}{2m_\downarrow} - \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\right] i\omega_m}$$

Feynman diagrammatic theory of Fermi polaron

Therefore, we must have,

$$\mu_{\downarrow} = \Sigma(0,0)$$

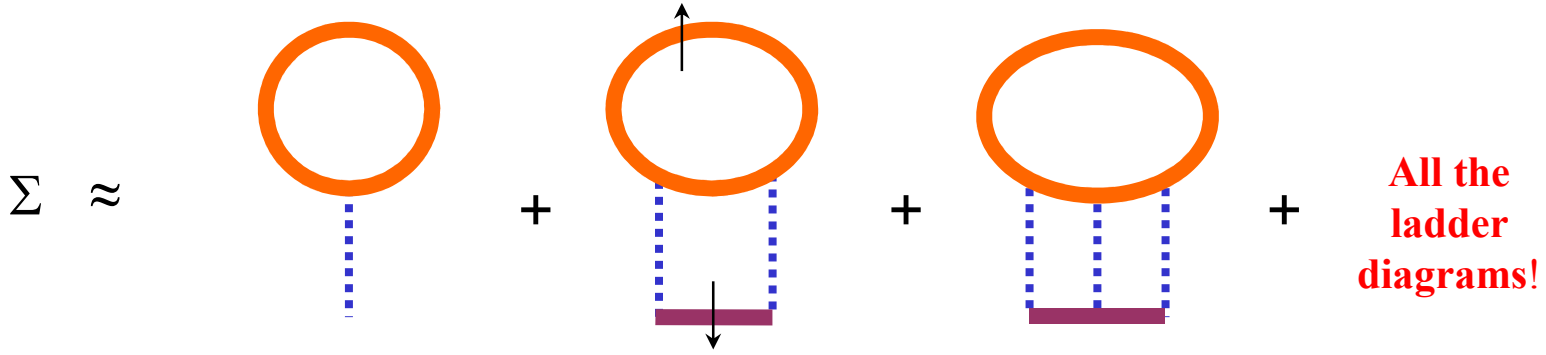
$$Z = \frac{1}{1 - \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\right]}$$

$$\frac{m^*}{m_{\downarrow}} = \frac{1 - \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\right]}{1 + \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \varepsilon_{\downarrow \boldsymbol{k}}}\right]}$$

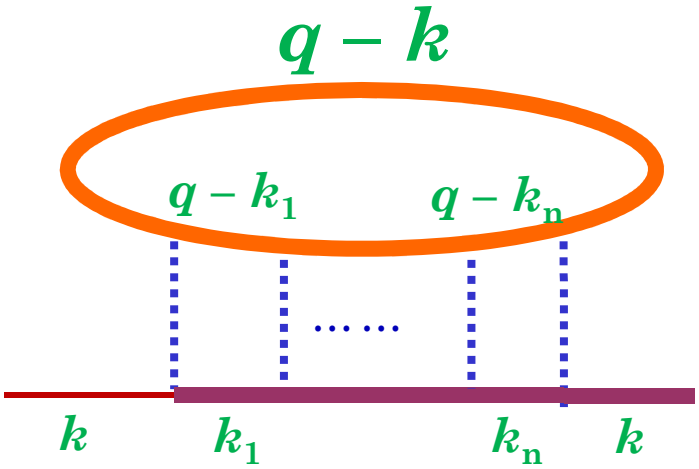
The first equation determines the spin-down chemical potential, since here **ω** is the energy measured from the chemical potential, and physically, the chemical potential corresponds to **the energy cost of adding a particle** with zero momentum ***k***=0.

Feynman diagrammatic theory of Fermi polaron

Now let us consider the following **ladder** diagrams for the self-energy:

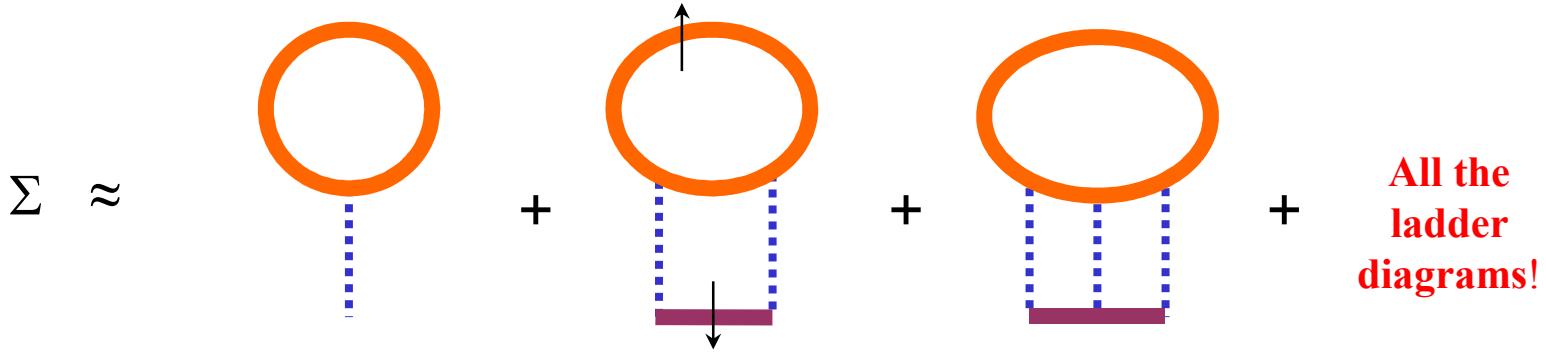


Can you write down the expressions of the above diagrams, say the *n*-th diagrams?

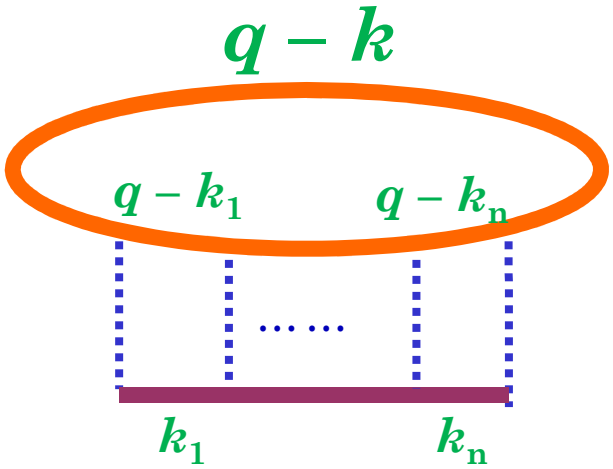


Feynman diagrammatic theory of Fermi polaron

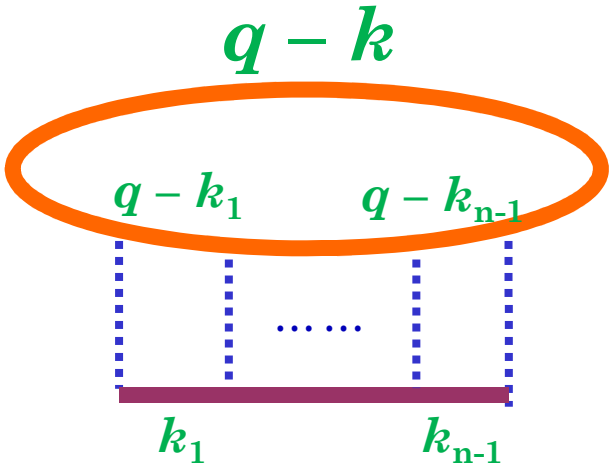
Now let us consider the following **ladder** diagrams for the self-energy:



Can you write down the expressions of the above diagrams, say the *n*-th diagrams?



Feynman diagrammatic theory of Fermi polaron



The answer is,

$$\Sigma^{(n)}(k) = (-)^{n+1} U_0^n \sum_q G_\uparrow(q-k) \sum_{k_1} \left[G_\uparrow(q-k_1) G_\downarrow(k_1) \right] \cdots \sum_{k_{n-1}} \left[G_\uparrow(q-k_{n-1}) G_\downarrow(k_{n-1}) \right]$$

or

$$\Sigma^{(n)}(k) = (-)^{n+1} U_0^n \sum_q G_\uparrow(q-k) \left[\chi(q) \right]^{n-1}$$

if we define

$$\chi(q) = \sum_p \left[G_\uparrow(q-p) G_\downarrow(p) \right] = \sum_{\mathbf{p}} k_B T \sum_{i\omega_p} G_\uparrow(\mathbf{q}-\mathbf{p}, i\nu_n - i\omega_p) G_\downarrow(\mathbf{p}, i\omega_p)$$

Feynman diagrammatic theory of Fermi polaron

How about the summation over "n"?

$$\Sigma(k)=\sum_{n=1,\ldots\infty}\Sigma^{(n)}(k)=\sum_{n=1,\ldots\infty}(-)^{n+1}U_0^n\sum_q G_\uparrow(q-k)\left[\chi(q)\right]^{n-1}$$

Recall the identity: $1/(1+x)=1-x+x^2-\cdots$

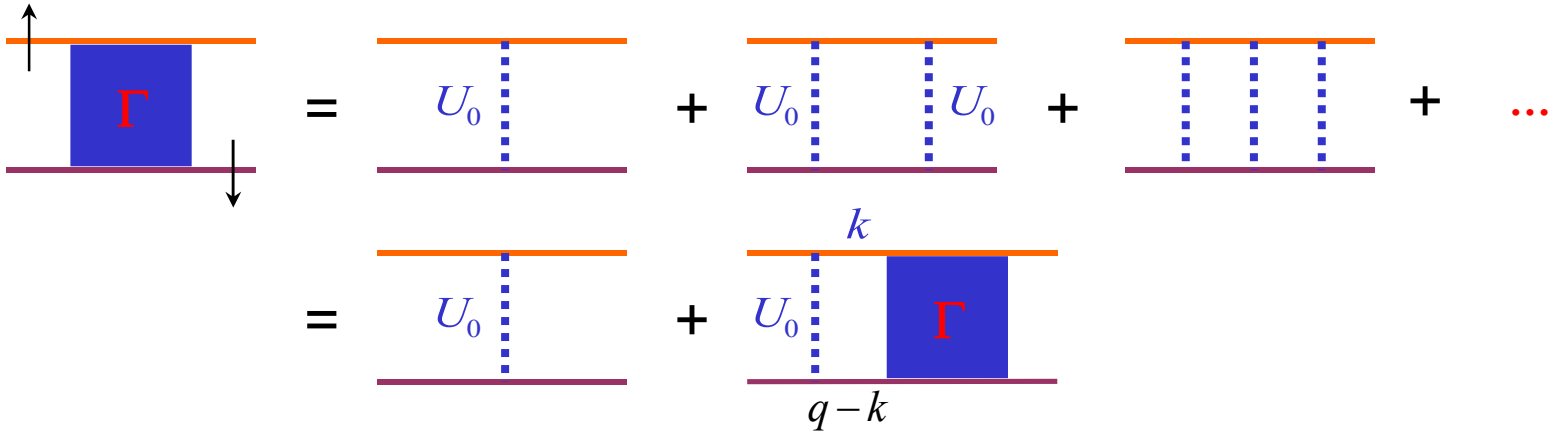
$$\Sigma(k)=\sum_q G_\uparrow(q-k)U_0/\left[1+U_0\chi(q)\right]=\sum_q G_\uparrow(q-k)\Gamma(q)$$

where, the two-particle vertex function (within ladder diagrams) is given by,

$$\Gamma(q)=\frac{U_0}{1+U_0\chi(q)}=\frac{1}{U_0^{-1}+\chi(q)}$$

Feynman diagrammatic theory of Fermi polaron

Actually we already define the vertex function (T-matrix) earlier,



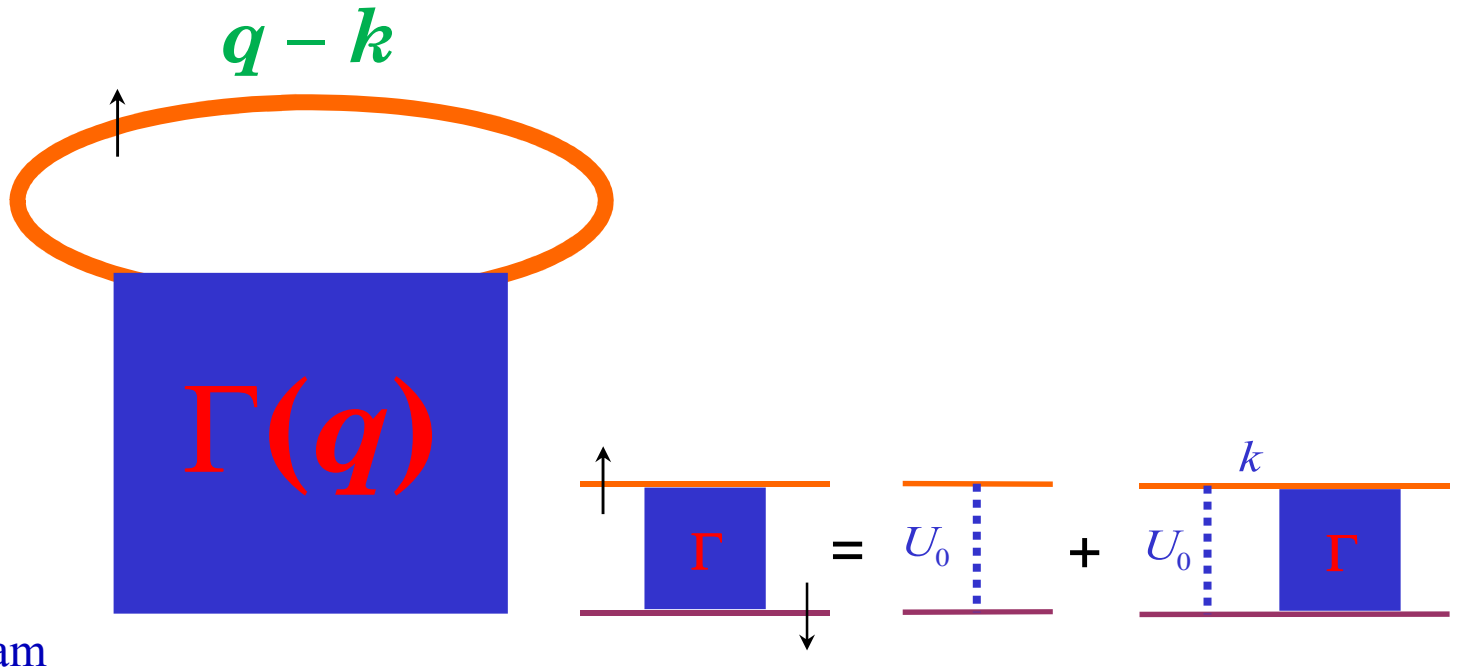
We thus obtain $\Gamma(q) = U_0 + (-1)U_0 \sum_k G^{(0)}(k)G^{(0)}(q-k)\Gamma(q)$

or

$$\Gamma^{-1}(q) = U_0^{-1} + \chi(q)$$

Feynman diagrammatic theory of Fermi polaron

This means we can directly calculate the self-energy by using the following diagram:

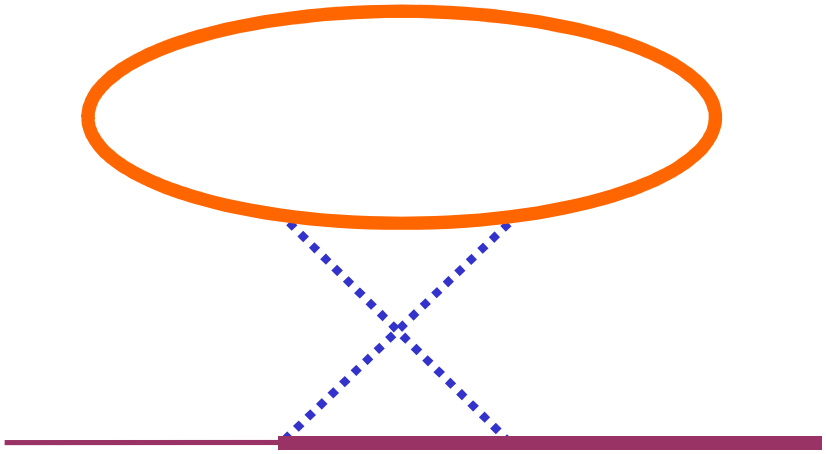


First order diagram

$$\Sigma(k) = (-)(-)\sum_{q} G_{\uparrow}(q-k)\Gamma(q) = \sum_{\mathbf{q}} k_B T \sum_{i\nu_n} G_{\uparrow}(\mathbf{q}-\mathbf{k}, i\nu_n - i\omega_m)\Gamma(\mathbf{q}, i\nu_n)$$

One Fermi loop

Can you give an example of ignored diagrams?



Feynman diagrammatic theory of Fermi polaron

Now our problem becomes, solving the coupled equations (within ladder diagrams),

$$\Sigma(\mathbf{k}, i\omega_m) = \sum_{\mathbf{q}} k_B T \sum_{i\nu_n} G_\uparrow(\mathbf{q}-\mathbf{k}, i\nu_n - i\omega_m)\Gamma(\mathbf{q}, i\nu_n)$$

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = U_0^{-1} + \sum_{\mathbf{p}} k_B T \sum_{i\omega_p} G_\uparrow(\mathbf{q}-\mathbf{p}, i\nu_n - i\omega_p) G_\downarrow(\mathbf{p}, i\omega_p)$$

together the Dyson equation,

$$G_\downarrow(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \varepsilon_{\downarrow\boldsymbol{k}} + \mu_\downarrow - \Sigma(\boldsymbol{k}, i\omega_m)}$$

and subjected to the constriction (i.e., single impurity):

$$\mu_\downarrow = \Sigma(0,0) < 0$$

Also, note the renormalisation for interaction,

$$\frac{m_r}{2\pi\hbar^2 a} = \frac{1}{U_0} + \sum_{\mathbf{k}} \frac{2m_r}{\hbar^2 \mathbf{k}^2}$$

Feynman diagrammatic theory of Fermi polaron

The coupled equation can be solved iteratively, but first, let us work out the **first-order iteration**, i.e., using the non-interacting spin-down Green function,

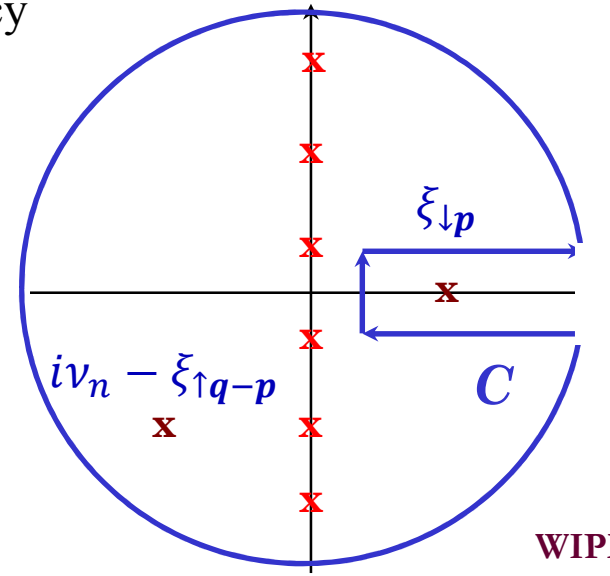
$$G_{\downarrow}(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \varepsilon_{\downarrow\boldsymbol{k}} + \mu_{\downarrow}}$$

Let us also focus on the **zero temperature** case. What is the vertex function?

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = U_0^{-1} + \sum_{\mathbf{p}} k_B T \sum_{i\omega_p} \frac{1}{i\nu_n - i\omega_p - \xi_{\uparrow\mathbf{q}-\mathbf{p}}} \frac{1}{i\omega_p - \xi_{\downarrow\mathbf{p}}}$$

Let us sum over the fermionic Matsubara frequency at ***zero* temperature**!

$$\oint \frac{e^{\omega 0^+}}{e^{\beta\omega} + 1} \left[\frac{1}{i\nu_n - \omega - \xi_{\uparrow\boldsymbol{q}-\boldsymbol{p}}} \right] \frac{1}{\omega - \xi_{\downarrow\boldsymbol{p}}} = 0?$$



Feynman diagrammatic theory of Fermi polaron

You may only need to take care of the pole at $i\nu_n - \xi_{\uparrow \boldsymbol{q}-\boldsymbol{p}}$, so the result after Matsubara frequency summation is,

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = U_0^{-1} + \sum_{\mathbf{p}} \left[\frac{-1}{e^{-\beta \xi_{\uparrow \mathbf{q}-\mathbf{p}}} + 1} \right] \frac{1}{i\nu_n - \xi_{\uparrow \mathbf{q}-\mathbf{p}} - \xi_{\downarrow \mathbf{p}}}$$

$$= \frac{m_r}{2\pi\hbar^2 a} - \sum_{\mathbf{p}} \left[\frac{2m_r}{\hbar^2 \mathbf{p}^2} + \frac{\theta(\xi_{\uparrow \mathbf{q}-\mathbf{p}})}{i\nu_n - \xi_{\uparrow \mathbf{q}-\mathbf{p}} - \xi_{\downarrow \mathbf{p}}} \right]$$

Here, we have already replaced the bare interaction with the scattering length ***a*** and $\theta(x)$ is the step function.

Remarks:

(1) For a given set of ($\mathbf{q}$, $i\nu_n$), it is a **two-dimensional** integral to calculate $\Gamma(\boldsymbol{q}, i\nu_n)$.
(2) We may and may not have a **two-particle bound state**. But from now on, let us **assume** there is no bound state, which means that the vertex function does not have poles in the left complex plane.

Feynman diagrammatic theory of Fermi polaron

Let us move on to calculate the self-energy,

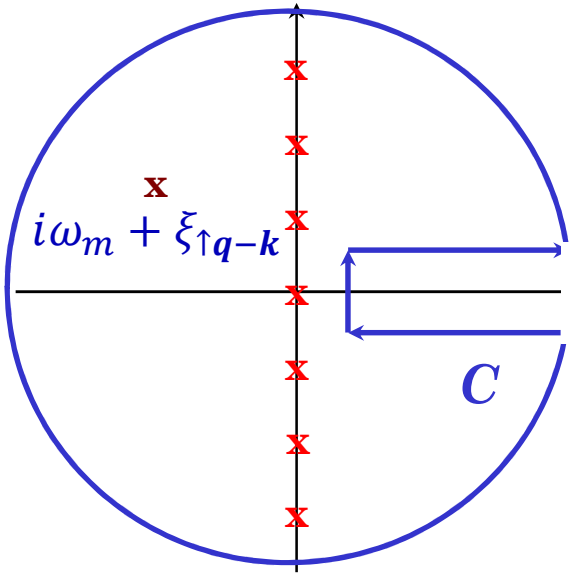
$$\Sigma(\mathbf{k}, i\omega_m) = \sum_{\mathbf{q}} k_B T \sum_{i\nu_n} \frac{1}{i\nu_n - i\omega_m - \xi_{\uparrow \mathbf{q}-\mathbf{k}}} \Gamma(\mathbf{q}, i\nu_n)$$

with the assumption that there is no pole in $\Gamma(\boldsymbol{q}, i\nu_n)$ in the left complex plane.

We need to sum over the bosonic Matsubara frequency at *zero* temperature…

Let us consider the contour integral,

$$\oint \frac{e^{\omega 0^+}}{e^{\beta\omega} - 1} \left[\frac{1}{\omega - i\omega_m - \xi_{\uparrow \boldsymbol{q}-\boldsymbol{k}}} \right] \Gamma(q, \omega) = 0 \, !$$



Feynman diagrammatic theory of Fermi polaron

We only need to take care of the pole at $i\omega_m + \xi_{\uparrow \boldsymbol{q-k}}$, so the result after Matsubara frequency summation is,

$$\Sigma(\mathbf{k}, i\omega_m) = \sum_{\mathbf{q}} (-) \frac{1}{e^{\beta(i\omega_m + \xi_{\uparrow \mathbf{q-k}})} - 1} \Gamma(\mathbf{q}, i\omega_m + \xi_{\uparrow \mathbf{q-k}}) = \sum_{\mathbf{q}} \theta(-\xi_{\uparrow \mathbf{q-k}}) \Gamma(\mathbf{q}, i\omega_m + \xi_{\uparrow \mathbf{q-k}})$$

This is another two-dimensional integral! Together with

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = \frac{m_r}{2\pi\hbar^2 a} - \sum_{\mathbf{p}} \left[\frac{2m_r}{\hbar^2 \mathbf{p}^2} + \frac{\theta(\xi_{\uparrow \mathbf{q-p}})}{i\nu_n - \xi_{\uparrow \mathbf{q-p}} - \xi_{\downarrow \mathbf{p}}} \right]$$

and $\mu_\downarrow = \Sigma(0,0)$, we solve the Fermi polaron!

Remarks:

(1) This is the first-order iteration result (involving a **four-dimensional** integral).
(2) We **assume** that $\Gamma(\boldsymbol{q}, i\nu_n)$ does not have poles in the left complex plane.
(3) It is possible to solve the full coupled equation! (Hu *et al.*, arXiv:1708.03410).

Feynman diagrammatic theory of Fermi polaron

By the way, the **self-consistent** coupled equation will be,

$$\Sigma(\mathbf{k}, i\omega_m) = \sum_{\mathbf{q}} \theta(-\xi_{\uparrow \mathbf{q}-\mathbf{k}}) \Gamma(\mathbf{q}, i\omega_m + \xi_{\uparrow \mathbf{q}-\mathbf{k}})$$

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = \frac{m_r}{2\pi\hbar^2 a} - \sum_{\mathbf{p}} \left[\frac{2m_r}{\hbar^2 \mathbf{p}^2} + \theta(\xi_{\uparrow \mathbf{q}-\mathbf{p}}) G_{\downarrow}(\mathbf{p}, i\nu_n - \xi_{\uparrow \mathbf{q}-\mathbf{p}}) \right]$$

$$G_{\downarrow}(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \varepsilon_{\downarrow \boldsymbol{k}} + \mu_{\downarrow} - \Sigma(\boldsymbol{k}, i\omega_m)}$$

Here, $\mu_{\downarrow} = \Sigma(0,0)$.

Remarks:

(1) This is the zero-temperature result, so there are two step functions.
(2) We **assume** that $\Gamma(\boldsymbol{q}, i\nu_n)$ does not have poles in the left complex plane.

Feynman diagrammatic theory of Fermi polaron

Once we solve the **first-order iteration** equation, we may immediately obtain the effective mass and residue by using,

$$\frac{m^*}{m_\downarrow} = \frac{1 - \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\right]}{1 + \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \varepsilon_{\downarrow \boldsymbol{k}}}\right]} \qquad Z = \frac{1}{1 - \left[\frac{\partial \mathrm{Re}\Sigma}{\partial \omega}\right]}$$

Before we present the numerical result, it is useful to consider weak-coupling limit, where the scattering length $\boldsymbol{a} \to 0$,

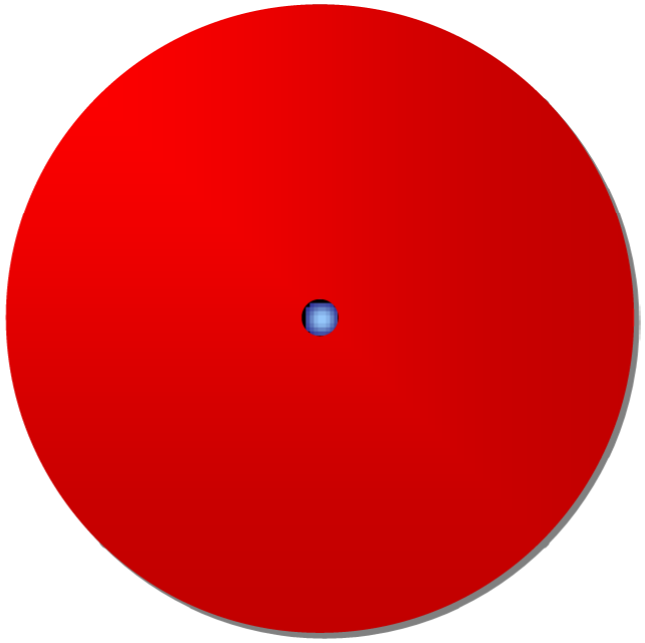
$$\Gamma^{-1}(\mathbf{q}, i\nu_n) \to \frac{m_r}{2\pi\hbar^2 a} \qquad \Sigma(\mathbf{k}, i\omega_m) \to \sum_{\mathbf{q}} \theta(-\xi_{\uparrow \mathbf{q}-\mathbf{k}}) \frac{2\pi\hbar^2 a}{m_r} = \frac{2\pi\hbar^2 a}{m_r} n_\uparrow$$

This is simply the mean-field result.

Okay, we see the derivation of the Fermi polaron equations; but, what is the simple physical picture of the above diagrammatic theory?

Swimming in the Fermi sea

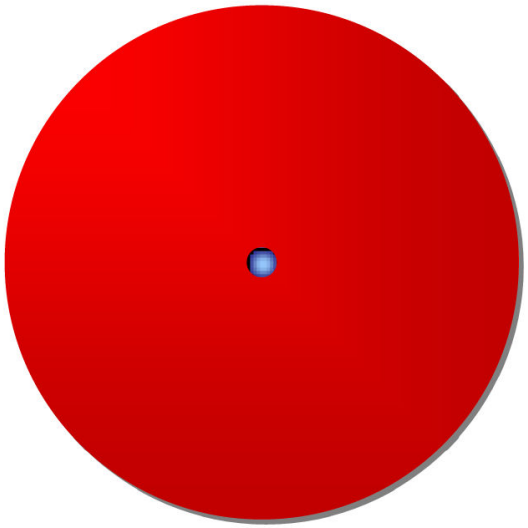
A single $|\downarrow\rangle$ atom immersed in a $|\uparrow\rangle$ cloud
with unitarity limited interactions



credit to Martin Zwierlein WIPM, CAS

Swimming in the Fermi sea

$$|\Psi\rangle = \phi_0 |\mathbf{0}\rangle_\downarrow |FS\rangle_\uparrow$$

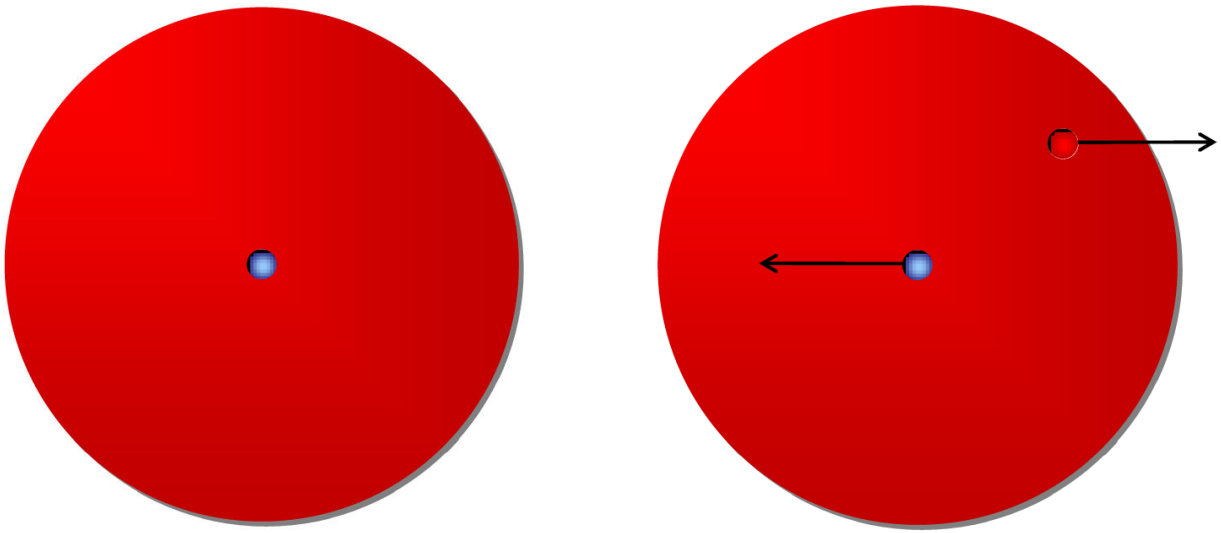


F. Chevy PRA **74**, 063628 (2006), Variational Cooper pair Ansatz

credit to Martin Zwierlein WIPM, CAS

Swimming in the Fermi sea

$$|\Psi\rangle = \phi_0 |\mathbf{0}\rangle_\downarrow |FS\rangle_\uparrow + \sum_{\substack{k>k_F \\ q<k_F}} \phi_{\mathbf{qk}} |\mathbf{q}-\mathbf{k}\rangle_\downarrow c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{q}\uparrow} |FS\rangle_\uparrow$$



F. Chevy PRA **74**, 063628 (2006), Variational Cooper pair Ansatz

credit to Martin Zwierlein WIPM, CAS

Feynman diagrammatic theory of Fermi polaron

That is the reason why you have two step functions at zero temperature!!!

$$\Sigma(\mathbf{k}, i\omega_m) = \sum_{\mathbf{q}} \theta(-\xi_{\uparrow \mathbf{q}-\mathbf{k}}) \Gamma(\mathbf{q}, i\omega_m + \xi_{\uparrow \mathbf{q}-\mathbf{k}})$$

$$\Gamma^{-1}(\mathbf{q}, i\nu_n) = \frac{m_r}{2\pi\hbar^2 a} - \sum_{\mathbf{p}} \left[\frac{2m_r}{\hbar^2 \mathbf{p}^2} + \theta(\xi_{\uparrow \mathbf{q}-\mathbf{p}}) G_{\downarrow}(\mathbf{p}, i\nu_n - \xi_{\uparrow \mathbf{q}-\mathbf{p}}) \right]$$

$$G_{\downarrow}(\boldsymbol{k}, i\omega_m) = \frac{1}{i\omega_m - \varepsilon_{\downarrow \boldsymbol{k}} + \mu_{\downarrow} - \Sigma(\boldsymbol{k}, i\omega_m)}$$

Here, $\mu_{\downarrow} = \Sigma(0,0)$.

Remarks:

(1) This is the zero-temperature result, so there are two step functions.
(2) We **assume** that $\Gamma(\boldsymbol{q}, i\nu_n)$ does not have poles in the left complex plane.

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Feynman diagrammatic theory of Fermi polaron

PRL **98,** 180402 (2007) PHYSICAL REVIEW LETTERS week ending 4 MAY 2007

Normal State of Highly Polarized Fermi Gases: Simple Many-Body Approaches

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(Received 16 February 2007; published 1 May 2007)

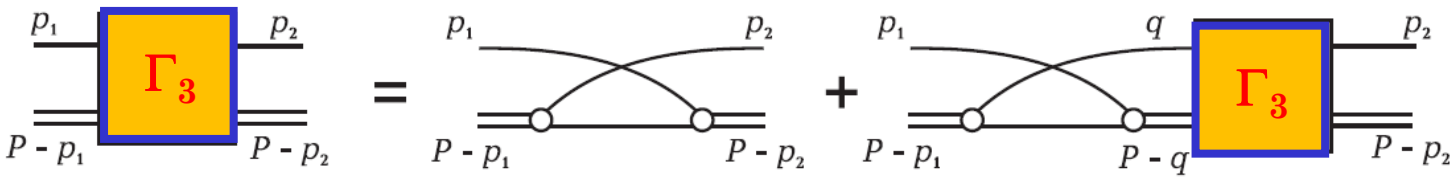
We consider the problem of a single $\downarrow$ atom in the presence of a Fermi sea of $\uparrow$ atoms, in the vicinity of a Feshbach resonance. We calculate the chemical potential and the effective mass of the $\downarrow$ atom using two simple approaches: a many-body variational wave function and a T-matrix approximation. These two methods lead to the same results and are in good agreement with existing quantum Monte Carlo calculations performed at unitarity and, in one dimension, with the known exact solution. Surprisingly, our results suggest that, even at unitarity, the effect of interactions is fairly weak and can be accurately described using single particle-hole excitations. We also consider the case of unequal masses.

Bored with equations? But,

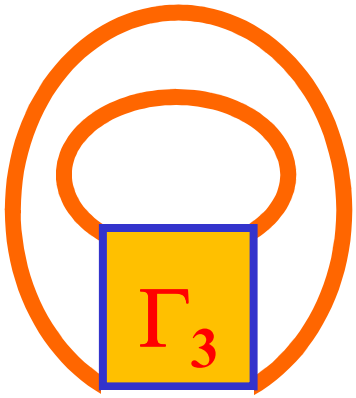
Congratulations: **You reproduce the above titled seminal PRL paper**! Is research easy?

Feynman diagrammatic theory of Fermi polaron

We may find the three-particle vertex function:



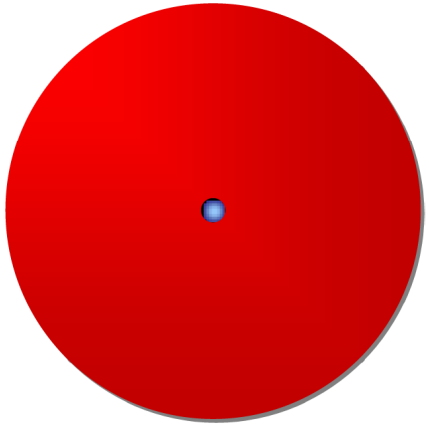
And then the self-energy (corresponding to the two-particle-hole excitations):



Want to go beyond the ladder approximation?

Feynman diagrammatic theory of Fermi polaron

A single $|\downarrow\rangle$ atom immersed in a $|\uparrow\rangle$ cloud
with unitarity limited interactions



In the unitary limit, where the scattering length a diverges, the energy of Fermi polaron (i.e., $E_\infty = \mu_\downarrow$) should be universal! i.e.,

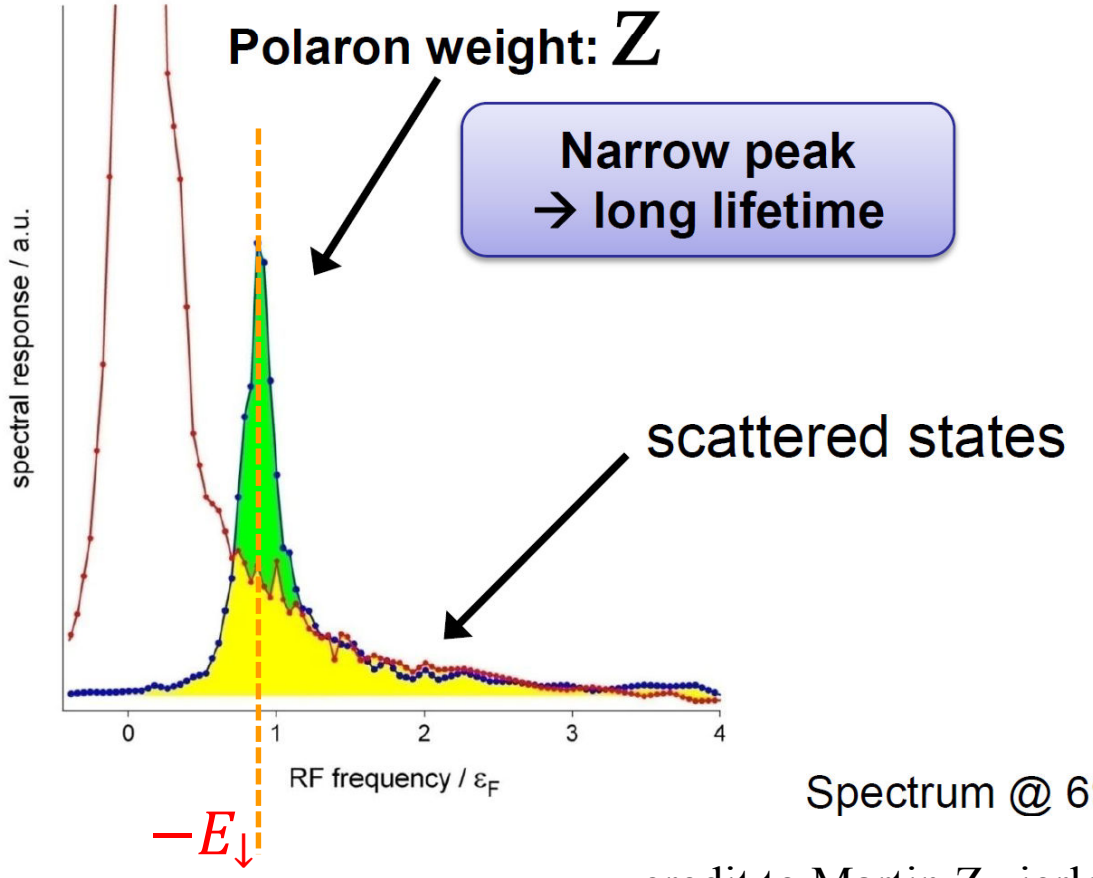
$$E_\infty = -AE_{\uparrow F}$$

Remarkably, A(T-matrix) = **0.6066**, agrees very well with quantum Monte Carlo simulations: A(QMC) = **0.615**.

MIT experiment on attractive Fermi polaron

Polaron = $\sqrt{Z}$ Free particle + $\sqrt{1-Z}$ scattered states

RF Spectrum: $I(\omega) \propto Z\ \delta(\hbar\omega + E_{\downarrow}) + \Gamma_{incoherent}(\omega)$



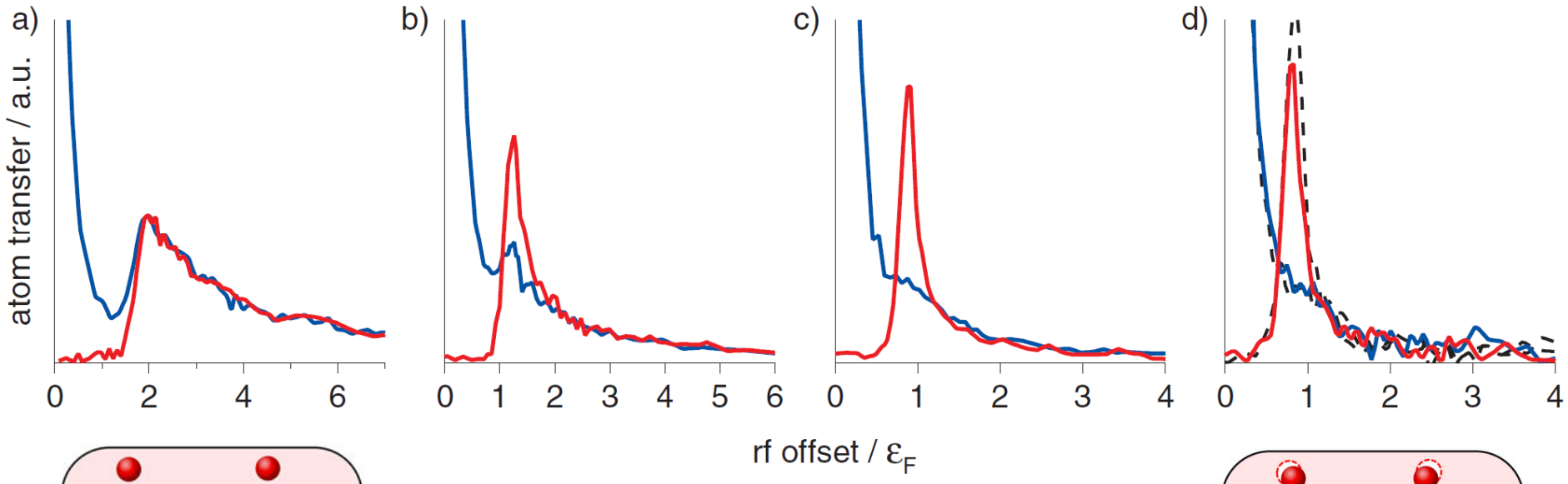
Spectrum @ 690 G

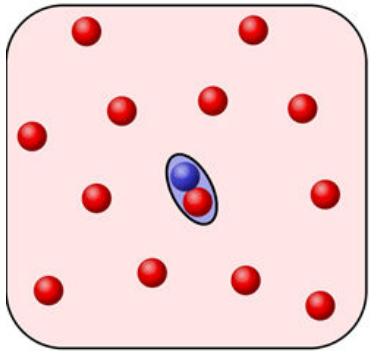
credit to Martin Zwierlein WIPM, CAS

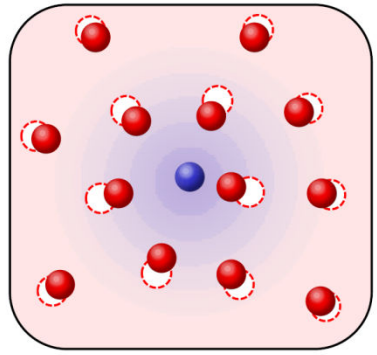
MIT experiment on attractive Fermi polaron

molecular regime

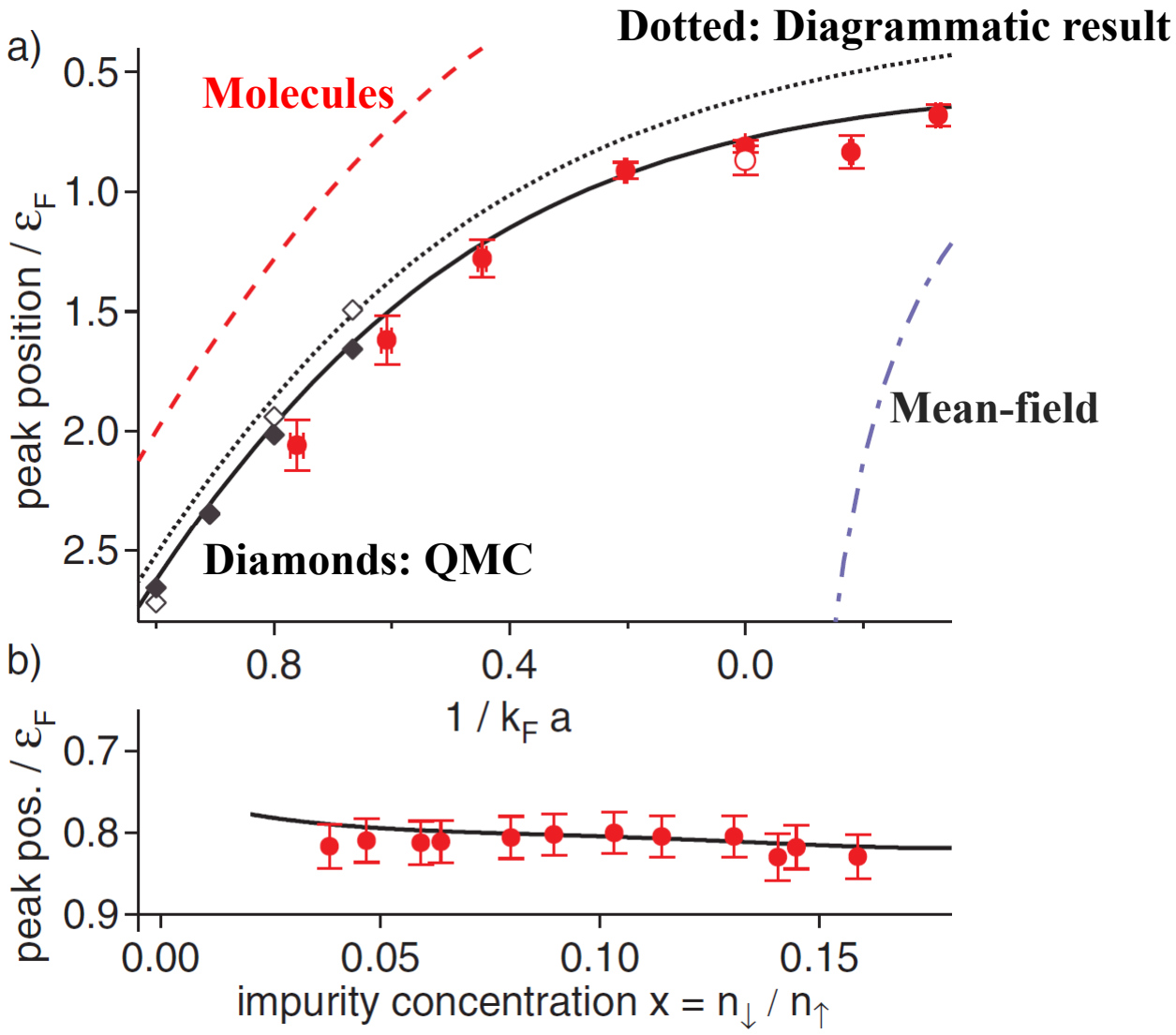
unitary limit







MIT experiment on attractive Fermi polaron



Repulsive Fermi polaron?

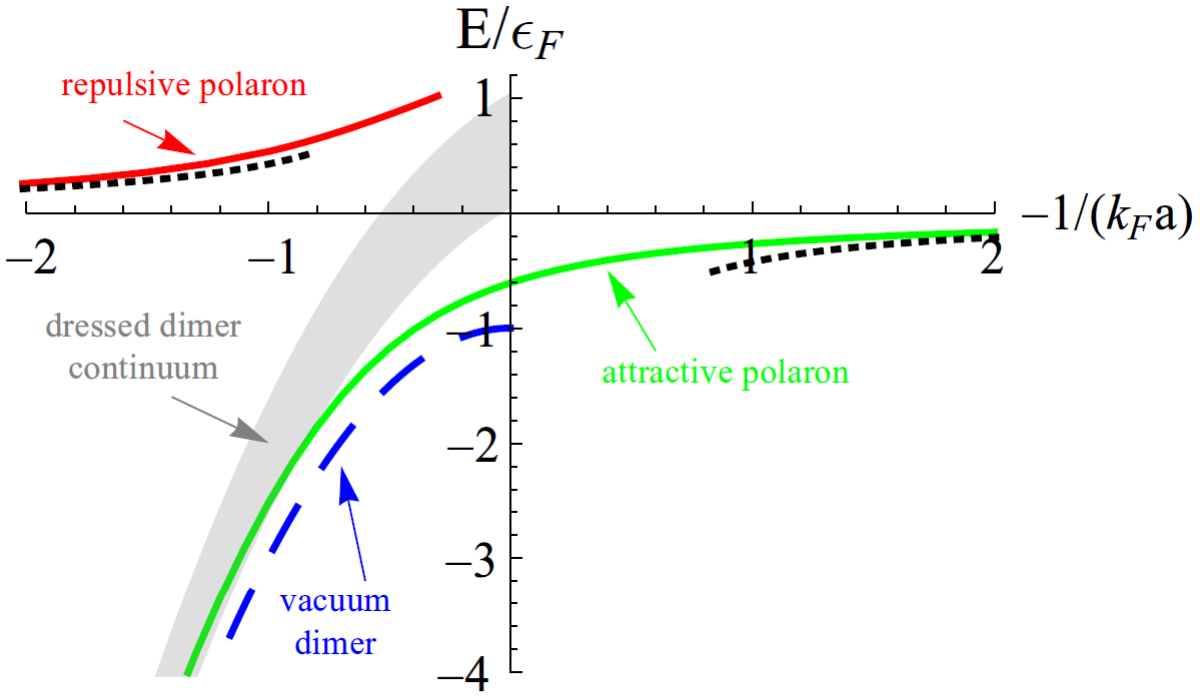
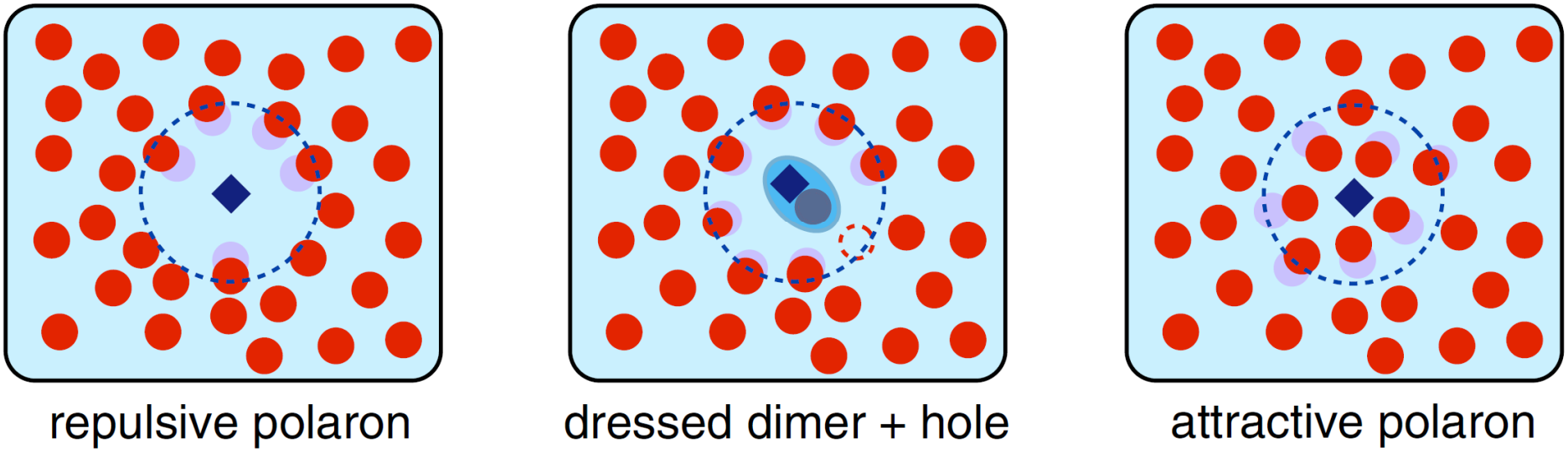


Figure 4. The energy spectrum of a zero-momentum impurity in a Fermi sea contains a repulsive polaron, a continuum of dressed dimers and an attractive polaron. The dotted black lines are the mean-field result, and the dashed line is the dimer energy in the absence of the Fermi sea. The spectrum is generic but the quantitative details correspond here to the case $m_\uparrow = m_\downarrow$ and $R^* = 0$. The curves are obtained from the 1PH approximation described in section 2.3.

Repulsive Fermi polaron?



P. Massignan and G. M. Bruun, *Eur. Phys. J D* **65**, 83 (2011).

LETTER

doi:10.1038/nature11065

Metastability and coherence of repulsive polarons in a strongly interacting Fermi mixture

C. Kohstall[1,2], M. Zaccanti[1], M. Jag[1,2], A. Trenkwalder[1], P. Massignan[3], G. M. Bruun[4], F. Schreck[1] & R. Grimm[1,2]

Ultracold Fermi gases with tunable interactions provide a test bed for exploring the many-body physics of strongly interacting quantum systems[1–4]. Over the past decade, experiments have investigated many intriguing phenomena, and precise measurements of ground-state properties have provided benchmarks for the development of theoretical descriptions. Metastable states in Fermi gases with

parameterized by the scattering length, a, using a magnetic field, B. The interaction strength is described by the dimensionless parameter $-1/\kappa_F a$, where $\kappa_F = \hbar^{-1}\sqrt{2m_{Li}\varepsilon_F} = 1/2{,}850a_0$ is the Fermi wave-number. Here $\hbar = h/2\pi$, a_0 is the Bohr radius and m_{Li} is the mass of a ^{6}Li atom. Near the centre of the FR, the linear approximation $-1/\kappa_F a \approx (B - B_0)/(20\,\mathrm{mG})$ holds. The momentum dependence of

But, narrow Feshbach resonance…

[1]Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria. [2]Institut für Experimentalphysik und Zentrum für Quantenphysik, Universität Innsbruck, 6020 Innsbruck, Austria. [3]Institut de Ciències Fotòniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain. [4]Department of Physics and Astronomy, University of Aarhus, 8000 Aarhus C, Denmark.

Repulsive Fermi polaron!

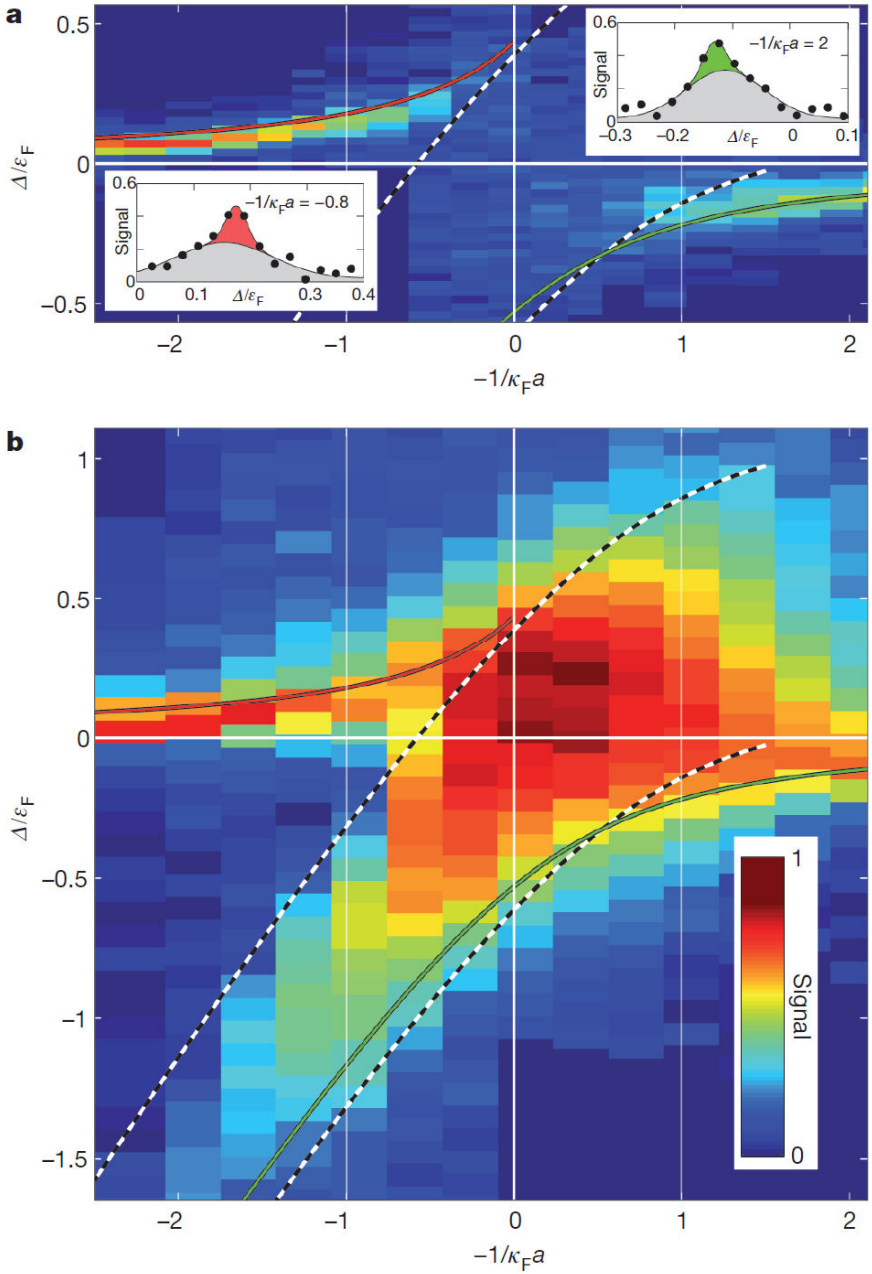


Figure 2 | Spectral response of ^{40}K impurities in a ^{6}Li Fermi sea. The false-colour plots show the fraction of ^{40}K atoms transferred from the non-interacting spin state, $|0\rangle$, to the interacting state, $|1\rangle$, for different values of the radio-frequency detuning parameter, $\Delta = h(\nu_{rf} - \nu_0)$, and for variable interaction strength, $-1/\kappa_F a$: low radio-frequency power (**a**); high radio-frequency power (**b**). For comparison, the lines correspond to the theoretical predictions for E_+, E_-, E_m and $E_m - \varepsilon_F$ as shown in Fig. 1. In **a**, the two insets show the signals for $-1/\kappa_F a = -0.8$ and 2, respectively, corresponding to vertical cuts through the signal data.

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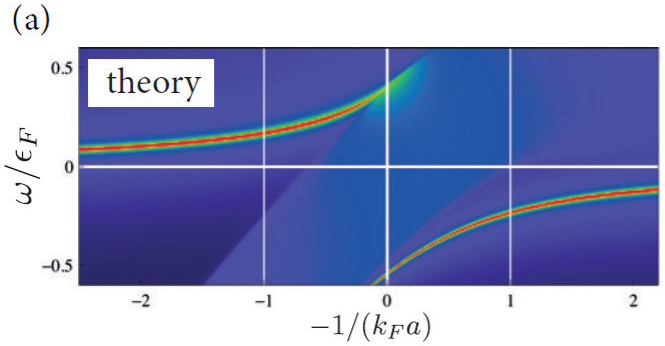
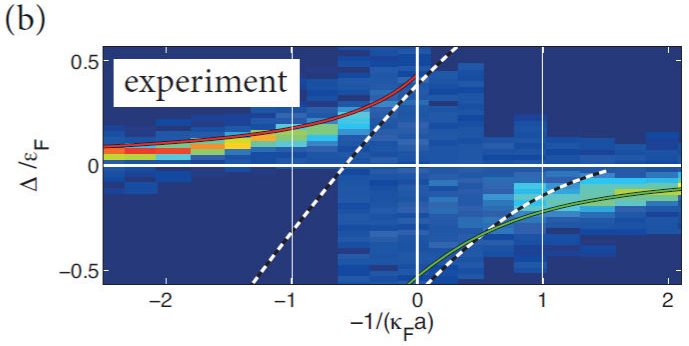
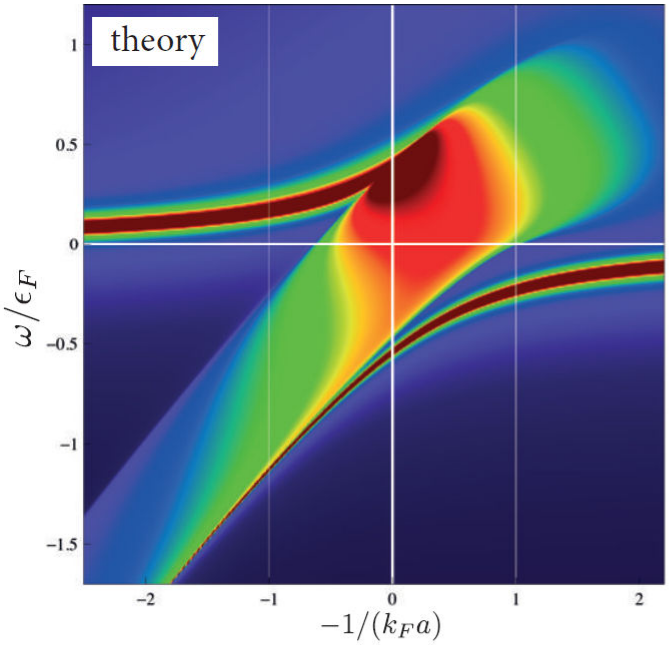
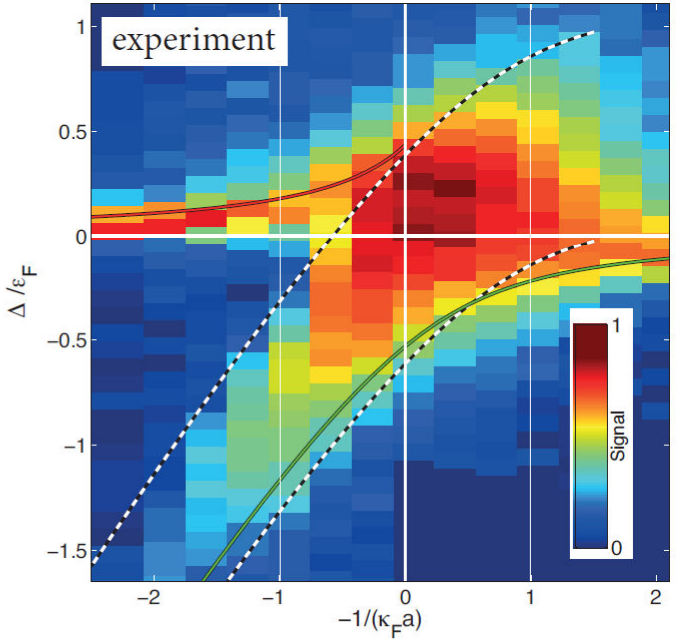
Repulsive Fermi polaron!









RS, Enss, PRA 83 (2011)
Massignan, Bruun, EPJD 65 (2011)
RS, PhD Thesis (2013)

Kohstall et al., Nature 485 (2012)

Repulsive Fermi polaron!

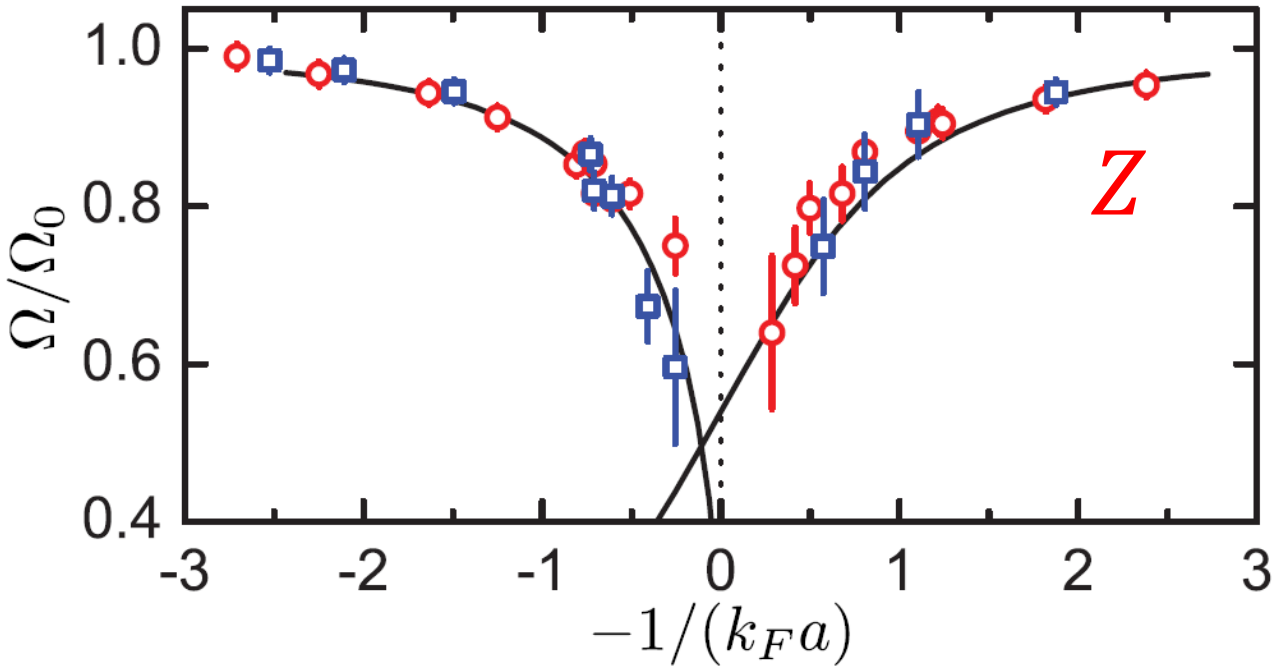


Figure 11. The square root of the polaron residues for the ^{40}K–^{6}Li mixture with $k_F R^* = 1$ extracted by the normalized Rabi frequency, Ω/Ω_0. Lines are the 1PH results for $\sqrt{Z_\pm}$ and symbols are the experimental measurements. Reprinted from [14]. Copyright 2012 Nature.

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **85**, 021602(R) (2012)

Fermi polarons in two dimensions

Richard Schmidt and Tilman Enss
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Ville Pietilä and Eugene Demler
Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA

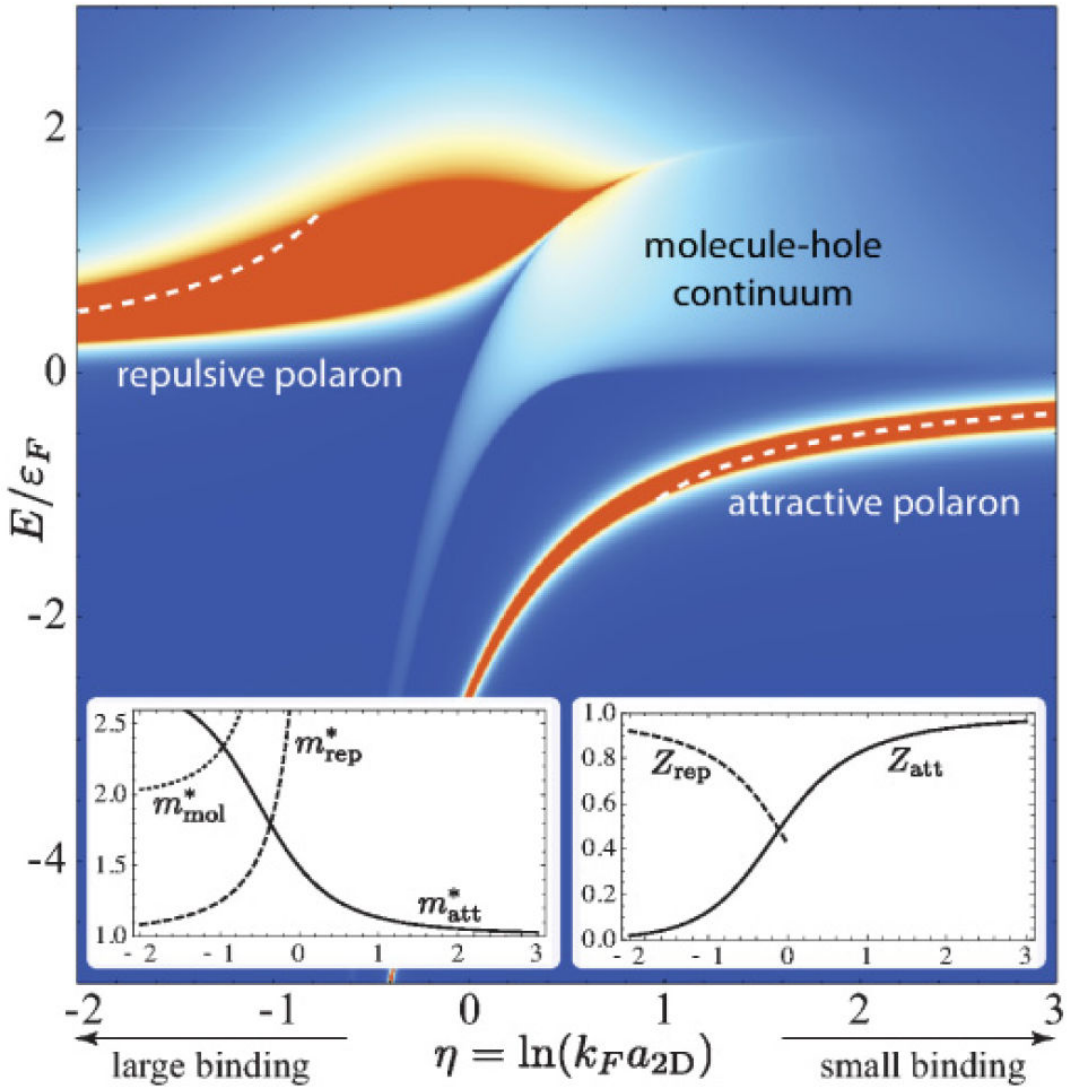
(Received 18 October 2011; published 7 February 2012)

We theoretically analyze inverse radio-frequency (rf) spectroscopy experiments in two-component Fermi gases. We consider a small number of impurity atoms interacting strongly with a bath of majority atoms. In two-dimensional geometries we find that the main features of the rf spectrum correspond to an attractive polaron and a metastable repulsive polaron. Our results suggest that the attractive polaron has been observed in a recent experiment [B. Fröhlich *et al.*, Phys. Rev. Lett. **106**, 105301 (2011)].

DOI: 10.1103/PhysRevA.85.021602 PACS number(s): 67.85.Lm, 03.65.Ge, 32.30.Bv, 68.65.−k

Diagrammatic theory of 2D Fermi polaron

Fermi polaron in 2D



Diagrammatic theory of 2D Fermi polaron

LETTER

doi:10.1038/nature11151

Attractive and repulsive Fermi polarons in two dimensions

Marco Koschorreck[1]*, Daniel Pertot[1]*, Enrico Vogt[1], Bernd Fröhlich[1], Michael Feld[1] & Michael Köhl[1]

The dynamics of a single impurity in an environment is a fundamental problem in many-body physics. In the solid state, a well known case is an impurity coupled to a bosonic bath (such as lattice vibrations); the impurity and its accompanying lattice distortion form a new entity, a polaron. This quasiparticle plays an important role in the spectral function of high-transition-temperature superconductors, as well as in colossal magnetoresistance in manganites[1]. For impurities in a fermionic bath, studies

fermionic bath is notably different. For short-range interactions, strong repulsion between particles can only be achieved if the underlying interaction potential is attractive, which implies a two-particle bound state with binding energy E_B. Repulsive impurities are therefore metastable and eventually decay either into a bound state or into an attractive polaron with the simultaneous creation of particle and hole excitations. It has very recently been theoretically proposed[15,16] that a repulsively interacting impurity, despite its metastability, still forms a

A two-component ^{40}K Fermi gas (broad MFR): no final-state effect

[1]Cavendish Laboratory, University of Cambridge, JJ Thomson Avenue, Cambridge CB3 0HE, UK.
*These authors contributed equally to this work.

Fermi polaron in 2D

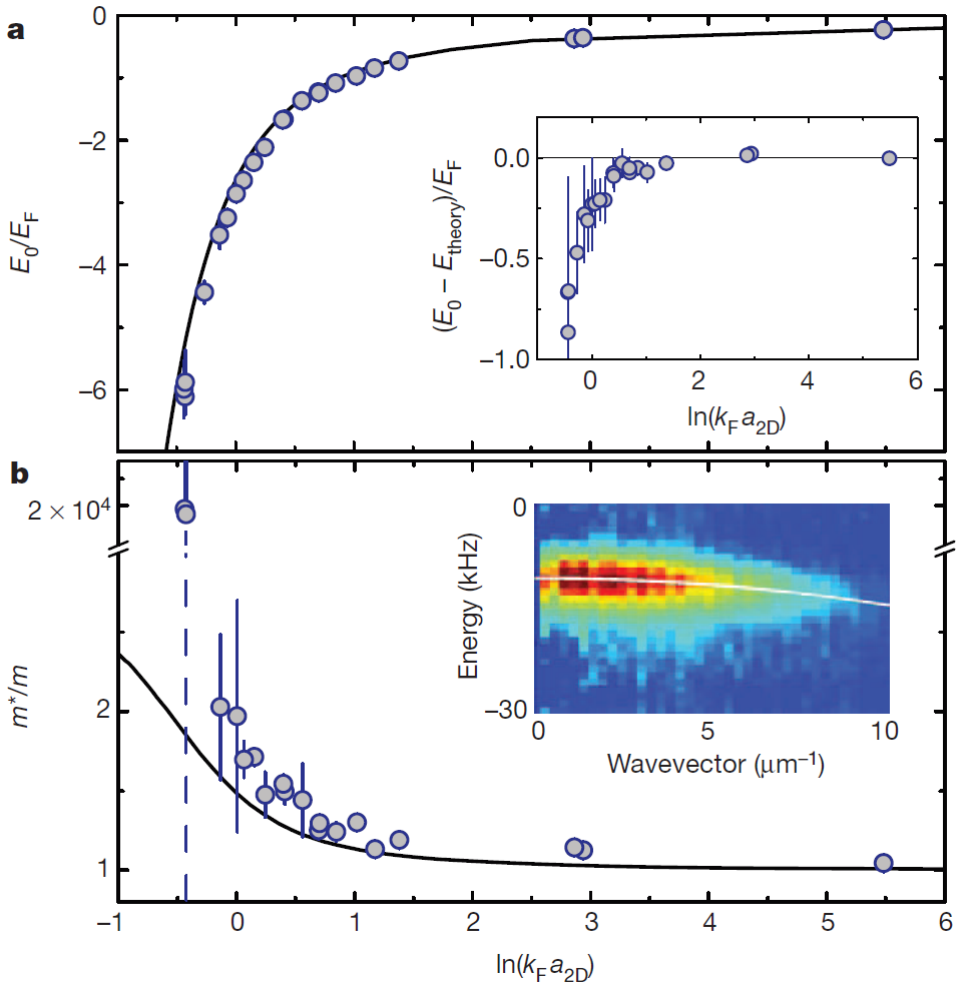


Figure 2 | Attractive polaron. **a**, **b**, Energy (**a**) and effective mass (**b**) of the quasiparticle peak compared to the theoretical prediction[15] (solid line). The dashed vertical line indicates the limit for reliably determining the effective quasiparticle mass. The inset in **a** shows the difference between experiment and theory. The inset in **b** shows an example of a fit used to determine the effective mass at $\ln(k_F a_{2D}) = 0.7$.

Fermi polaron in 2D

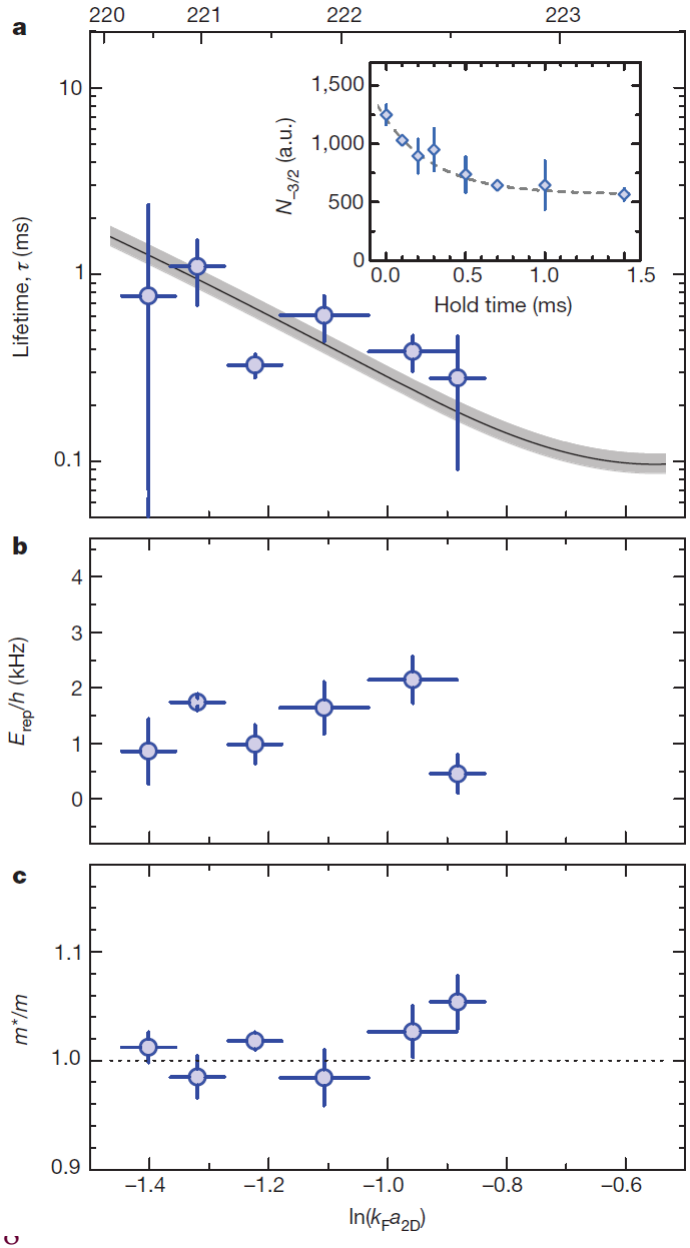
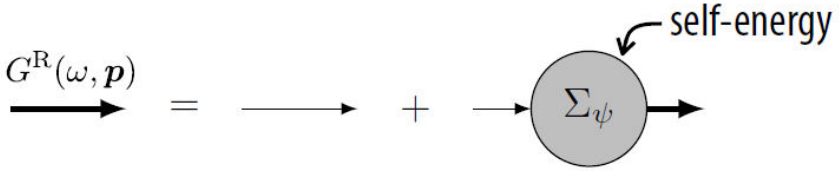


Figure 4 | Repulsive polaron. **a**, Lifetime of the repulsive branch. The inset shows an example of a time-resolved measurement at $\ln(k_F a_{2D}) = -1.2$ from which we extract the lifetime by a fit (dashed line). The solid line is the theoretical prediction from ref. 16. The grey shaded area reflects the range of Fermi energies of (11 ± 1) kHz. **b**, **c**, Measured energy (**b**) and effective mass (**c**). Error bars correspond to 1 s.d. uncertainty in the fit.

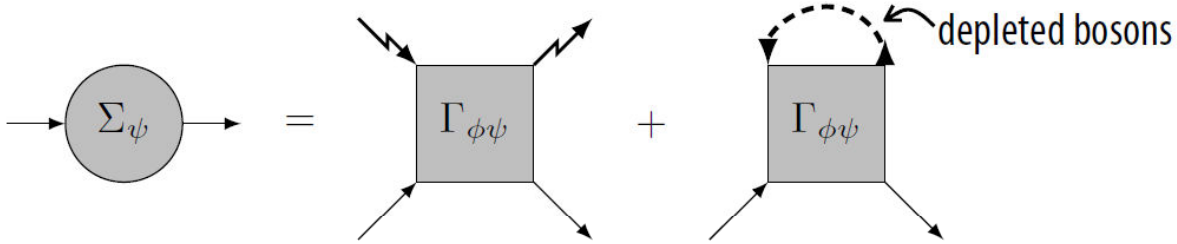
Bose polaron

impurity spectral function $A_{\rm pol}(\omega, \boldsymbol{p}) = -2\,\mathrm{Im}\,G^{\rm R}(\omega, \boldsymbol{p})$ **from Dyson equation**

full Green's function:



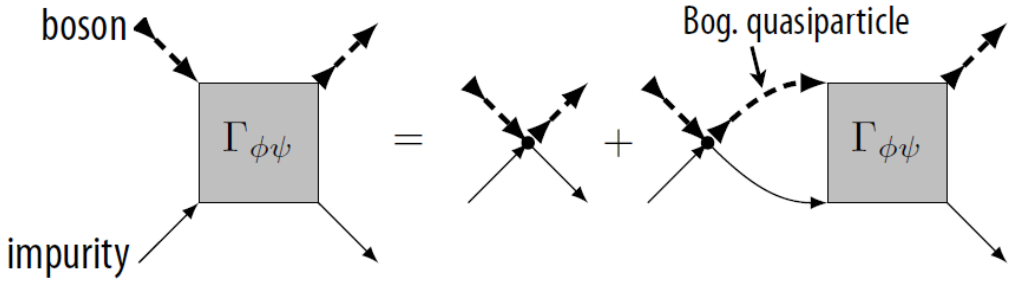
self-energy:



prerequisite: recover exact two-body solution [unlike previous works]

resummed perturbation theory

T-matrix equation:



PRA 88, 053632 (2013)

credit to Richard Schmidt WIPM, CAS

PRL 117, 055301 (2016)

Selected for a Viewpoint in *Physics*
PHYSICAL REVIEW LETTERS

week ending
29 JULY 2016

Bose Polarons in the Strongly Interacting Regime

Ming-Guang Hu, Michael J. Van de Graaff, Dhruv Kedar, John P. Corson, Eric A. Cornell, and Deborah S. Jin
JILA, NIST, and University of Colorado, Boulder, Colorado 80309, USA
and Department of Physics, University of Colorado, Boulder, Colorado 80309, USA
(Received 3 May 2016; revised manuscript received 26 May 2016; published 28 July 2016)

When an impurity is immersed in a Bose-Einstein condensate, impurity-boson interactions are expected to dress the impurity into a quasiparticle, the Bose polaron. We superimpose an ultracold atomic gas of ^{87}Rb with a much lower density gas of fermionic ^{40}K impurities. Through the use of a Feshbach resonance and radio-frequency spectroscopy, we characterize the energy, spectral width, and lifetime of the resultant polaron on both the attractive and the repulsive branches in the strongly interacting regime. The width of the polaron in the attractive branch is narrow compared to its binding energy, even as the two-body scattering length diverges.

DOI: 10.1103/PhysRevLett.117.055301

Bose polaron

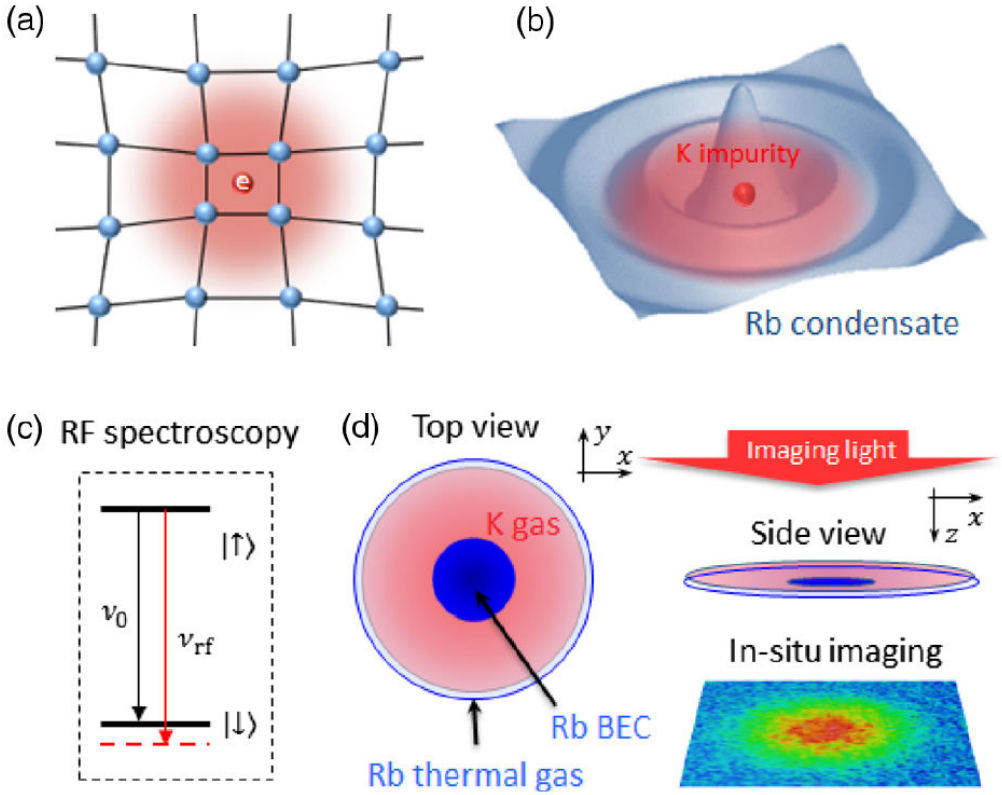
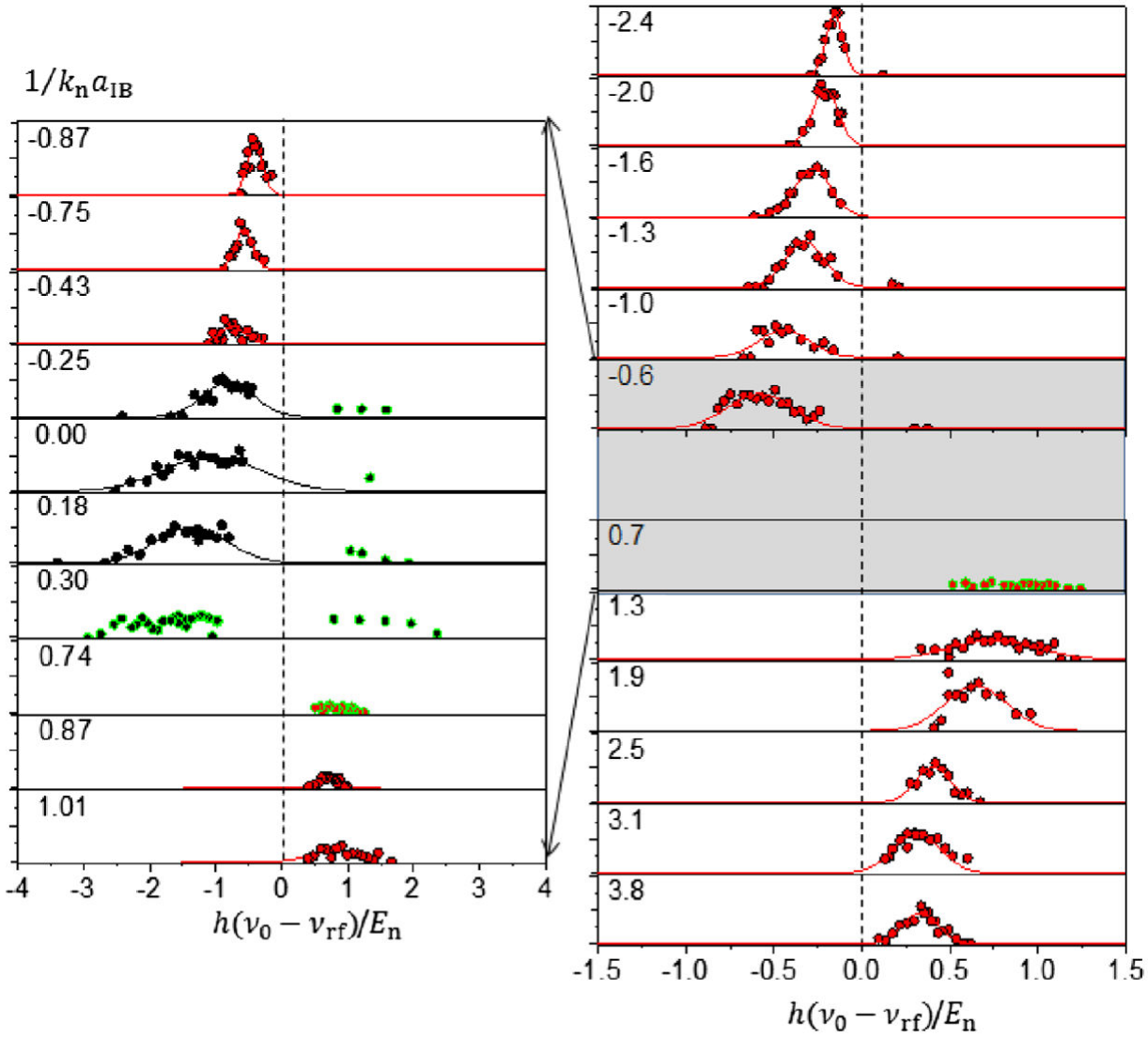


FIG. 1. Impurities immersed in a bosonic bath. (a) Cartoon depictions of the Bose polaron formed by an electron moving in a crystal lattice and (b) its counterpart of an impurity in a continuous system. (c) Radio-frequency (rf) spectroscopy of ^{40}K impurities in a ^{87}Rb Bose-Einstein condensate (BEC). The black lines denote two hyperfine states of bare K atoms and the red dashed line is the shifted energy level due to interactions with the BEC. (d) Geometry of the trapped BEC and impurity clouds. The dark blue represents the Rb BEC cloud, the light blue shows the Rb thermal cloud, and the red shows the K impurity cloud. The imaging light propagates from top to bottom along z.

Bose polaron



Radio-frequency spectroscopy

Bose polaron

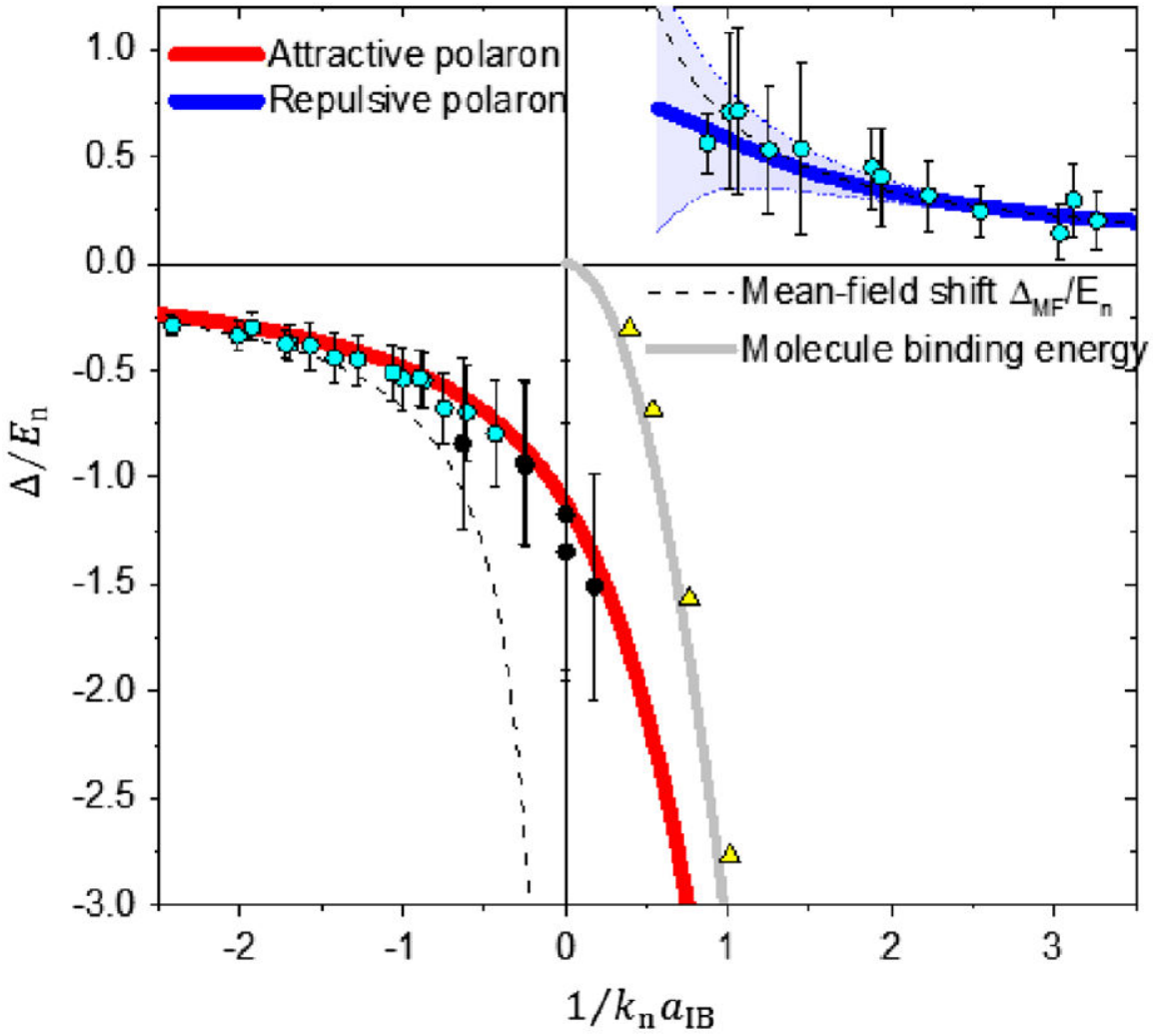


FIG. 3. Energy shift, Δ, and spectral width of Bose polarons. Cyan and black dots show data taken with long and short rf pulses, respectively. Error bars indicate the measured rms spectral width. The mean-field prediction is shown with dashed lines. Recent $T = 0$ Bose polaron energy predictions [11,12] for the attractive and repulsive branches are shown with red and blue lines, respectively. The gray line shows the universal two-body prediction for the energy of KRb Feshbach molecules. Triangles show separate measurements of this energy using rf association of molecules performed in a very low-density Rb gas. This two-body result is not valid for the high-density regime and is shown only for reference. The blue shaded area indicates the predicted spectral width of the repulsive branch.

SWIN BURNE CENTRE FOR QUANTUM AND OPTICAL SCIENCE

Latest experiment: Repulsive Fermi polaron

PRL 118, 083602 (2017) PHYSICAL REVIEW LETTERS week ending 24 FEBRUARY 2017

Repulsive Fermi Polarons in a Resonant Mixture of Ultracold ^{6}Li Atoms

F. Scazza,[1,2,*] G. Valtolina,[1,2] P. Massignan,[3] A. Recati,[4,5] A. Amico,[2] A. Burchianti,[1,2] C. Fort,[2] M. Inguscio,[1,2] M. Zaccanti,[1,2] and G. Roati[1,2]

[1]*Istituto Nazionale di Ottica del Consiglio Nazionale delle Ricerche (INO-CNR), 50019 Sesto Fiorentino, Italy*
[2]*LENS and Dipartimento di Fisica e Astronomia, Università di Firenze, 50019 Sesto Fiorentino, Italy*
[3]*ICFO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels, Spain*
[4]*INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, 38123 Povo, Italy*
[5]*Ludwig-Maximilians-Universität München, 80333 München, Germany*

(Received 30 September 2016; published 21 February 2017)

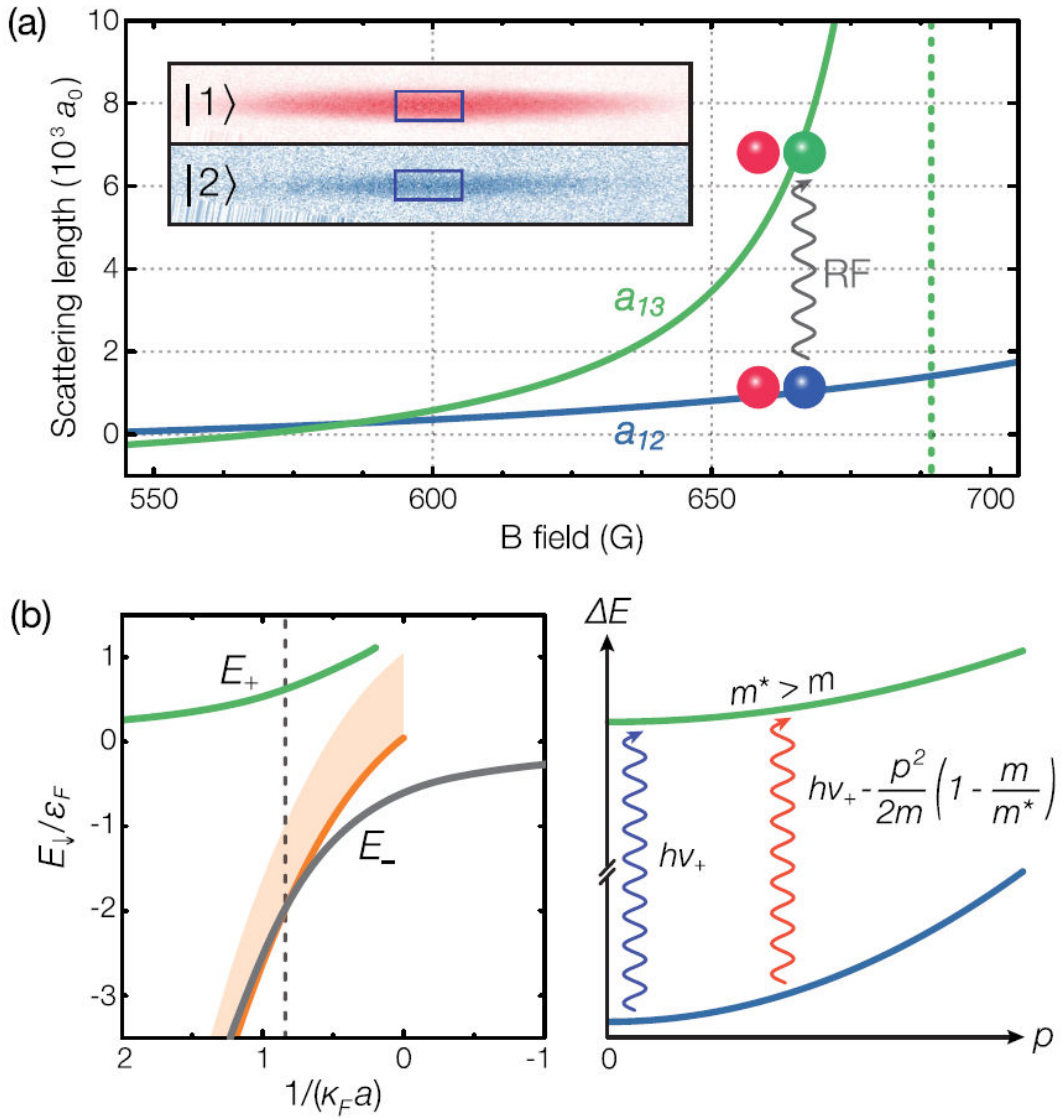
We employ radio-frequency spectroscopy to investigate a polarized spin mixture of ultracold ^{6}Li atoms close to a broad Feshbach scattering resonance. Focusing on the regime of strong repulsive interactions, we observe well-defined coherent quasiparticles even for unitarity-limited interactions. We characterize the many-body system by extracting the key properties of repulsive Fermi polarons: the energy E_+, the effective mass m^*, the residue Z, and the decay rate Γ. Above a critical interaction, E_+ is found to exceed the Fermi energy of the bath, while m^* diverges and even turns negative, thereby indicating that the repulsive Fermi liquid state becomes energetically and thermodynamically unstable.

DOI: 10.1103/PhysRevLett.118.083602

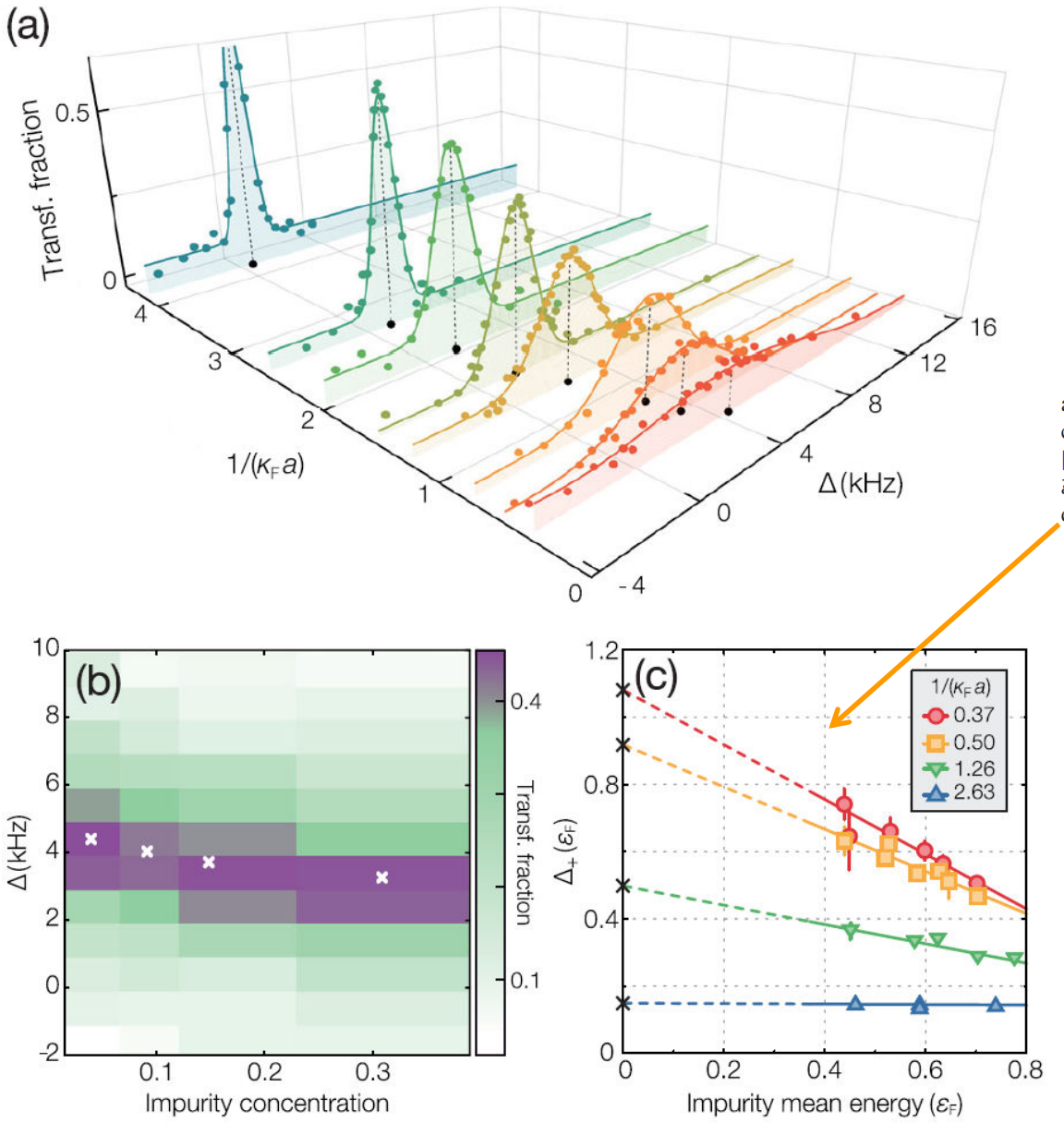
A two-component ^{6}Li Fermi gas (broad Feshbach resonance)

Reverse radio frequency: no final-state effect

Latest experiment: Repulsive Fermi polaron



Latest experiment: Repulsive Fermi polaron



On the other hand, polaron-polaron effective interactions are expected, within an equilibrium Fermi liquid, to contribute with a positive resonance shift $\propto x \sim \tilde{\varepsilon}^{3/2}$ [2,38,54,55], leading to a nonlinear increase of Δ_+ with $\tilde{\varepsilon}$. Such a trend is incompatible with the observed linear decrease.

Latest experiment: Repulsive Fermi polaron

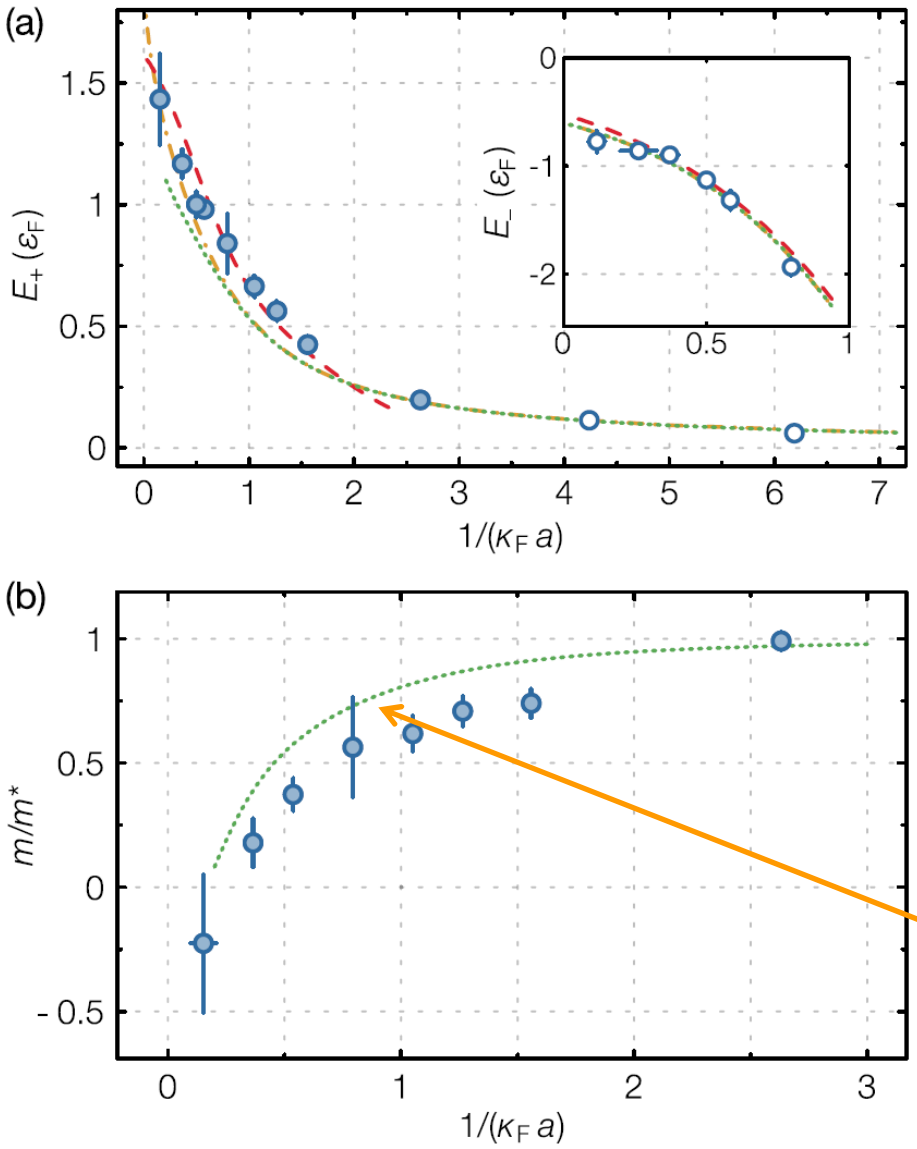


FIG. 3. (a) Zero-momentum repulsive polaron energy E_+ as a function of $1/(\kappa_F a)$ (symbols). Inset: Attractive polaron energy E_-. Theory predictions from Refs. [23] (dot-dashed yellow line), [25] (dotted green line), and [26] (dashed red line) are shown in both panels. Empty symbols denote points obtained by averaging measurements at different $\bar{\varepsilon}$ rather than by zero-energy extrapolation [38]. (b) Inverse effective mass m/m^* of the repulsive polaron as a function of $1/\kappa_F a$ (symbols), together with theory predictions from Ref. [25] (dotted green line). Error bars combine the linear fit parameter errors with the standard error of the mean (s.e.m.) of binned data.

Diagrammatic theory should be refined for repulsive polaron!

Latest experiment: Repulsive Fermi polaron

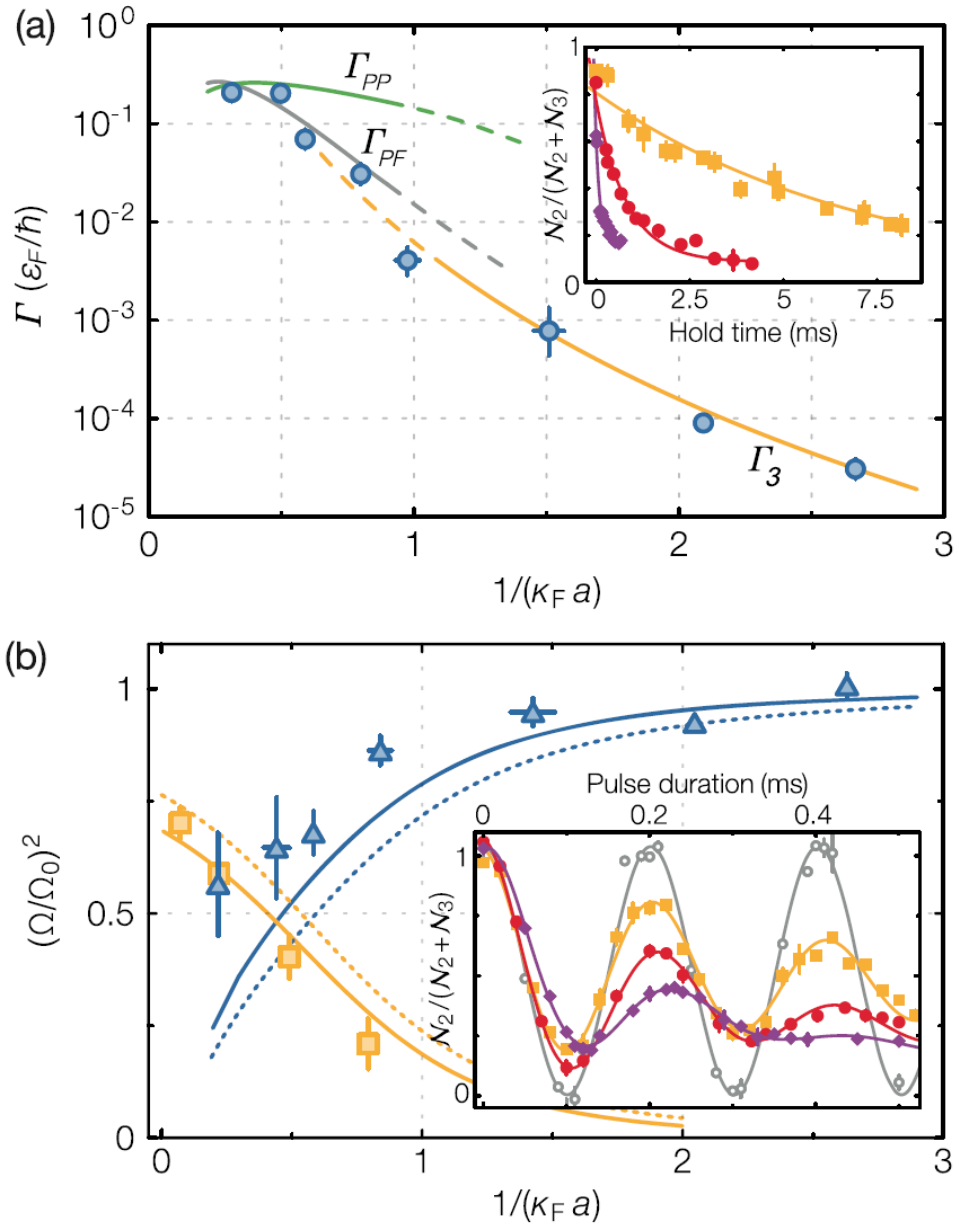


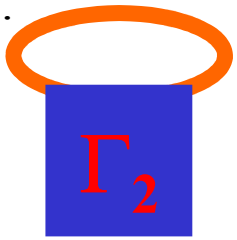
FIG. 4. (a) Decay rate Γ of the repulsive branch population measured as a function of $1/(\kappa_F a)$. Theory predictions for three-body recombination Γ_3 [56] (yellow line), polaron-to-polaron Γ_{PP} [25] (green line), and polaron-to-bare atom Γ_{PF} [38] (gray line) decay processes are plotted within their respective regimes of validity. Inset: Examples of polaron population decay for $\kappa_F a \simeq 1$ (yellow squares), 1.3 (red circles), and 3 (purple diamonds), together with the exponential fits. (b) $(\Omega/\Omega_0)^2$ for the repulsive (blue triangles) and attractive (yellow squares) polarons at various $1/(\kappa_F a)$. Solid curves are our theory predictions for $(\Omega/\Omega_0)^2$ obtained within the ladder approximation [38], while dotted curves depict the lowest-order results $\sqrt{Z_{+2}Z_{\pm 3}}$. Inset: Repulsive polaron Rabi oscillations at $x = 0.15(3)$ for $\kappa_F a \simeq 0$ (empty gray circles), 1.1 (yellow squares), 1.3 (red circles), and 1.7 (purple diamonds). Error bars combine the fit parameter errors with binned data s.e.m.

Fermi/Bose polaron: Summary and outlooks

A simple many-body problem; yet, **enormous** experimental and theoretical efforts. This is exactly the beauty of many-body systems.

Rapid experimental advances over the past ten years. Yet, **we may anticipate new big surprises!**

Many theoretical attempts; however, the **simple** *T*-matrix diagrammatic theory works very well for **attractive** polaron (which is amazing!) ☺.



Outlooks (we are clearly at the stage of making important contributions):

(1) How about the full self-consistent solution of the *T*-matrix theory?
(2) What is the temperature effect?
(3) How about two-particle-hole excitations (Efimov physics)?
(4) …