Introduction to CFT

Aims: Principles of CFT, illustrated by free boson, free fermion and Ising model.
These CFTs are relation - bosonisation lets us compute physical quantities (correlation functions) exactly!

1. Conformal transformations in 2 dimensions

Conf. trans. preserve angles. eg. in $\mathbb{R}^n$, translations + rotations. (Mink - boosts are conformal as well). $\therefore$ Lorentz (Poincaré) trans. are conformal. But, conf. is bigger: dilations (rescalings). Scale inv. observed in 2nd order phase transitions.

Moral: Conf inv. is much more special than Lorentz inv.

Consider $\mathbb{R}^2$ equipped with metric $g_{\mu\nu}$, so that length squared of $x = x^\mu$ is

$$x \cdot x = x^\mu g_{\mu\nu} x^\nu .$$

Define angle between x and y by $\cos\theta = \dfrac{x^\mu g_{\mu\nu} y^\nu}{(x\cdot x)^{1/2}(y\cdot y)^{1/2}}$

when $x \cdot x,\ y \cdot y \neq 0$.

Recall: $x_\mu = g_{\mu\nu} x^\nu$, $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$, $\partial^\mu = g^{\mu\nu}\partial_\nu$,

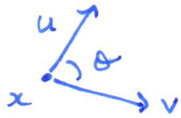
$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$ etc...

Def: A conf. trans. $x^\mu \longrightarrow x'^\mu$ is one satisfying

$$g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x),$$

where $\Lambda(x) > 0$ is a scalar.

Thus, the angle θ between u and v is preserved:



$$\cos\theta' = \frac{u^\mu g'_{\mu\nu}(x') v^\nu}{(u^\mu g'_{\mu\nu}(x') u^\nu)^{1/2} (v^\mu g'_{\mu\nu}(x') v^\nu)^{1/2}}$$

$$= \frac{u^\mu \Lambda(x) g_{\mu\nu}(x) v^\nu}{(u^\mu \Lambda(x) g_{\mu\nu}(x) u^\nu)^{1/2} (v^\mu \Lambda(x) g_{\mu\nu}(x) v^\nu)^{1/2}}$$

$$= \cos\theta.$$

& Note: $\Lambda(x) > 0$ so we can take the square root!

Consider now an infinitesimal ~~tra~~ trans. $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$.
Expand $\Lambda(x) = 1 - \Omega(x)$, where $\Omega(x)$ is infinitesimal.

Since $ds^2 = g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu$, we have

$$g'_{\mu\nu} dx'^\mu dx'^\nu = g'_{\mu\nu}(dx^\mu + \partial_\rho \varepsilon^\mu dx^\rho)(dx^\nu + \partial_\sigma \varepsilon^\nu dx^\sigma)$$

$$= g'_{\mu\nu} dx^\mu dx^\nu + \partial_\rho \underbrace{\varepsilon^\mu g'_{\mu\nu}}_{\varepsilon_\nu} dx^\rho dx^\nu + \partial_\sigma \underbrace{\varepsilon^\nu g'_{\mu\nu}}_{\varepsilon_\mu} dx^\mu dx^\sigma$$

$$= (g'_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)\, dx^\mu dx^\nu$$

$$\Rightarrow \qquad g_{\mu\nu} = g'_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu .$$

This is for all infinitesimal transformations. For infinitesimal conformal trans.,

$$g'_{\mu\nu} = \Lambda g_{\mu\nu} = g_{\mu\nu} - \Omega g_{\mu\nu}$$

$$\Rightarrow \quad \boxed{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \Omega g_{\mu\nu}}.$$

$\times$ by $g^{\nu\mu}$ gives

$$\partial_\mu \varepsilon^\mu + \partial_\nu \varepsilon^\nu = \Omega g^\mu_\mu = 2\Omega \qquad (\text{dim.} = 2)$$

$$\Rightarrow \quad \boxed{\Omega = \partial \cdot \varepsilon}$$

$$\therefore \quad \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \partial_\rho \varepsilon^\rho g_{\mu\nu}$$

$$= g_{\mu\nu} g^{\rho\sigma} \partial_\rho \varepsilon_\sigma .$$

Take $\mathbb{R}^2$ with <u>euclidean</u> metric $g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$\mu = \nu = 1$: $2\partial_1 \varepsilon_1 = \partial_1 \varepsilon_1 + \partial_2 \varepsilon_2$ $\qquad$ $\mu = 1, \nu = 2$: $\partial_1 \varepsilon_2 + \partial_2 \varepsilon_1 = 0$

$\mu = \nu = 2$: $2\partial_2 \varepsilon_2 = \partial_1 \varepsilon_1 + \partial_2 \varepsilon_2$ $\qquad$ $\mu = 2, \nu = 1$: $\partial_2 \varepsilon_1 + \partial_1 \varepsilon_2 = 0$

$$\therefore \quad \partial_1 \varepsilon_1 = \partial_2 \varepsilon_2 \quad \underline{\text{AND}} \quad \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1 .$$

[Cauchy-Riemann equations from complex analysis!]

Change to complex coords ($\mathbb{R}^2 \cong \mathbb{C}$) : $\quad z = x^1 + ix^2 \quad \varepsilon = \varepsilon^1 + i\varepsilon^2$

$\bar{z} = x^1 - ix^2 \quad \bar{\varepsilon} = \varepsilon^1 - i\varepsilon^2$.

$$\Rightarrow \quad \bar{\partial}\varepsilon = 0 \quad \underline{\text{AND}} \quad \partial\bar{\varepsilon} = 0 \qquad \begin{aligned} \partial \equiv \partial_z &= \tfrac{1}{2}(\partial_1 - i\partial_2) \\ \bar{\partial} \equiv \partial_{\bar{z}} &= \tfrac{1}{2}(\partial_1 + i\partial_2) \end{aligned}$$

ie $\quad \varepsilon = \varepsilon(z) \qquad \bar{\varepsilon} = \bar{\varepsilon}(\bar{z})$

ε holomorphic, $\quad \bar{\varepsilon}$ antiholomorphic.

$\therefore$ An inf. conf. transformation of euclidean $\mathbb{R}^2 = \mathbb{C}$ has form

$$z' = z + \varepsilon(z), \quad \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z})$$

where $\varepsilon, \bar{\varepsilon}$ are independent hol, antihol. functions.

Generators of inf. conf. trans: Let ϕ be a function.

$$\phi(x') = \phi(x+\varepsilon) = \phi(x) + \underbrace{\varepsilon^\mu \partial_\mu}_{\text{generator!}} \phi(x) \qquad \text{(Taylor)}$$

A <u>basis</u> for the generators of the inf. conf. trans. is

$$\{ l_n = -z^{n+1}\partial \, , \; \bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \; : \; n \in \mathbb{Z} \}.$$

These generators form a Lie algebra. eg,

$$[l_m, l_n] = -z^{m+1}\partial(-z^{n+1}\partial) + z^{n+1}\partial(-z^{m+1}\partial)$$
$$= (n+1) z^{m+n+1}\partial + z^{m+n+2}\partial^2 - (m+1) z^{m+n+1}\partial - z^{m+n+2}\partial^2$$
$$= (n-m) z^{m+n+1}\partial$$
$$= (m-n) l_{m+n}.$$

Also, $[l_m, \bar{l}_n] = 0$ and $[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}$.

This Lie algebra is two commuting copies of the Witt algebra.
It is the Lie algebra of (classical) inf. conf. trans.

<u>Def</u>:

$P_1 = -(l_{-1} + \bar{l}_{-1})$	$D = -(l_0 + \bar{l}_0)$	$K_1 = -(l_1 + \bar{l}_1)$
$P_2 = -i(l_{-1} - \bar{l}_{-1})$	$M = -i(l_0 - \bar{l}_0)$	$K_2 = -i(l_1 - \bar{l}_1)$
inf. translations	inf. dilation inf. rotation	inf. special conformal trans.

eg. $P_1 = \partial + \bar{\partial} = \partial_1$ and the inf. translation $x'^1 = x^1 + \varepsilon^1$, $x'^2 = x^2$ indeed has generator P_1 for $\varepsilon^1 = 1$:

$$\phi(x'^1, x'^2) = \phi(x^1 + \varepsilon^1, x^2) = \phi(x^1, x^2) + \varepsilon^1 \partial_1 \phi(x^1, x^2)$$
$$= (1 + \varepsilon^1 P_1)\phi(x^1, x^2).$$

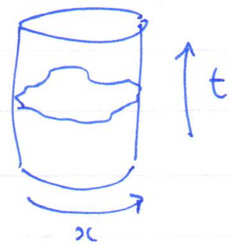
The span of the ℓ_n and $\bar{\ell}_n$ with $n = -1, 0, +1$ are called <u>global</u> conf. trans. because they integrate to give genuine conf. trans. (not inf.)

The ℓ_n and $\bar{\ell}_n$ for general n are called local conf. trans.

2. The Free Boson

Let φ be a scalar bosonic field on a cylinder:

$$\varphi(t, x) = \varphi(t, x+L).$$



The cylinder has lorentzian metric $\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} = g_{\mu\nu}$

Describes a closed bosonic string!

<u>Action</u>: $$S[\varphi] = \underset{\uparrow \text{ coupling constant}}{\frac{1}{2g}} \int_{S^1 \times \mathbb{R}} \underbrace{\partial_\mu \varphi \, \partial^\mu \varphi}_{\partial_\mu \varphi \, g^{\mu\nu} \partial_\nu \varphi \, = \, -(\partial_t \varphi)^2 + (\partial_x \varphi)^2} \, dx \, dt \qquad \mu \in \{t, x\}.$$

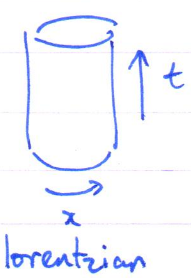
No mass term, no spin coupling, no interactions.
This is a free, massless, spinless bosonic string!

<u>Equations of Motion</u>: $\varphi' = \varphi + \eta$ (η infinitesimal)

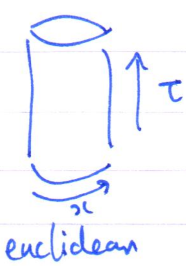
$$S[\varphi'] - S[\varphi] = \frac{1}{g} \int \partial_\mu \eta \, \partial^\mu \varphi \, dx \, dt = -\frac{1}{g} \int \eta \, \partial_\mu \partial^\mu \varphi \, dx \, dt$$

$\therefore$ Euler-Lagrange eqns are $\partial_\mu \partial^\mu \varphi = 0$, ie $\boxed{\partial_t^2 \varphi = \partial_x^2 \varphi}$

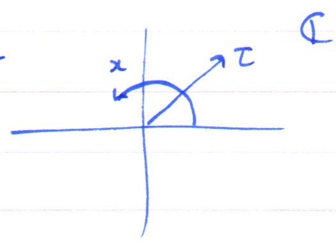
Conformal invariance? Switch to complex coords. & Wick rotate:



$$\xrightarrow{\boxed{\tau = -it}}$$



$$\overset{z = e^{2\pi(\tau + ix)/L}}{\underset{\bar{z} = e^{2\pi(\tau - ix)/L}}{\longrightarrow}}$$



$$\Rightarrow S[\varphi] = \frac{1}{g}\int_{\mathbb{C}} \partial\varphi\,\bar\partial\varphi\, dz\, d\bar z .$$

EOM become $\partial\bar\partial\varphi = 0$, ie $\partial\varphi = \partial\varphi(z)$, $\bar\partial\varphi = \bar\partial\varphi(\bar z)$.
hol! antihol!

<u>Classical Conformal Invariance</u>: The action will not change under
$z \to z' = z + \varepsilon(z)$, $\bar z \to \bar z' = \bar z + \bar\varepsilon(\bar z)$.

Boson is <u>spinless</u>: $\varphi'(z', \bar z') = \varphi(z, \bar z)$ [scalar field].

If $z' = z + \varepsilon(z)$, $\partial = (1 + \partial\varepsilon)\partial'$ and $dz' = (1 + \partial\varepsilon)dz$

$$\Rightarrow S'[\varphi'] = \frac{1}{g}\int_{\mathbb{C}} \cancel{(1-\partial\varepsilon)\partial\varphi\,\bar\partial\varphi\cdot(1+\partial\varepsilon)}\ \partial'\varphi'(z',\bar z')\,\bar\partial'\varphi'(z',\bar z')\, dz' d\bar z'$$

$$= \frac{1}{g}\int_{\mathbb{C}} (1-\partial\varepsilon)\partial\varphi(z,\bar z)\,\bar\partial\varphi(z,\bar z)\cdot(1+\partial\varepsilon)\, dz\, d\bar z$$

$$= \frac{1}{g}\int_{\mathbb{C}} \partial\varphi(z,\bar z)\,\bar\partial\varphi(z,\bar z)\, dz\, d\bar z$$

$$= S[\varphi].$$

Action is invariant under ~~con~~ conf. trans. $\Rightarrow$ classical conf. inv.

Not inv. under general trans., but these characterise the stress-energy ~~-momentum~~ tensor $T^{\mu\nu}$:

$$x'^{\mu} = x^{\mu} + \eta^{\mu} \quad \Longrightarrow \quad S' - S = \int T^{\mu\nu}\,\partial_\mu \eta_\nu\, dx\, dt .$$

We take $z' = z + \eta(z,\bar z)$, $\bar z' = \bar z + \bar\eta(z, \bar z)$ and compute that

$$S' - S = \int \left[-\frac{1}{g}\partial\varphi\,\partial\varphi\,\bar\partial\eta - \frac{1}{g}\bar\partial\varphi\,\bar\partial\varphi\,\partial\bar\eta \right] dz\, d\bar z$$

ie $T^{zz} = 0$, $T^{\bar z z} = \underbrace{-\frac{1}{g}\partial\varphi\,\partial\varphi}_{\text{hol.}}$, $T^{z\bar z} = \underbrace{-\frac{1}{g}\bar\partial\varphi\,\bar\partial\varphi}_{\text{antihol.}}$, $T^{\bar z\bar z} = 0$.

Renormalise: Define $T(z) = \frac{1}{2}\partial\varphi(z)\partial\varphi(z)$, $\bar T(\bar z) = \frac{1}{2}\bar\partial\varphi(\bar z)\bar\partial\varphi(\bar z)$.

This is the (scaled) ~~EM~~ Stress-energy tensor.

<u>Canonical Quantisation:</u> 1) Determine "degrees of freedom".
2) Compute conjugate momenta.
3) Impose canonical commutation rules (<u>equal-time</u>)

1) Fourier decomposition (since $\varphi(t,x) = \varphi(t, x+L)$):

$$\varphi(t,x) = \sum_{n\in\mathbb{Z}} \underbrace{\varphi_n(t)}_{\text{(degrees of freedom)}} e^{2\pi i n x/L}.$$

$$\Rightarrow \quad S = \frac{1}{2g}\int_{-\infty}^{\infty}\int_0^L \left[-(\partial_t\varphi)^2 + (\partial_x\varphi)^2\right] dx\, dt$$

$$= -\frac{L}{2g}\int_{-\infty}^{\infty}\sum_{m\in\mathbb{Z}}\left[\dot\varphi_m(t)\,\dot\varphi_{-m}(t) - \frac{4\pi^2 m^2}{L^2}\varphi_m(t)\varphi_{-m}(t)\right] dt.$$

2) Conjugate momentum to $\varphi_n(t)$ is $\pi_n(t) = \dfrac{\partial \mathcal{L}}{\partial\dot\varphi_n(t)}$, ie

$$\pi_n(t) = -\frac{L}{g}\dot\varphi_{-n}(t).$$

3) We promote both $\varphi_n(t)$ and $\pi_n(t)$ to operators with commutation relations ($\hbar = 1$):

$$\boxed{[\varphi_m(t), \varphi_n(t)] = [\pi_m(t), \pi_n(t)] = 0, \qquad [\varphi_m(t), \pi_n(t)] = i\,\delta_{m,n}.}$$

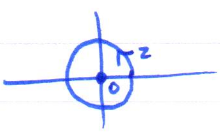
Thus, $\quad [\varphi_m(t), \dot\varphi_n(t)] = \dfrac{-ig}{L}\delta_{m+n,0}.$

Now change back to complex coords: $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$,

$\bar\partial\varphi(\bar z) = \sum_{n\in\mathbb{Z}} \bar a_n \bar z^{-n-1}$.

The a_n and $\bar a_n$ are the degrees of freedom in these coords. Thus, they become operators in the quantum theory.

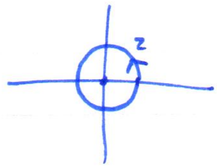
<u>Note</u>: Cauchy's theorem from complex analysis lets us write



$$a_n = \oint_0 \partial\varphi(z)\, z^n \frac{dz}{2\pi i} \qquad \left(\begin{matrix}\text{\underline{any} anticlockwise}\\ \text{contour around } 0\end{matrix}\right)$$

Cauchy's theorem: If $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}$, then

$$\oint_0 f(z)\, z^n \frac{dz}{2\pi i} = f_n. \qquad (n \in \mathbb{Z})$$



Proof: The contour is any (simple) curve encircling the origin, so we may take it to be a circle of radius 1. We parametrise $z = e^{2\pi i \theta}$, $\theta \in [0, 1)$, so $dz = 2\pi i e^{2\pi i\theta} d\theta = 2\pi i z\, d\theta$.

We compute $\oint_0 z^m \frac{dz}{2\pi i} = \int_0^1 e^{2\pi i m\theta} \cdot z\, d\theta$, for $m \in \mathbb{Z}$,

$$= \int_0^1 e^{2\pi i (m+1)\theta}\, d\theta$$

$$= \left[\frac{e^{2\pi i (m+1)\theta}}{2\pi i (m+1)} \right]_0^1 \qquad (m \neq -1)$$

$$= \frac{1}{2\pi i (m+1)} \left(e^{2\pi i (m+1)} - 1 \right)$$

$$= 0 \quad \text{since } m \in \mathbb{Z}.$$

If $m = -1$, we get $\oint_0 z^{-1} \frac{dz}{2\pi i} = \int_0^1 1\, d\theta = 1$ instead.

$$\therefore \oint_0 z^m \frac{dz}{2\pi i} = \delta_{m,-1}.$$

Now, $\oint_0 f(z)\, z^n \frac{dz}{2\pi i} = \sum_{m\in\mathbb{Z}} f_m \oint_0 z^{n-m-1} \frac{dz}{2\pi i} = \sum_{m\in\mathbb{Z}} f_m \delta_{n,m} = f_n$. $\blacksquare$

Generalisation: We may translate the contour to any other point $w \in \mathbb{C}$:

$$f(z) = \sum_{n\in\mathbb{Z}} f_n (z-w)^{-n-1} \implies \oint_w f(z)(z-w)^n \frac{dz}{2\pi i} = f_n. \quad (n\in\mathbb{Z}).$$

Alternatively: $\oint_w \frac{f(z)}{(z-w)^{n+1}} \frac{dz}{2\pi i} = \frac{1}{n!} \partial^n f(w)$, $n \in \mathbb{Z}$.

Take the contour to be a circle of fixed radius. Since $z = e^{2\pi(\tau + ix)/L}$, this circle is constant in time! Thus,

$$a_n = \oint_0 \partial\varphi(z)\, z^n \frac{dz}{2\pi i} = \frac{1}{4\pi i}\int_0^L (\partial_x \varphi - \partial_t \varphi)\, e^{2\pi i n(x-t)/L}\, dx$$

using $\partial = \frac{L}{4\pi i} z^{-1}(\partial_x - \partial_t)$ and $dz = \frac{2\pi i}{L} z\, dx$.

$$= -\left(\frac{n}{2}\varphi_{-n}(t) + \frac{L}{4\pi i}\dot{\varphi}_{-n}(t)\right) e^{-2\pi i n t/L}$$

by subs. $\varphi(t,x) = \sum_n \varphi_n(t)\, e^{2\pi i n x/L}$.

The commutators of the a_m are thus

$$[a_m, a_n] = m\,\delta_{m+n,0}\,\frac{g}{4\pi^2}.$$

We set $g = 4\pi^2$ to simplify this. Similarly,

$$[a_m, \bar{a}_n] = 0 \quad \text{and} \quad [\bar{a}_m, \bar{a}_n] = m\,\delta_{m+n,0}.$$

<u>Exercise:</u> Show that the a_n are conserved charges: $\dot{a}_n = 0$. One way involves rewriting the classical EOMs in terms of the $\varphi_n(t)$.

The Lie algebra generated by the a_n is called the <u>Heisenberg algebra</u>. We have two commuting copies!

Fock Spaces

Given the symmetry <u>algebra</u> of the a_n and $\bar{a}_n$, we can consider those with $n > 0$ to be <u>annihilation</u> operators, those with $n < 0$ to be <u>creation</u> operators, and those with $n = 0$ to be <u>zero-modes</u>. Let $a_n^\dagger = a_{-n}$ and $\bar{a}_n^\dagger = \bar{a}_{-n}$, so the zero-modes are self-adjoint:

$$a_0^\dagger = a_0, \quad \bar{a}_0^\dagger = a_0.$$

Their eigenvalues are real – they are ~~the~~ physical observables! In fact, the eigenvalues are the momenta in the z or $\bar{z}$ directions.

So, consider a vacuum (ground state) of momentum $p, \bar{p}$

$$|p,\bar{p}\rangle, \qquad p,\bar{p} \in \mathbb{R}.$$

It is annihilated by the $a_n, \bar{a}_n$ with $n>0$, but those with $n<0$ act on it to give excited states, eg.

$$a_{-1}|p,\bar{p}\rangle \quad a_{-1}^3|p,\bar{p}\rangle, \quad \bar{a}_{-2}|p,\bar{p}\rangle, \quad a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}|p,\bar{p}\rangle, \ldots$$

The span of ~~these~~ all excited states of a vacuum $|p,\bar{p}\rangle$ is called the Fock space $F_{p,\bar{p}}$

Note that these excited states have the same momentum:

$$[a_0, a_n] = 0, \quad [\bar{a}_0, \bar{a}_n] = 0 \quad \Rightarrow \quad a_0 a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}|p,\bar{p}\rangle$$
$$= a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}a_0|p,\bar{p}\rangle$$
$$= a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}\, p\,|p,\bar{p}\rangle$$
$$= p\left(a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}|p,\bar{p}\rangle\right).$$

Why are they excited? They have more energy! So, we turn to the ~~energy-momentum~~ stress-energy tensor $T(z), \bar{T}(\bar{z})$.

Recall: $T(z) = \frac{1}{2}\partial\varphi(z)\partial\varphi(z)$ (forget about antihol. stuff)

$$= \frac{1}{2}\sum_{r,s\in\mathbb{Z}} a_r a_s z^{-r-s-2} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{r\in\mathbb{Z}} a_r a_{n-r}\right] z^{-n-2}.$$

Problem: The Fourier mode $L_n \equiv \sum_{r\in\mathbb{Z}} a_r a_{n-r}$ is an infinite sum, so it might diverge when we act on some quantum state. ~~in a~~ ~~Fock~~

Divergences are common in quantum field theory, ~~but people know how to deal with them.~~

eg. $$L_0|p\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r}|p\rangle \cancel{= \frac{1}{2}\sum a_0^2|p\rangle}$$

$$= \frac{1}{2}\left[\sum_{r=-\infty}^{-1} \cancel{a_r a_{-r}|p\rangle} + a_0^2|p\rangle + \sum_{r=1}^{\infty} a_r a_{-r}|p\rangle\right]$$

$$= \frac{1}{2}p^2|p\rangle + \frac{1}{2}\sum_{r=1}^{\infty}\left(\cancel{a_{-r}a_r} + [a_r, a_{-r}]\right)|p\rangle$$

$$= \frac{1}{2}p^2|p\rangle + \frac{1}{2}\underbrace{\sum_{r=1}^{\infty} r}_{\text{diverges!}}|p\rangle$$

To fix this, note that any excited state $a_{-n_1}\cdots a_{-n_k}|p\rangle$ will be annihilated by a_n for n sufficiently large because

$$[a_n, a_{-n_i}] = 0 \quad \text{whenever } n \neq n_i.$$

We regularise the above divergence by introducing "<u>normal-ordering</u>" in which annihilators are moved to the right of creators:

More precisely, $$:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \leq -1 \quad (a_m \text{ is a creator}) \\ a_n a_m & \text{if } m \geq 0 \quad (a_m \text{ is not a creator}) \end{cases}$$

<u>Justification</u>: Classically, the a_n are number so they commute. When we quantise, there is an ambiguity in the ordering chosen. The naïve ordering gives divergences, so we replace it by normal-ordering.

Quantum stress-energy tensor:

$$T(z) = \frac{1}{2}:\partial\varphi(z)\partial\varphi(z): = \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big[\overbrace{\sum_{r\in\mathbb{Z}} :a_r a_{n-r}:}^{L_n}\Big] z^{-n-2}$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{r\leq -1} a_r a_{n-r} + \sum_{r\geq 0} a_{n-r}a_r\right] z^{-n-2}.$$

$$\left(T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\right)$$

These L_n act on excited states without divergences, eg.

$$L_0 |p\rangle = \frac{1}{2}\sum_{r\leq -1} a_r \cancel{a_{-r} |p\rangle} + \frac{1}{2}\sum_{r\geq 0} a_{-r}a_r |p\rangle = \frac{1}{2}a_0^2 |p\rangle = \frac{1}{2}p^2 |p\rangle .$$

The eigenvalue of L_0 is called the <u>energy</u> (actually, it is the sum of the eigenvalues of L_0 and $\overline{L}_0$ which is the total energy. — their difference is the angular momentum or spin!)

<u>Exercise</u>: • Check that $L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}$ $(n\neq 0)$

and $L_0 = \frac{1}{2}a_0^2 + \sum_{r=1}^{\infty} a_{-r}a_r$.

We compute for $m\neq 0$

$$[L_m, a_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}} [a_r a_{m-r}, a_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}} \left(a_r [a_{m-r}, a_n] + [a_r, a_n] a_{m-r}\right)$$
$$= \frac{1}{2}\sum_{r\in\mathbb{Z}} \left((m-r)\,\delta_{r,m+n}\, a_r + r\,\delta_{r,-n}\, a_{m-r}\right)$$
$$= \frac{1}{2}\left(-n\, a_{m+n} - n\, a_{m+n}\right) = -n\, a_{m+n} \qquad \text{(nice!)}$$

<u>Exercise</u>: Check that $[L_0, a_n] = -n a_n$ so that

$[L_m, a_n] = -n a_{m+n}$ for all $m,n\in\mathbb{Z}$.

<u>Exercise</u>: (hard!) Check that

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3-m}{12}\,\delta_{m+n,0}\, c\,, \quad \text{where } c=1.$$

These commutation relations are similar to those of the (classical) infinitesimal conformal generators ℓ_n:

$$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n} .$$

The extra term with $c=1$ is due to the quantisation. c is called the <u>central charge</u> or <u>conformal anomaly</u>.

Moral: Quantum CFTs may have $c \neq 0$. Classical CFTs must have $c=0$. The free boson is a quantum CFT with $c=1$. Actually,

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}, \quad \bar{L}_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} :\bar{a}_r \bar{a}_{n-r}:, \quad \bar{c}=1,$$

so we actually have $c=\bar{c}=1$.

Note that $[L_0, a_{-n}] = n a_{-n}$ means that acting with a creator a_{-n} increases the energy by n units.

eg. if $|\psi\rangle$ has energy E : $L_0|\psi\rangle = E|\psi\rangle$

then $a_{-n}|\psi\rangle$ has energy $E+n$:

$$\begin{aligned} L_0 a_{-n}|\psi\rangle &= (a_{-n}L_0 + [L_0, a_{-n}])|\psi\rangle \\ &= (a_{-n}E + n a_{-n})|\psi\rangle \\ &= (E+n)\, a_{-n}|\psi\rangle. \end{aligned}$$

So excited states do have more energy, as claimed!

Last time: Canonical Quantisation: $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$

$$[a_m, a_n] = m\,\delta_{m+n,0}$$

Fock Spaces: $|p\rangle$ vacuum of momentum $p \in \mathbb{R}$.

F_p $\qquad a_0|p\rangle = p|p\rangle, \quad a_n|p\rangle = 0 \quad \forall n>0.$

$$\boxed{\begin{array}{ccc} \vdots & \vdots & \vdots \\ a_{-1}^3|p\rangle & a_{-2}a_{-1}|p\rangle & a_{-3}|p\rangle \\ a_{-1}^2|p\rangle & a_{-2}|p\rangle & E=\tfrac{1}{2}p^2+2 \\ & a_{-1}|p\rangle & E=\tfrac{1}{2}p^2+1 \\ & |p\rangle & E=\tfrac{1}{2}p^2 \end{array}}$$

a_{-n} act on $|p\rangle$ to give excited states.

Excited states have same momentum.

Normal-Ordering: $T(z) = \frac{1}{2}\partial\varphi(z)^2$

$$= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$\Rightarrow \quad L_0|p\rangle = \infty.$$

$$:a_m a_n: \;=\; \begin{cases} a_m a_n & m \leq -1 \\ a_n a_m & m \geq 0 \end{cases}$$

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}:$$

$$T(z) = \frac{1}{2} :\partial\varphi(z)\,\partial\varphi(z): \;=\; \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$\Rightarrow \quad L_0|p\rangle = \frac{1}{2}p^2.$$

a_0 & $\bar{a}_0$ = momentum operators

$L_0 + \bar{L}_0$ = energy operator, $\quad L_0 - \bar{L}_0$ = angular momentum operator.

$$[L_m, a_n] = -n\,a_{m+n}, \qquad \underbrace{[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\,\delta_{m+n,0}\,c}_{\text{Virasoro algebra}}$$

with central charge $\underline{c=1}$.

Virasoro algebra = quantisation of infinitesimal conformal generators

$\Rightarrow$ Free Boson is a quantum conformal field theory.

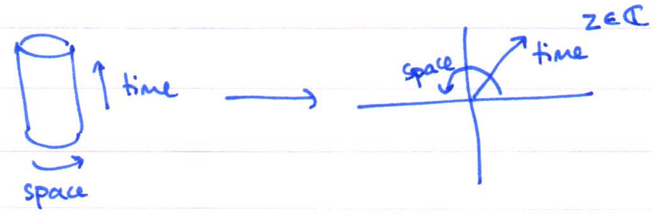
State-Field Correspondence

Since $E_p = \frac{1}{2}p^2$, the vacuum $|0\rangle$ has minimal energy. It is the "true vacuum".

Given a field $\psi(z)$ eg. $\partial\varphi(z)$, $T(z)$, etc..., there is a corresponding quantum state $|\psi\rangle$ defined by

$$|\psi\rangle = \lim_{z\to 0} \psi(z)|0\rangle . \qquad \text{(AND VICE-VERSA!)}$$

Recall:



$z \to 0$ means $t \to -\infty$.

In scattering language, $|\psi\rangle$ is an "asymptotic in-state".

Examples:
- The identity field $\Omega(z) = 1 = \sum_{n\in\mathbb{Z}} \delta_{n,0} z^{-n}$ satisfies

$$\lim_{z\to 0}\Omega(z)|0\rangle = \lim_{z\to 0} 1\,|0\rangle = |0\rangle .$$

Thus, the corresponding state is $|\Omega\rangle = |0\rangle$.

- The holomorphic derivative of the boson field $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ satisfies

$$\begin{aligned}\lim_{z\to 0}\partial\varphi(z)|0\rangle &= \lim_{z\to 0}\sum_{n\in\mathbb{Z}} a_n z^{-n-1}|0\rangle \\ &= \lim_{z\to 0}\sum_{n\leq -1} a_n z^{-n-1}|0\rangle \qquad \left[\begin{array}{l}\text{since } a_n|0\rangle = 0 \\ \text{for } n=0,1,2,\dots\end{array}\right. \\ &= \lim_{z\to 0}\left[a_{-1}|0\rangle + a_{-2}z|0\rangle + a_{-3}z^2|0\rangle + \dots\right] \\ &= a_{-1}|0\rangle .\end{aligned}$$

ie $|\partial\varphi\rangle$ is the excited state $a_{-1}|0\rangle$.

- $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ corresponds to $L_{-2}|0\rangle$ because

$$L_{-1}|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r-1}|0\rangle = 0 \quad \left[\begin{array}{l} a_r|0\rangle = 0 \text{ for } r\geq 0 \\ a_{-r-1}|0\rangle = 0 \text{ for } r\leq -1\end{array} \quad [a_r, a_{-r-1}] = 0\right].$$

- $L_{-2}|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r-2}|0\rangle = \frac{1}{2}a_{-1}^2|0\rangle$ $\quad \begin{bmatrix} a_r|0\rangle = 0 \ \forall r \geq 0 \\ a_{-r-2}|0\rangle = 0 \ \forall r \leq -2 \end{bmatrix}$

so $T(z) = \frac{1}{2}:\partial\varphi(z)\partial\varphi(z):$ corresponds to $\frac{1}{2}a_{-1}^2|0\rangle$,

ie $:\partial\varphi(z)\partial\varphi(z):$ corresponds to $a_{-1}^2|0\rangle$.

<u>Exercise</u>: Show that $\partial^2\varphi(z) = \partial(\partial\varphi(z)) = \partial\sum_{n\in\mathbb{Z}} a_n z^{-n-1} = -\sum_{n\in\mathbb{Z}}(n+1)a_n z^{-n-2}$ corresponds to $a_{-2}|0\rangle$.

Show that $\partial^j\varphi(z)$ corresponds to $(j-1)!\, a_{-j}|0\rangle$.

<u>OPEs</u>

<u>Recall</u>: $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $[a_m, a_n] = m\,\delta_{m+n,0}$.

$$\Rightarrow [a_m, \partial\varphi(w)] = \sum_{n\in\mathbb{Z}} [a_m, a_n]\, w^{-n-1} = \sum_{n\in\mathbb{Z}} m\,\delta_{m+n,0}\, w^{-n-1}$$

$$= m\,w^{m-1}.$$

But,
$$[\partial\varphi(z), \partial\varphi(w)] = \cancel{\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}[a_m, a_n] z^{-m-1} w^{-n-1}}$$
$$= \sum_{m\in\mathbb{Z}} [a_m, \partial\varphi(w)]\, z^{-m-1}$$
$$= \sum_{m\in\mathbb{Z}} m\,w^{m-1} z^{-m-1} = \frac{1}{zw}\overbrace{\sum_{m\in\mathbb{Z}} m\left(\frac{w}{z}\right)^m}^{\text{DIVERGENT!}}$$

We again have to remove such divergences. This is done, not by normal-ordering, but by "time-ordering". Because time on the cylinder is the radial direction on $\mathbb{C}$, we call it "radial-ordering":

$$R\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z| > |w|, \text{ z is \underline{later} in time than w} \\ B(w)A(z) & \text{if } |z| < |w|. \text{ z is \underline{earlier} in time than w} \end{cases}$$

Why is this reasonable? These fields act on quantum states $|C\rangle$ which are fields at $z=0$ (ie time $=-\infty$). Thus,

$$A(z)B(w)|C\rangle$$

is interpreted as:

- $|C\rangle$ is a state at time $-\infty$
- Act with $B(w)$ at time $|w|$
- Then act with $A(z)$ at time $|z|$.

This only makes sense if $|z|>|w|$! If $|z|<|w|$, then $A(z)$ acts before $B(w)$ and we must write

$$B(w)A(z)|C\rangle.$$

In all cases, physics requires that the fields are radially-ordered.

This removes the divergence in $[\partial\varphi(z), \partial\varphi(w)]$:

$$R\{[\partial\varphi(z),\partial\varphi(w)]\} = R\{\partial\varphi(z)\partial\varphi(w)\} - R\{\partial\varphi(w)\partial\varphi(z)\} = 0.$$

But in a boring way! More interesting is that we can compute for $|z|>|w|$:

$$R\{\partial\varphi(z)\partial\varphi(w)\} = \partial\varphi(z)\partial\varphi(w) = \sum_{r,s\in\mathbb{Z}} a_r a_s z^{-r-1} w^{-s-1}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} a_r a_{n-r} z^{-r-1} w^{-n-1+r}.$$

For $n\neq 0$, $a_r a_{n-r} = :a_r a_{n-r}:$ (since $[a_r, a_{n-r}]=0$).

For $n=0$, $a_r a_{-r} = :a_r a_{-r}:$ if $r\leq -1$

and $a_r a_{-r} = a_{-r}a_r + [a_r, a_{-r}] = :a_r a_{-r}: + r$ if $r\geq 0$.

ie $$R\{\partial\varphi(z)\partial\varphi(w)\} = \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}: z^{-r-1} w^{-n-1+r} + \sum_{r=0}^{\infty} r z^{-r-1} w^{r-1}$$

$$= :\partial\varphi(z)\partial\varphi(w): + \frac{1}{z^2}\sum_{r=0}^{\infty} r\left(\frac{w}{z}\right)^{r-1}$$

$$= \quad '' \quad + \frac{1}{z^2}\frac{1}{(1-w/z)^2} \qquad \text{(since } |z|>|w|\text{)}$$

$$\Rightarrow \quad \mathcal{R}\{\partial\varphi(z)\partial\varphi(\omega)\} = :\partial\varphi(z)\partial\varphi(\omega): + \frac{1}{(z-\omega)^2},$$

at least when $|z|>|\omega|$. If $|z|<|\omega|$, then we get

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(\omega)\} = \partial\varphi(\omega)\partial\varphi(z) = :\partial\varphi(\omega)\partial\varphi(z): + \frac{1}{(\omega-z)^2}.$$

<u>Claim</u>: $:a_r a_s: = :a_s a_r:$, so $:\partial\varphi(z)\partial\varphi(\omega): = :\partial\varphi(\omega)\partial\varphi(z):$

<u>Proof</u>: • $r+s \neq 0 \Rightarrow :a_r a_s: = a_r a_s = a_s a_r = :a_s a_r:$ as $[a_r, a_s] = 0$.

• $r+s=0$. (1) $r \leq -1 \Rightarrow :a_r a_{-r}: = a_r a_{-r}$ and

$:a_{-r} a_r: = a_r a_{-r}$ since $-r \geq 1$.

(2) $r \geq 1 \Rightarrow :a_r a_{-r}: = a_{-r} a_r = :a_{-r} a_r:$ since $-r \leq 1$.

(3) $r=0 \Rightarrow :a_0 a_0: = :a_0 a_0:$. ■

$\therefore$ For $|z|>|\omega|$ and $|z|<|\omega|$,

$$\boxed{\mathcal{R}\{\partial\varphi(z)\partial\varphi(\omega)\} = \frac{1}{(z-\omega)^2} + :\partial\varphi(z)\partial\varphi(\omega):.}$$

We can Taylor-expand the RHS about $z=\omega$:

$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(\omega)\} = \underbrace{\frac{1}{(z-\omega)^2} + :\partial\varphi(\omega)\partial\varphi(\omega): + :\partial^2\varphi(\omega)\partial\varphi(\omega):(z-\omega) + \cdots}_{\text{Laurent series expansion in } z-\omega.}$$

<u>NB</u>: • There is a pole for $z=\omega$, so $\mathcal{R}\{\partial\varphi(z)\partial\varphi(z)\}$ diverges.

• $:\partial\varphi(\omega)\partial\varphi(\omega): = \oint_\omega \frac{\mathcal{R}\{\partial\varphi(z)\partial\varphi(\omega)\}}{z-\omega}\frac{dz}{2\pi i}$

• This Laurent series expansion is called an <u>OPE</u>.

In general, an OPE always has the form

$$\mathcal{R}\{A(z)B(w)\} = [\text{singular terms as } z\to w] + :A(z)B(w):.$$

[In fact, the <u>general definition</u> of normal-ordering is

$$:A(w)B(w): = \oint_w \frac{\mathcal{R}\{A(z)B(w)\}}{z-w}\frac{dz}{2\pi i}.$$]

Because of this, lazy physicists just write the singular terms:

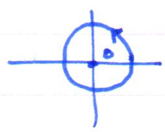
$$\mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} \sim \frac{1}{(z-w)^2}.$$

Usually they leave out the radial-ordering symbol too:

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}.$$

The OPE contains all information about commutators in its singular terms! Recall that

$$a_n = \oint_0 \partial\varphi(z) z^n \frac{dz}{2\pi i}.$$



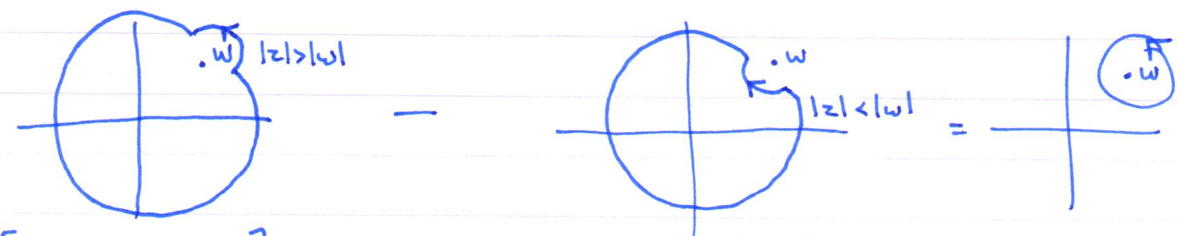
$$\therefore [a_m, a_n] = a_m a_n - a_n a_m$$

$$= \oint_0\oint_0 \partial\varphi(z)\partial\varphi(w) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0\oint_0 \partial\varphi(w)\partial\varphi(z) z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

(these products must be radially-ordered to make sense!)

$$= \oint_0 \oint_{0, |z|>|w|} \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz\,dw}{} - \oint_0 \oint_{0, |z|<|w|} \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz\,dw}{}$$

~~zw~~ z contour:



$$= \oint_0 \left[\oint_{0, |z|>|w|} - \oint_{0, |z|<|w|}\right] \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0 \oint_w \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0 \oint_w \left[\frac{z^m w^n}{(z-w)^2} + \cancel{\overbrace{:\partial\varphi(z)\,\partial\varphi(w): z^m w^n}^{\text{no pole at } z=w}} \right] \frac{dz}{2\pi i} \frac{dw}{2\pi i}$$

$$= \oint_0 m w^{m+n-1} \frac{dw}{2\pi i}$$

$$= m\,\delta_{m+n,0} .$$

☺

Equivalently, the commutation rules determine the ~~singular part of the~~ OPE using the state-field correspondence:

If $\mathcal{R}\{\partial\varphi(z)\,\partial\varphi(w)\} = \sum_n \psi_n(w)(z-w)^{-n-1}$ for some unknown $\psi_n(w)$, then apply both sides to $|0\rangle$ and let $w \to 0$:

LHS: $\lim_{w\to 0} \mathcal{R}\{\partial\varphi(z)\,\partial\varphi(w)\}|0\rangle = \partial\varphi(z) \lim_{w\to 0} \partial\varphi(w)|0\rangle = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} |\partial\varphi\rangle$

$$= \sum_{n\in\mathbb{Z}} a_n a_{-1} |0\rangle z^{-n-1} \qquad \left(\text{recall } |\partial\varphi\rangle = a_{-1}|0\rangle\right).$$

RHS: $\lim_{w\to 0} \sum_n \psi_n(w)(z-w)^{-n-1}|0\rangle = \sum_n \left[\lim_{w\to 0} \psi_n(w)|0\rangle\right] z^{-n-1} = \sum_n |\psi_n\rangle z^{-n-1}$.

Comparing gives $|\psi_n\rangle = a_n a_{-1}|0\rangle$ for all $n\in\mathbb{Z}$.

For $n \geq 2$, $a_n a_{-1}|0\rangle = a_{-1}a_n|0\rangle = 0$, so $|\psi_n\rangle = 0$, hence $\psi_n(w) = 0$.

For $n=1$, $a_1 a_{-1}|0\rangle = (\cancel{a_{-1}a_1} + [a_1, a_{-1}])|0\rangle = |0\rangle$, so $|\psi_1\rangle = |0\rangle$ and $\psi_1(w) = 1$.

For $n=0$, $a_0 a_{-1}|0\rangle = a_{-1}a_0|0\rangle = 0$, so $|\psi_0\rangle = 0$, hence $\psi_0(w) = 0$

For $n=-1$, $a_{-1}^2|0\rangle$ is $|\psi_{-1}\rangle$, so $\psi_{-1}(w)$ is $:\partial\varphi(w)\,\partial\varphi(w):$

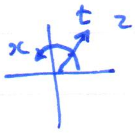
etc...

$$\therefore \quad \mathcal{R}\{\partial\varphi(z)\,\partial\varphi(w)\} = \frac{\psi_1(w)}{(z-w)^2} + \psi_{-1}(w) + \psi_{-2}(w)(z-w) + \cdots$$

$$= \frac{1}{(z-w)^2} + :\partial\varphi(w)\,\partial\varphi(w): + \cdots$$

<u>Exercise</u>: Use $[L_m, a_n] = -n a_{m+n}$ to compute the singular terms of $\mathcal{R}\{T(z)\,\partial\varphi(w)\}$.

<u>Recap:</u> State-Field Correspondence: $|\psi\rangle = \lim_{z\to 0} \psi(z)|0\rangle$



(field in the infinite past) $\nearrow$ $|\psi\rangle$; $z\to 0$ $\uparrow$ $t\to-\infty$; $|0\rangle$ $\uparrow$ true vacuum

Radial Ordering: (Time) $R\{A(z)B(\omega)\} = \begin{cases} A(z)B(\omega) & \text{if } |z|>|\omega|, \\ B(\omega)A(z) & \text{if } |z|<|\omega|. \end{cases}$

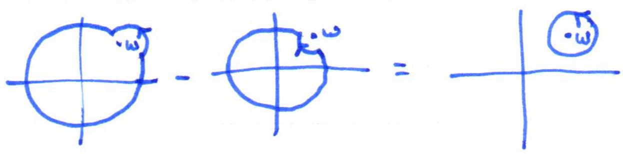
(Physical products must be radially-ordered!)

Operator Product Expansions:

$$R\{\partial\varphi(z)\partial\varphi(\omega)\} = \frac{1}{(z-\omega)^2} + :\partial\varphi(z)\partial\varphi(\omega):$$

<u>or</u> $R\{\partial\varphi(z)\partial\varphi(\omega)\} \sim \frac{1}{(z-\omega)^2}$

OPEs $\Longleftrightarrow$ Commutators $\left[\oint_{\substack{0 \\ |z|>|\omega|}} - \oint_{\substack{0 \\ |z|<|\omega|}}\right] \cdots \frac{dz}{2\pi i} = \oint_{\omega} \cdots \frac{dz}{2\pi i}$



$R\{\partial\varphi(z)\partial\varphi(\omega)\} \sim \frac{1}{(z-\omega)^2} \Longleftrightarrow [a_m, a_n] = m\,\delta_{m+n,0}.$

Wick's Theorem for OPEs

~~The OPE of any normally~~

Wick's theorem lets us compute the OPE of normally-ordered products of $\partial\varphi(z)$. eg. we can compute $R\{T(z)\partial\varphi(\omega)\} = \frac{1}{2}R\{:\partial\varphi(z)\partial\varphi(z):\partial\varphi(\omega)\}$.
To do this, we introduce the contraction

$$\partial\varphi(z)\,\partial\varphi(\omega) = \frac{1}{(z-\omega)^2} \quad \text{(singular term).}$$

The singular terms of the OPE ~~is~~ are computed by taking all possible contractions and normal-ordering what remains:

$$R\{T(z)\partial\varphi(\omega)\} = \frac{1}{2}R\{:\partial\varphi(z)\partial\varphi(z):\partial\varphi(\omega)\}$$

$$\sim \frac{1}{2}\Big[:\partial\varphi(z)\partial\varphi(z):\partial\varphi(\omega) + :\partial\varphi(z)\partial\varphi(z):\partial\varphi(\omega)\Big]$$

$$= \frac{1}{2}\left[\frac{:\partial\varphi(z)}{(z-\omega)^2} + \frac{:\partial\varphi(z):}{(z-\omega)^2}\right] = \frac{\partial\varphi(z)}{(z-\omega)^2}$$

$$= \frac{\partial\varphi(\omega)}{(z-\omega)^2} + \frac{\partial^2\varphi(\omega)}{z-\omega} + \frac{1}{2}\partial^3\varphi(\omega) + \cdots$$

$$\sim \frac{\partial\varphi(\omega)}{(z-\omega)^2} + \frac{\partial^2\varphi(\omega)}{z-\omega}.$$

Another example:

$$R\{T(z)T(\omega)\} = \frac{1}{4}R\{:\partial\varphi(z)\partial\varphi(z)::\partial\varphi(\omega)\partial\varphi(\omega):\}$$

$$\sim \frac{1}{4}\Big[:\partial\varphi(z)\partial\varphi(z)\partial\varphi(\omega)\partial\varphi(\omega): + :\partial\varphi(z)\partial\varphi(z)\partial\varphi(\omega)\partial\varphi(\omega):$$

$$+ :\partial\varphi(z)\partial\varphi(z)\partial\varphi(\omega)\partial\varphi(\omega): + :\partial\varphi(z)\partial\varphi(z)\partial\varphi(\omega)\partial\varphi(\omega):$$

$$+ :\partial\varphi(z)\partial\varphi(z)\partial\varphi(\omega)\partial\varphi(\omega): + :\partial\varphi(z)\partial\varphi(z)\partial\varphi(\omega)\partial\varphi(\omega):\Big]$$

$$= \frac{:\partial\varphi(z)\partial\varphi(\omega):}{(z-\omega)^2} + \frac{1/2}{(z-\omega)^4}$$

$$\sim \frac{1/2}{(z-\omega)^4} + \frac{:\partial\varphi(\omega)\partial\varphi(\omega):}{(z-\omega)^2} + \frac{:\partial^2\varphi(\omega)\partial\varphi(\omega):}{z-\omega}$$

$$= \frac{1/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}.$$

Here, we use $\partial T(\omega) = \frac{1}{2}\partial :\partial\varphi(\omega)\partial\varphi(\omega): = \frac{1}{2}:\partial^2\varphi(\omega)\partial\varphi(\omega): + \frac{1}{2}:\partial\varphi(\omega)\partial^2\varphi(\omega):$

$$= :\partial^2\varphi(\omega)\partial\varphi(\omega): \qquad (\text{since } :a_r a_s: = :a_s a_r:).$$

<u>Exercise</u>: Use $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, and so $L_n = \oint_0 T(z) z^{n+1}\frac{dz}{2\pi i}$, to compute

$[L_m, L_n]$ using the double contour trick:

$$\left[\oint_{\substack{0\\|z|>|\omega|}} - \oint_{\substack{0\\|z|<|\omega|}}\right]\frac{dz}{2\pi i} = \oint_{\omega} \cdots \frac{dz}{2\pi i}.$$

<u>Primary Fields</u> Define a primary field to be any field that corresponds to a vacuum under the state-field correspondence. (*) The ~~field~~ primary field corresponding to the momentum p vacuum $|p\rangle$ is denoted by

$$V_p(z), \quad \text{~~and~~}$$

We must have $\lim_{z\to 0} V_p(z)|0\rangle = |p\rangle$. Aside from $V_0(z) = 1$, we haven't found any primary fields yet.

~~The $V_p(z)$ are called~~ Being primary, or being a vacuum, depends upon the algebra.

- A free boson vacuum has $a_0|p\rangle = p|p\rangle$ and $a_n|p\rangle = 0 \ \forall n>0$.

- A conformal vacuum has $L_0|h\rangle = h|h\rangle$ and $L_n|h\rangle = 0 \ \forall n>0$.

A free boson primary field $V_p(z)$ is automatically a conformal primary field, because $|p\rangle$ satisfies $L_0|p\rangle = \frac{1}{2}p^2|p\rangle$ and $L_n|p\rangle = 0 \ \forall n>0$. But, conformal primaries need not be free boson primaries.

(*) All other fields are said to be secondary.

Proof that $L_n|p\rangle = 0 \ \forall n>0$: If $L_n|p\rangle \neq 0$, then

$$L_0 L_n|p\rangle = (L_n L_0 + [L_0, L_n])|p\rangle$$
$$= (L_n \cdot \tfrac{1}{2}p^2 - nL_n)|p\rangle$$
$$= (\tfrac{1}{2}p^2 - n)L_n|p\rangle,$$

so the quantum state $L_n|p\rangle$ would have energy $\frac{1}{2}p^2 - n$. But, the states of the Fock space have energy at least $\frac{1}{2}p^2$, a contradiction. $\therefore L_n|p\rangle = 0$. ■

Examples.

- The field $\partial\varphi(z)$ is a conformal primary because the corresponding state $|\partial\varphi\rangle = a_{-1}|0\rangle$ is a conformal vacuum:

$$L_0|\partial\varphi\rangle = L_0 a_{-1}|0\rangle = (a_{-1}\cancel{L_0} + [L_0, a_{-1}])|0\rangle = a_{-1}|0\rangle = |\partial\varphi\rangle \ (E=1),$$

$$L_1|\partial\varphi\rangle = L_1 a_{-1}|0\rangle = (a_{-1}\cancel{L_1} + [L_1, a_{-1}])|0\rangle = a_0|0\rangle = 0,$$

and $L_n|\partial\varphi\rangle = 0$ for $n \geq 2$ because $L_n|\partial\varphi\rangle$ would have energy $1-n < 0$.

$\partial\varphi(z)$ is not a free boson primary because

$$a_1|\partial\varphi\rangle = a_1 a_{-1}|0\rangle = (a_{-1}\cancel{a_1} + [a_1, a_{-1}])|0\rangle = |0\rangle \neq 0,$$

ie $|\partial\varphi\rangle$ is not a free boson vacuum.

- $T(z)$ is a conformal primary if and only if the central charge is $c=0$:

$$L_0|T\rangle = L_0 L_{-2}|0\rangle = (L_{-2}\cancel{L_0} + [L_0, L_{-2}])|0\rangle = 2L_{-2}|0\rangle = 2|T\rangle,$$

$$L_1|T\rangle = L_1 L_{-2}|0\rangle = (L_{-2}\cancel{L_1} + [L_1, L_{-2}])|0\rangle = 3L_{-1}|0\rangle = 0, \qquad *$$

$$L_2|T\rangle = L_2 L_{-2}|0\rangle = (L_{-2}\cancel{L_2} + [L_2, L_{-2}])|0\rangle = (4L_0 + \tfrac{1}{2}c)|0\rangle = \tfrac{1}{2}c|0\rangle,$$

$L_n|T\rangle = 0$ for $n \geq 3$ as $L_n|T\rangle$ would have energy $2-n < 0$.

(*) $L_{-1}|0\rangle = 0$ in any conformal field theory so that $\lim_{z\to 0} T(z)|0\rangle$ exists (and is $L_{-2}|0\rangle$).

How to charaterise a primary field using OPEs?

- Let $V_p(w)$ be a free boson primary, so $a_0|p\rangle = p|p\rangle$, $a_n|p\rangle = 0\ \forall n>0$.
  Write: $\mathcal{R}\{\partial\varphi(z)V_p(w)\} = \sum_{n\in\mathbb{Z}} \underset{\substack{\uparrow\\ \text{unknown fields}}}{\psi_n(w)} (z-w)^{-n-1}$,

  apply to $|0\rangle$ and let $w\to 0$. We get

  $$\text{LHS} = \partial\varphi(z)|p\rangle = \sum_{n\in\mathbb{Z}} a_n|p\rangle z^{-n-1} \text{ and RHS} = \sum_{n\in\mathbb{Z}} |\psi_n\rangle z^{-n-1},$$

  $$\text{ie } |\psi_n\rangle = a_n|p\rangle.$$

  $$\therefore\ |\psi_0\rangle = a_0|p\rangle = p|p\rangle \implies \psi_0(w) = pV_p(w),$$
  $$|\psi_n\rangle = a_n|p\rangle = 0 \implies \psi_n(w) = 0 \quad \forall n\geq 1.$$

  $$\implies \mathcal{R}\{\partial\varphi(z)V_p(w)\} = \frac{pV_p(w)}{z-w} + \text{terms regular as } z\to w.$$

- Let $\phi_h(w)$ be a conformal primary, so $L_0|h\rangle = h|h\rangle$, $L_n|h\rangle = 0\ \forall n>0$.
  By writing
  $$\mathcal{R}\{T(z)\phi_h(w)\} = \sum_{n\in\mathbb{Z}} \psi_n(w)(z-w)^{-n-2},$$
  show that the OPE is
  $$\mathcal{R}\{T(z)\phi_h(w)\} \sim \frac{h\phi_h(w)}{(z-w)^2} + \frac{\partial\phi_h(w)}{z-w}.$$

  This requires the following fact:

  In any CFT, ~~as the~~ if $A(z)$ corresponds to $|A\rangle$ then

  $\partial A(z)$ ...... $L_{-1}|A\rangle$.

Correlation Functions

In quantum mechanics, all physical quantities may be expressed in terms of amplitudes $\langle \phi | \psi \rangle$, (eg. $\langle \phi | H | \phi \rangle$ is the energy of the state ϕ) In quantum field theory, this is generalised so that the physical quantities are related to

$$\langle \phi | A(z) B(w) \cdots C(x) | \psi \rangle .$$

By the state-field correspondence, states $|\psi\rangle$ can be replaced by $\psi(z)|0\rangle$ if the limit $z \to 0$ is taken. Therefore, we consider the

<u>correlation functions</u> $\quad \langle 0 | \overbrace{A(z) B(w) \cdots C(x)}^{\text{radially-ordered}} | 0 \rangle \equiv \langle A(z) B(w) \cdots C(x) \rangle$

We interpret $|0\rangle$ as being at $t = -\infty$, $C(x)$ as acting at time $|x|$, ... $A(z)$ as acting at time $|z| > |w| > \cdots > |x|$. $\langle 0|$ then projects the result onto the true vacuum — so we interpret bras $\langle \phi |$ as being at $t = +\infty$.

Recall that operators act on bras using the adjoint:

$$\langle 0 | a_n = (a_n^\dagger |0\rangle)^\dagger = (a_{-n}|0\rangle)^\dagger .$$

Thus, $\langle 0 | a_n = 0$ for $n \leq 0$ because $a_n |0\rangle = 0$ for $n \geq 0$.

<u>Examples:</u>

- Correlators involving only the identity field are always 1 (by convention):
$$\langle 0 | 1 \cdots 1 | 0 \rangle = \langle 0 | 0 \rangle = 1 .$$
- $\langle 0 | \partial \varphi(z) | 0 \rangle = \sum_{n \in \mathbb{Z}} \langle 0 | a_n | 0 \rangle z^{-n-1} = 0$ as $a_n |0\rangle = 0 \ \forall n \geq 0$ and $\langle 0 | a_n = 0 \ \forall n \leq 0$.

- $\langle 0| :\partial\varphi(z)\partial\varphi(w): |0\rangle = \sum_{r,s\in\mathbb{Z}} \langle 0| :a_r a_s: |0\rangle z^{-r-1} w^{-s-1}$

$$= \sum_{\substack{r\le -1 \\ s\in\mathbb{Z}}} \underbrace{\langle 0| a_r}_{=0} a_s |0\rangle z^{-r-1} w^{-s-1} + \sum_{\substack{r\ge 0 \\ s\in\mathbb{Z}}} \langle 0| a_s \underbrace{a_r |0\rangle}_{=0} z^{-r-1} w^{-s-1}$$

$$= 0.$$ (normally-ordered correlators always vanish!)

- Show that $\langle 0| T(z) |0\rangle = 0$.

- $\langle 0| \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} |0\rangle = \langle 0| \frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w): |0\rangle = \frac{1}{(z-w)^2}$.

Correlators of n fields are called <u>n-point functions</u>.

- Show that $\langle 0| \mathcal{R}\{T(z)T(w)\} |0\rangle = \frac{c/2}{(z-w)^4}$.

Correlators involving only $\partial\varphi(z)$ and its derivatives can be computed using Wick's theorem:

$$\langle 0| \mathcal{R}\{\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\} |0\rangle$$

$$= \langle 0| :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3): |0\rangle + \langle 0| :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3) |0\rangle$$

$$+ \langle 0| :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3) |0\rangle$$

$$= \frac{1}{(z_1-z_3)^2} \langle 0| \partial\varphi(z_2) |0\rangle + \frac{1}{(z_1-z_2)^2} \langle \partial\varphi(z_3)\rangle + \frac{1}{(z_2-z_3)^2} \langle \partial\varphi(z_1)\rangle$$

$$= 0.$$

$$\langle 0| \mathcal{R}\{\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4)\} |0\rangle$$

$$= \quad + \quad +$$

$$= \frac{1}{(z_1-z_2)^2(z_3-z_4)^2} + \frac{1}{(z_1-z_3)^2(z_2-z_4)^2} + \frac{1}{(z_1-z_4)^2(z_2-z_3)^2}.$$

Constraints on correlators

Let $V_{p_1}(z_1), \ldots, V_{p_n}(z_n)$ be free boson primaries. ~~Since~~

~~anyhow~~

Compute: $$[a_m, V_p(w)] = \oint_{\substack{0 \\ |z|>|w|}} \partial\varphi(z) V_p(w) z^m \frac{dz}{2\pi i} - \oint_{\substack{0 \\ |z|<|w|}} V_p(w) \partial\varphi(z) z^m \frac{dz}{2\pi i}$$
$$= \oint_w R\{\partial\varphi(z) V_p(w)\} z^m \frac{dz}{2\pi i}$$
$$= \oint_w \frac{p V_p(w)}{z-w} z^m \frac{dz}{2\pi i}$$
$$= p w^m V_p(w).$$

In particular, $\underline{[a_0, V_p(w)] = p V_p(w)}$. Since $a_0|0\rangle = 0$ and
$$\langle 0| a_0 = [a_0^\dagger |0\rangle]^\dagger = [a_0|0\rangle]^\dagger = 0,$$

we compute
$$0 = \langle 0| a_0 V_{p_1}(z_1) \cdots V_{p_n}(z_n)|0\rangle$$
$$= \sum_{j=1}^n \langle 0| V_{p_1}(z_1) \cdots [a_0, V_{p_j}(z_j)] \cdots V_{p_n}(z_n)|0\rangle + \langle 0|V_{p_1}(z_1) \cdots V_{p_n}(z_n) \cancel{a_0|0\rangle}$$
$$= \sum_{j=1}^n p_j \cdot \langle 0| V_{p_1}(z_1) \cdots V_{p_n}(z_n)|0\rangle.$$

__Conclusion:__ A correla~~ta~~tor of free boson primaries is 0 unless their momenta sum to zero.

__Interpretation:__ Momentum is conserved: $|0\rangle$ has $p=0$ and we want to compare with $\langle 0|$ which likewise has $p=0$.

Let $\phi_{h_1}(z_1), \ldots, \phi_{h_n}(z_n)$ be conformal primaries!

__Ex:__ $[L_m, \phi_h(w)] = h(m+1) w^m \phi_h(w) + w^{m+1} \partial\phi_h(w)$.

We consider L_{-1}, L_0 and L_1 as they each annihilate $|0\rangle$ and $\langle 0|$.

$\cancel{i[L_{-1}, \phi_h(w)] = \partial \phi_h(w)}$ $(m = 0, \pm 1)$

$$\Rightarrow \quad 0 = \langle 0 | L_m \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n) | 0 \rangle$$

$$= \sum_{j=1}^{n} \langle 0 | \phi_{h_1}(z_1) \cdots [L_m, \phi_{h_j}(z_j)] \cdots \phi_{h_n}(z_n) | 0 \rangle \quad + \quad 0$$

$$= \sum_{j=1}^{n} \langle 0 | \phi_{h_1}(z_1) \cdots \left[h(m+1) z_j^m \phi_{h_j}(z_j) + z_j^{m+1} \partial_j \phi_{h_j}(z_j) \right] \cdots \phi_{h_n}(z_n) | 0 \rangle$$

$$(\partial_j \equiv \partial_{z_j})$$

$$= \sum_{j=1}^{n} \left[h_j (m+1) z_j^m + z_j^{m+1} \partial_j \right] \cdot \langle 0 | \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n) | 0 \rangle .$$

These are PDEs for the n-point function! Explicitly:

$$m = -1 \quad \Rightarrow \quad \sum_{j=1}^{n} \partial_j \langle 0 | \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n) | 0 \rangle = 0, \qquad (-1)$$

$$m = 0 \quad \Rightarrow \quad \sum_{j=1}^{n} (z_j \partial_j + h_j) \langle 0 | \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n) | 0 \rangle = 0, \qquad (0)$$

$$m = 1 \quad \Rightarrow \quad \sum_{j=1}^{n} (z_j^2 \partial_j + 2 h_j z_j) \langle 0 | \phi_{h_1}(z_1) \cdots \phi_{h_n}(z_n) | 0 \rangle = 0. \qquad (1)$$

Let's solve them:

<u>$n=1$</u>: $(-1) \Rightarrow \partial_1 \langle 0 | \phi_{h_1}(z_1) | 0 \rangle = 0 \Rightarrow \langle 0 | \phi_{h_1}(z_1) | 0 \rangle$ is constant.

$(0) \Rightarrow (\cancel{z_1 \partial_1} + h_1) \langle 0 | \phi_{h_1}(z_1) | 0 \rangle = 0 \Rightarrow \langle 0 | \phi_{h_1}(z_1) | 0 \rangle = 0$ or $h_1 = 0$.

<u>Note</u>: $\langle 0 | \underline{1} | 0 \rangle = 1$ and, indeed, $1 = \phi_0(z_1)$ has $h = 0$.

<u>$n=2$</u>: $(-1) \Rightarrow (\partial_1 + \partial_2) \langle 0 | \phi_{h_1}(z_1) \phi_{h_2}(z_2) | 0 \rangle = 0$.

Change coordinates to $z = z_1 + z_2$ and $z_{12} = z_1 - z_2$.

$(-1) \Rightarrow \partial \langle 0 | \phi_{h_1}(z_1) \phi_{h_2}(z_2) | 0 \rangle = 0 \Rightarrow \langle 0 | \phi_{h_1}(z_1) \phi_{h_2}(z_2) | 0 \rangle = f(z_{12})$.

$(0) \Rightarrow (z_{12} \partial_{12} + h_1 + h_2) f(z_{12}) = 0 \Rightarrow f(z_{12}) = \dfrac{C_{12}}{z_{12}^{h_1 + h_2}}$.

① $\Rightarrow$ $(z z_{12} \partial_{12} + (h_1+h_2) z + (h_1-h_2) z_{12}) f(z_{12}) = 0$.

Substitute for f to get $(h_1-h_2) \dfrac{C_{12}}{z_{12}^{h_1+h_2-1}} = 0 \Rightarrow C_{12}=0$ OR $h_1=h_2$.

$$\boxed{\therefore \langle 0| \phi_{h_1}(z_1) \phi_{h_2}(z_2) |0\rangle = \frac{C_{12}\, \delta_{h_1=h_2}}{(z_1-z_2)^{2h_1}}.}$$

n=3: Show (tricky) that

$$\langle 0| \phi_{h_1}(z_1) \phi_{h_2}(z_2) \phi_{h_3}(z_3) |0\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3} (z_1-z_3)^{h_1-h_2+h_3} (z_2-z_3)^{-h_1+h_2+h_3}}.$$

If we include the antiholomorphic contributions, then we would get, eg,

$$\langle 0| \phi_{h_1,\bar{h}_1}(z_1,\bar{z}_1) \phi_{h_2,\bar{h}_2}(z_2,\bar{z}_2) |0\rangle = \frac{C_{12}\, \delta_{h_1=h_2}\, \delta_{\bar{h}_1=\bar{h}_2}}{(z_1-z_2)^{2h_1} (\bar{z}_1-\bar{z}_2)^{2\bar{h}_1}}$$
$$= \frac{C_{12}\, e^{2i(h-\bar{h}) \arg z}}{|z_1-z_2|^{2(h_1+\bar{h}_1)}} \delta_{\bar{h}_1=\bar{h}_2} \delta_{h_1=h_2}.$$

This power law decay is the basis for CFT predictions of universal scaling laws and critical exponents!

<u>Review</u>: The free boson $\displaystyle S[\varphi] = \frac{1}{8\pi^2}\int_{\text{cylinder}} \partial_\mu\varphi\,\partial^\mu\varphi \; dx\,dt$

$$\varphi(t,x) = \varphi(t,x+L)$$

$\Rightarrow$ EOM : $\partial\varphi = \partial\varphi(z)$, $\bar\partial\varphi = \bar\partial\varphi(\bar z)$.

Spinless $\Rightarrow$ $\varphi'(z',\bar z') = \varphi(z,\bar z)$ $\Rightarrow$ classically conformal invariance.

Stress-energy tensor : $T(z) = \frac{1}{2}\partial\varphi(z)^2 \quad \bar T(\bar z) = \frac{1}{2}\bar\partial\varphi(\bar z)^2$.

Canonical quantisation : $\displaystyle \partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad [a_m, a_n] = m\,\delta_{m+n,0}$.

Fock spaces : Vacuum of momentum p is $|p\rangle$:

$$\underset{\substack{\text{(holomorphic)}\\ \text{momentum}\\ \text{operator}}}{a_0}|p\rangle = p|p\rangle, \qquad \underset{\text{annihilation operators}}{a_n}|p\rangle = 0 \quad \text{for } n>0.$$

$$\begin{array}{lll}
\vdots & \vdots & \vdots \\
a_{-1}^3|p\rangle & a_{-2}a_{-1}|p\rangle & a_{-3}|p\rangle \\
a_{-1}^2|p\rangle & a_{-2}|p\rangle & \\
a_{-1}|p\rangle & & \\
|p\rangle & &
\end{array}$$

Excited states are created by the creation operators a_n, $n<0$. eg. $a_{-4}a_{-1}^3|p\rangle$.

Fock space = vacuum + its excited states.

Normal ordering : $:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \le -1 \\ a_n a_m & \text{if } m \ge 0 \end{cases}$

$\Rightarrow$ quantisation of stress-energy tensor $\displaystyle T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

$$T(z) = \frac{1}{2}:\partial\varphi(z)\partial\varphi(z): \quad \Rightarrow \quad L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}: .$$

$\Rightarrow$ $\underset{\text{energy operator}}{L_0}|p\rangle = \frac{1}{2}p^2|p\rangle$.

$$[L_m, a_n] = -n\,a_{m+n},$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\,c, \quad c=1.$$

State-Field Correspondence : $\displaystyle |\psi\rangle = \lim_{z\to 0}\psi(z)|0\rangle$.

Radial Ordering : $\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z|>|w|, \\ B(w)A(z) & \text{if } |z|<|w|. \end{cases}$

Operator product expansion : $\displaystyle \mathcal{R}\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w):$

$$\sim \frac{1}{(z-w)^2}.$$

Wick's theorem $\Rightarrow$ OPEs $\quad \mathcal{R}\{T(z)\,\partial\varphi(w)\} \sim \dfrac{\partial\varphi(w)}{(z-w)^2} + \dfrac{\partial^2\varphi(w)}{z-w}$,

$$\mathcal{R}\{T(z)T(w)\} \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}.$$

Primary fields: Correspond to a vacuum!

- Free boson primaries $\equiv$ vertex operators $V_p(z) \leftrightarrow |p\rangle$ $(a_0|p\rangle = p|p\rangle,\ a_n|p\rangle \underset{n>0}{=} 0)$
- Conformal primaries $\phi_h(z) \leftrightarrow |\phi_h\rangle$ $(L_0|\phi_h\rangle = h|\phi_h\rangle,\ L_n|\phi_h\rangle = 0$ if $n>0)$.

Correlation Functions: $\langle A(z)\cdots B(w)\rangle \equiv \langle 0|\mathcal{R}\{A(z)\cdots B(w)\}|0\rangle$.

eg. $\langle \partial\varphi(z_1)\partial\varphi(z_2)\rangle = \dfrac{1}{(z_1-z_2)^2}$, $\quad \langle T(z_1)T(z_2)\rangle = \dfrac{c/2}{(z_1-z_2)^4}$.

Derived: $\langle V_{p_1}(z_1)\cdots V_{p_n}(z_n)\rangle = 0$ unless $p_1+\cdots+p_n = 0$. momentum conservation

$\langle \phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)\rangle = f(z_1,\dots,z_n)$ satisfies

- $\sum_{j=1}^n \partial_j f(z_1,\dots,z_n) = 0$ (translation invariance)
- $\sum_{j=1}^n (z_j\partial_j + h_j) f(z_1,\dots,z_n) = 0$
- $\sum_{j=1}^n (z_j^2\partial_j + 2h_j z_j) f(z_1,\dots,z_n) = 0$.

$$\Rightarrow \quad \langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\rangle = \frac{C_{12}\,\delta_{h_1=h_2}}{(z_1-z_2)^{2h_1}}, \qquad \langle \phi_{h_1}(z_1)\rangle = C_1\,\delta_{h_1=0},$$

$$\langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1-h_2+h_3}(z_2-z_3)^{-h_1+h_2+h_3}}.$$

The conformal dimensions h_i of the primary field describe the asymptotic behaviour of the correlators; they fix the <u>critical exponents</u>!

The Free Fermion

Just as the free boson describes a massless <u>spinless</u> bosonic string, the free fermion describes a massless <u>spin $\frac{1}{2}$</u> fermionic string. Thus, the fermion field has a ~~spin~~ $j = \frac{1}{2}$ and a $j = -\frac{1}{2}$ component:

$$\Psi(t,x) = \begin{pmatrix} \psi(t,x) \\ \bar{\psi}(t,x) \end{pmatrix}.$$

~~The action is~~ Being fermionic, we may impose periodic <u>or</u> antiperiodic boundary conditions on the cylinder, eg.

$$\psi(t, x+L) = \begin{cases} +\psi(t,x) & \text{periodic (Ramond sector)} \\ -\psi(t,x) & \text{antiperiodic (Neveu-Schwarz sector)} \end{cases}$$

With $\bar{\psi}$, there are therefore four sectors in total!

The action is
$$S[\Psi] = \frac{1}{4\pi} \int_{cyl.} \Psi^{\dagger}(t,x)\, \gamma^t \gamma^\mu \partial_\mu \Psi(t,x)\, dx\, dt$$
$$= \frac{1}{2\pi} \int_{cyl.} \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right) dx\, dt,$$

where the gamma matrices are $\gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^x = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The equations of motion are $\partial \bar{\psi} = \bar{\partial} \psi = 0$, ie $\psi = \psi(z)$ and $\bar{\psi} = \bar{\psi}(\bar{z})$.

Transforming to complex coordinates requires that we account for the fact that ψ and $\bar{\psi}$ are spinor fields, so

$$\psi(z) = \left(\frac{\partial z}{\partial x} \right)^{1/2} \psi(t,x) = \left(\frac{2\pi i z}{L} \right)^{1/2} \psi(t,x).$$

Since $x \to x + L$ amounts to $z \to e^{2\pi i} z$ on the plane, the nature of the boundary conditions swaps as we transform:

$$\psi(e^{2\pi i} z) = e^{i\pi} \left(\frac{2\pi i z}{L} \right)^{1/2} \psi(t, x+L) = \begin{cases} -\psi(z) & \text{antiperiodic (Ramond sector)} \\ +\psi(z) & \text{periodic (Neveu-Schwarz sector).} \end{cases}$$

As the string is spin $\frac{1}{2}$, the angular momentum operator $L_0 - \bar{L}_0$ should have eigenvalue $+\frac{1}{2}$ on $\psi(z)$ and $-\frac{1}{2}$ on $\bar{\psi}(\bar{z})$. Thus, the eigenvalues of $L_0 + \bar{L}_0$ must be $+\frac{1}{2}$ and $+\frac{1}{2}$, respectively, so we propose the decompositions

$$\psi(z) = \begin{cases} \sum_{n\in\mathbb{Z}} b_n z^{-n-1/2} & \text{(Ramond)} \\ \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{-n-1/2} & \text{(Neveu-Schwarz)} \end{cases} . \qquad b_n = \oint_0 \psi(z) z^{n-1/2} \frac{dz}{2\pi i}$$

Canonical quantisation gives anticommutation relations:

$$b_m b_n + b_n b_m = \{b_m, b_n\} = \delta_{m+n,0} .$$

<u>Note</u>: $b_n^2 = \frac{1}{2}\delta_{n,0}$.

Fock spaces: In each sector, let the b_m with $n>0$ be annihilators and the b_n with $n<0$ be creators.

In the Ramond sector, we have b_0. It is not a zero-mode as $b_0|\lambda\rangle = \lambda|\lambda\rangle$ ~~requires~~ would require that $|\lambda\rangle$ be both bosonic <u>and</u> fermionic! It is not an annihilator as $b_0|\lambda\rangle = 0$ contradicts $\frac{1}{2}|\lambda\rangle = b_0^2|\lambda\rangle = b_0(b_0|\lambda\rangle)$. $\therefore$ <u>b_0 is a creator</u>.

Without any zero-modes, we cannot distinguish vacua. $\therefore$ there is a single vacuum $|NS\rangle$ in the Neveu-Schwarz sector and a single vacuum $|R\rangle$ in the Ramond sector:

$$\begin{array}{llll} & \vdots & \vdots & \vdots \\ & b_{-3/2}b_{-1/2}|NS\rangle & b_{-2}|R\rangle & b_{-2}b_0|R\rangle \\ (b_{-1/2}^2 = 0) & b_{-3/2}|NS\rangle & & \\ & \cdot & b_{-1}|R\rangle & b_{-1}b_0|R\rangle \qquad (b_{-1}^2=0) \\ & b_{-1/2}|NS\rangle & & \\ & |NS\rangle & |R\rangle & b_0|R\rangle \quad \leftarrow \text{degenerate!} \end{array}$$

Which one is the true vacuum? It can't be $|R\rangle$ because

$$\lim_{z\to 0}\psi(z)|R\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}} b_n|R\rangle z^{-n-1/2} = \lim_{z\to 0}\left[b_0|R\rangle z^{-1/2} + b_{-1}|R\rangle z^{1/2} + \dots\right],$$

which doesn't exist! However,

$$\lim_{z\to 0}\psi(z)|NS\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}}' b_n|NS\rangle z^{-n-1/2} = \lim_{z\to 0}\left[b_{-1/2}|NS\rangle + b_{-3/2}|NS\rangle z + \dots\right] = b_{-1/2}|NS\rangle$$

does. We write $|0\rangle \equiv |NS\rangle$, and $|\psi\rangle = b_{-1/2}|0\rangle$.

<u>Radial-Ordering</u>: ~~Being fermionic~~ Swapping fermionic fields gives a sign. Let the parity of the field $A(z)$ be $\bar{A} = \begin{cases} 0 & \text{if } A(z) \text{ is bosonic}, \\ 1 & \text{if } A(z) \text{ is fermionic}. \end{cases}$

Define:
$$R\{A(z)B(\omega)\} = \begin{cases} A(z)\, B(\omega) & \text{if } |z|>|\omega|, \\ (-1)^{\bar{A}\bar{B}}\, B(\omega)\, A(z) & \text{if } |z|<|\omega|. \end{cases}$$

<u>OPE</u>: $R\{\psi(z)\psi(\omega)\} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \phi_n(\omega)(z-\omega)^{-n-1/2}$

$$\Rightarrow \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n|\psi\rangle z^{-n-1/2} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} |\phi_n\rangle z^{-n-1/2} \Rightarrow |\phi_n\rangle = b_n|\psi\rangle = b_n b_{-1/2}|0\rangle.$$

$n>\frac{1}{2} \Rightarrow |\phi_n\rangle = b_n b_{-1/2}|0\rangle = -b_{-1/2}b_n|0\rangle = 0 \Rightarrow \phi_n(\omega) = 0.$

$n=\frac{1}{2} \Rightarrow |\phi_{1/2}\rangle = b_{1/2}b_{-1/2}|0\rangle = (-b_{-1/2}b_{1/2}+1)|0\rangle = |0\rangle \Rightarrow \phi_{1/2}(\omega) = 1.$

$$\therefore \quad R\{\psi(z)\psi(\omega)\} \sim \frac{1}{z-\omega}.$$

Note that $R\{\psi(\omega)\psi(z)\} \sim \dfrac{1}{\omega-z} = -\dfrac{1}{z-\omega} = -R\{\psi(z)\psi(\omega)\}$.

<u>Stress-energy tensor</u>: For the free boson, $T(z)$ was a dimension 2 field: $|T\rangle = L_{-2}|0\rangle = \frac{1}{2}a_{-1}^2|0\rangle$, $\quad L_0|T\rangle = 2|T\rangle$.

We therefore <u>guess</u> that

$$|T\rangle = \alpha\, b_{-3/2}b_{-1/2}|0\rangle = -\alpha\, b_{-1/2}b_{-3/2}|0\rangle$$

$$\Rightarrow \quad T(z) = -\alpha :\psi(z)\,\partial\psi(z):,$$

where the normal-ordering is defined, <u>in the NS sector</u>, by

$$:b_m b_n: = \begin{cases} +b_m b_n & \text{if } m \le -\frac{1}{2}, \\ -b_n b_m & \text{if } m \ge \frac{1}{2}. \end{cases}$$

Since $\psi(z) = \sum_{n\in\mathbb{Z}+1/2} b_n z^{-n-1/2}$, $\quad \partial\psi(z) = -\sum_{n\in\mathbb{Z}+1/2} (n+\frac{1}{2})\, b_n z^{-n-3}$

$$\Rightarrow \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = T(z) = +\alpha \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} (n-r+\tfrac{1}{2}) :b_r b_{n-r}: z^{-n-2}$$

$$\Rightarrow L_n = \alpha \sum_{r\in\mathbb{Z}} (n-r+\tfrac{1}{2}) :b_r b_{n-r}: = \alpha \sum_{r\le -\frac{1}{2}} (n-r+\tfrac{1}{2})\, b_r b_{n-r} - \alpha \sum_{r\ge\frac{1}{2}} (n-r+\tfrac{1}{2})\, b_{n-r} b_r.$$

Wick's theorem works for OPEs of $\psi(z)$ and its derivatives, but we need to include a sign every time we exchange two fermions to contract:

$$\text{eg.} \quad \psi(z)\,\partial\psi(x)\,\psi(w) = \frac{\psi(w)}{(z-x)^2}, \qquad \psi(z)\,\partial\psi(x)\,\psi(w) = -\partial\psi(x)\,\psi(z)\psi(w) = \frac{-\partial\psi(x)}{z-w}.$$

$$\text{since } R\{\psi(z)\partial\psi(x)\} = \partial_x R\{\psi(z)\psi(x)\} \sim \partial_x \frac{1}{z-x} = \frac{1}{(z-x)^2}.$$

$$\therefore \quad R\{T(z)\psi(w)\} = -\alpha\, R\{:\psi(z)\partial\psi(z):\psi(w)\}$$

$$\sim -\alpha :\psi(z)\partial\psi(z)\psi(w): \; - \alpha :\psi(z)\partial\psi(z)\psi(w):$$

$$= \frac{\alpha\,\partial\psi(z)}{z-w} + \frac{\alpha\,\psi(z)}{(z-w)^2}$$

$$\sim \frac{\alpha\,\psi(w)}{(z-w)^2} + \frac{2\alpha\,\partial\psi(w)}{z-w},$$

$$\text{since } R\{\partial\psi(z)\psi(w)\} = \partial_z R\{\psi(z)\psi(w)\} \sim \partial_z \frac{1}{z-w} = \frac{-1}{(z-w)^2}.$$

We therefore take $\alpha = \frac{1}{2}$, so $T(z) = -\frac{1}{2}:\psi(z)\,\partial\psi(z):$.

<u>Ex</u>: $R\{T(z)T(w)\} = \frac{1}{4} R\{:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w):\}$

$$\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \qquad \therefore \quad \underline{c = \tfrac{1}{2}}.$$

This (finally) demonstrates that the free fermion is a CFT.

We can now compute the energy of the vacua $|0\rangle \equiv |NS\rangle$ and $|R\rangle$.

For $|NS\rangle$, this is easy and the answer is as expected:

$$L_0|NS\rangle = -\frac{1}{2}\sum_{r\leq -\frac{1}{2}} (r-\tfrac{1}{2})\, b_r \cancel{b_{-r}|NS\rangle} + \frac{1}{2}\sum_{r\geq\frac{1}{2}} (r-\tfrac{1}{2})\, b_{-r} \cancel{b_r|NS\rangle} = 0.$$

But, we cannot use this expression for L_0 in the Ramond sector (the normal-ordering is only valid in the NS sector!).

To compute the energy of $|R\rangle$, we derive a <u>generalised commutation relation</u>:

$$R\{\psi(z)\psi(w)\} = \frac{1}{z-w} + :\psi(z)\psi(w):$$

$$= \frac{1}{z-w} + \underbrace{:\psi(w)\psi(w):}_{\substack{= -:\psi(w)\psi(w): \\ \text{so } :: = 0.}} + \underbrace{:\partial\psi(w)\,\psi(w):}_{\substack{-:\psi(w)\,\partial\psi(w): \\ = 2T(w)}}(z-w) + \cdots$$

$$= \frac{1}{z-w} + 2T(w)(z-w) + \cdots .$$

We therefore compute $\oint_0 \oint_w R\{\psi(z)\psi(w)\}\, z^{m+3/2} w^{n-1/2} (z-w)^{-2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}$ in two ways:

$$① = \oint_0 \oint_w \left[\frac{z^{m+3/2} w^{n-1/2}}{(z-w)^3} + \frac{2T(w)\, z^{m+3/2} w^{n-1/2}}{z-w} \right] \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0 \left[\tfrac{1}{2}(m+\tfrac{1}{2})(m+\tfrac{3}{2})\, w^{m+n-1} + 2T(w)\, w^{m+n+1} \right] \frac{dw}{2\pi i}$$

$$= \tfrac{1}{2}(m+\tfrac{1}{2})(m+\tfrac{3}{2})\, \delta_{m+n,0} + 2L_{m+n}.$$

$$\frac{1}{(1-x)^2} = \partial_x \frac{1}{1-x} = \sum_{r=1}^{\infty} r x^{r-1} = \sum_{r=0}^{\infty} (r+1) x^r .$$

$$② = \oint_0 \oint_0_{|z|>|w|} \psi(z)\psi(w)\, z^{m\cancel{+3/2}-1/2} w^{n-1/2} (1 - w/z)^{-2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$- \oint_0 \oint_0_{|z|<|w|} -\psi(w)\psi(z)\, w^{n-5/2} z^{m+3/2} (1 - z/w)^{-2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \sum_{r=0}^{\infty} (r+1) \Big[\oint_0 \psi(z)\, z^{m-r-1/2} \frac{dz}{2\pi i} \oint_0 \psi(w)\, w^{n+r-1/2} \frac{dw}{2\pi i}$$
$$+ \oint_0 \psi(w)\, w^{n-r-5/2} \frac{dw}{2\pi i} \oint_0 \psi(z)\, z^{m+r+3/2} \frac{dz}{2\pi i} \Big]$$

$$= \sum_{r=0}^{\infty} (r+1) \left[\psi_{m-r}\psi_{n+r} + \psi_{n-r-2}\psi_{m+r+2} \right].$$

Thus, $$\sum_{r=0}^{\infty} (r+1)\left[\psi_{m-r}\psi_{n+r} + \psi_{n-r-2}\psi_{m+r+2}\right] = \frac{(2m+1)(2m+3)}{8}\, \delta_{m+n,0} + 2L_{m+n}.$$

We apply this, with $m = -\frac{1}{2}$, $n = +\frac{1}{2}$, to $|NS\rangle$:

$$\sum_{r=0}^{\infty} (r+1)\left[\psi_{-r-1/2}\cancel{\psi_{r+1/2}} + \psi_{-r-3/2}\cancel{\psi_{r+3/2}}\right] |NS\rangle = 2L_0 |NS\rangle \quad \Rightarrow \quad L_0|NS\rangle = 0. \quad ✓$$

With $m=-1$ and $n=1$, we can apply it to $|R\rangle$:

$$\sum_{r=0}^{\infty} (r+1)\left[\psi_{-r-1}\cancel{\psi_{r+1}} + \psi_{-r-1}\cancel{\psi_{r+1}}\right] |R\rangle = \frac{(-1)(1)}{8} + 2L_0|R\rangle \quad \Rightarrow \quad L_0|R\rangle = \frac{1}{16}|R\rangle .$$

The Ramond vacuum has energy $1/16$. ☺

<u>Last time:</u> Free fermion = massless spin $\frac{1}{2}$ fermionic string: $\psi(z)$, $\bar{\psi}(\bar{z})$.

$$\psi(e^{2\pi i}z) = \begin{cases} +\psi(z) & \text{periodic (Neveu-Schwarz)} \\ -\psi(z) & \text{antiperiodic (Ramond)} \end{cases}$$

$$\psi(z) = \sum_n b_n z^{-n-1/2}, \qquad n \in \begin{cases} \mathbb{Z}+\frac{1}{2} & (NS) \\ \mathbb{Z} & (R) \end{cases} ; \qquad \{b_m, b_n\} = \delta_{m+n,0}.$$

<u>Fock spaces:</u> There are only two!

$$\begin{array}{lll} & \vdots & \\ E = 3/2 & b_{-3/2}|NS\rangle & \\ E = 1/2 & b_{-1/2}|NS\rangle \equiv |\psi\rangle & \\ E = 0 & |NS\rangle \equiv |0\rangle & \end{array} \qquad \begin{array}{lll} \vdots & \vdots & \vdots \\ b_{-2}|R\rangle & b_{-2}b_0|R\rangle & E = \frac{33}{16} \\ b_{-1}|R\rangle & b_{-1}b_0|R\rangle & E = \frac{17}{16} \\ |R\rangle & b_0|R\rangle & E = \frac{1}{16} \end{array}$$

<u>OPE:</u> $\mathcal{R}\{\psi(z)\psi(w)\} \sim \frac{1}{z-w}$. $\qquad T(z) = -\frac{1}{2} :\psi(z)\partial\psi(z):$

$$\mathcal{R}\{T(z)\psi(w)\} \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}$$ $\Rightarrow$ ψ is a conformal primary of conformal dimension $\frac{1}{2}$.

$$\mathcal{R}\{T(z)T(w)\} \sim \frac{\frac{1}{4}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$ $\Rightarrow$ central charge is $c = \frac{1}{2}$.

Why is there no momentum operator for the free fermion?
Because it lives in a <u>fermionic</u> spacetime (ψ and $\bar{\psi}$ are anticommuting), not a normal bosonic spacetime.

Today, we will discuss CFTs that <u>only</u> have the conformal symmetry

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\, c.$$

Here, the Fock spaces are built by exciting a conformal vacuum $|\phi_h\rangle$ satisfying $L_0|\phi_h\rangle = h|\phi_h\rangle$ and $L_n|\phi_h\rangle = 0$ for $n>0$. Something interesting happens when $h=0$ because the <u>true</u> vacuum $|\phi_0\rangle = |0\rangle$ must be annihilated by L_{-1}:

$$\begin{array}{llll} \vdots & \vdots & \vdots & \vdots \\ L_{-1}^3|\phi_h\rangle & L_{-2}L_{-1}|\phi_h\rangle & L_{-3}|\phi_h\rangle & E = h+3 \\ L_{-1}^2|\phi_h\rangle & L_{-2}|\phi_h\rangle & & E = h+2 \\ L_{-1}|\phi_h\rangle & & & E = h+1 \\ |\phi_h\rangle & & & E = h \end{array} \qquad \begin{array}{lll} \vdots & \vdots & \vdots \\ L_{-2}^2|0\rangle & L_{-4}|0\rangle & E = 4 \\ & L_{-3}|0\rangle & E = 3 \\ & L_{-2}|0\rangle \equiv |T\rangle & E = 2 \\ & \cdot & \\ |0\rangle & & E = 0 \end{array}$$

- $L_{-1}|0\rangle = 0$ for the state-field correspondence to work: $\lim_{z\to 0} T(z)|0\rangle$ must exist!
- $L_{-1}|0\rangle$ corresponds to the derivative $(L_{-1} \leftrightarrow \partial)$ of the identity field $(|0\rangle \leftrightarrow 1)$.
- The true vacuum is maximally symmetric: $L_n|0\rangle = 0$ for all $n \geq -1$.

Here is a fourth way of thinking about this:

- The state $L_{-1}|0\rangle$ is <u>unphysical</u> meaning that it vanishes in any amplitude: $\langle\psi|L_{-1}|0\rangle = 0$, hence any physical measurement involving this state gives zero!

Physically, $L_{-1}|0\rangle$ is indistinguishable from the zero state!

Why is $\langle\psi|L_{-1}|0\rangle = 0$ for all $|\psi\rangle$?

1) L_0 is hermitian $(L_0^\dagger = L_0)$, so its eigenvectors are orthogonal.
   $\therefore$ $\langle\psi|L_{-1}|0\rangle = 0$ unless $|\psi\rangle$ has energy $E=1$.
2) Different Fock spaces are orthogonal (by definition).
   $\therefore$ $\langle\psi|L_{-1}|0\rangle = 0$ unless $|\psi\rangle$ is created from $|0\rangle$ with $E=1$.

There is therefore only one possibility for $|\psi\rangle$: $L_{-1}|0\rangle$ itself. But, then:

$$\begin{aligned}\langle\psi|L_{-1}|0\rangle &= \langle 0|L_1 L_{-1}|0\rangle = \langle 0|\{L_{-1}L_1 + [L_1, L_{-1}]\}|0\rangle \\ &= \langle 0|2L_0|0\rangle = 0.\end{aligned} \qquad \left[L_n^\dagger = L_{-n}\right]$$

There are <u>no</u> unphysical states in the Fock spaces of the free boson or the free fermion. [$\longrightarrow$ example on next page]

<u>FACT</u>: The Fock space built from the true vacuum by acting with conformal creators L_n $(n<0)$ has an unphysical state unrelated to $L_{-1}|0\rangle$ if and only if the central charge has the form

$$c = 1 - \frac{6(p'-p)^2}{pp'}$$

where $p, p' \in \{2,3,4,\dots\}$ and $\gcd\{p,p'\} = 1$.

<u>Clarifying Example:</u> In quantum mechanics, a spin $\frac{1}{2}$ particle is described by $su(2)$ where we have two states $|\frac{1}{2};\frac{1}{2}\rangle$ and $|\frac{1}{2};-\frac{1}{2}\rangle$ upon which the spin operator $S = J^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the ladder operators

$$J^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad J^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

act. Here, J^3 is a zero mode (it measures the spin) and, if $|\frac{1}{2};\frac{1}{2}\rangle$ is taken as a vacuum, then J^+ is an annihilation operator while J^- is a creation op.

But, the Fock space created from $|\frac{1}{2};\frac{1}{2}\rangle$ <u>should</u> look like this:

$$\begin{array}{l} \quad\vdots \\ |\tfrac{1}{2};-\tfrac{3}{2}\rangle \\ J^- \uparrow \\ |\tfrac{1}{2};-\tfrac{1}{2}\rangle \\ J^- \uparrow \quad \downarrow J^+ \\ |\tfrac{1}{2};\tfrac{1}{2}\rangle \end{array}$$

However, there are no states with spin $< -\frac{1}{2}$ because $|\frac{1}{2};-\frac{3}{2}\rangle$ is <u>unphysical</u>:

1) J^3 is hermitian, so it is orthogonal to all states with spin $\sigma \neq -\frac{3}{2}$. ie to all states but itself.

2) $\langle\frac{1}{2};-\frac{3}{2}|\frac{1}{2};-\frac{3}{2}\rangle = \langle\frac{1}{2};-\frac{1}{2}|J^+J^-|\frac{1}{2};-\frac{1}{2}\rangle \qquad (J^{-\dagger} = J^+)$

$$\begin{aligned}
&= \langle\tfrac{1}{2};-\tfrac{1}{2}|J^-J^+ + [J^+,J^-]|\tfrac{1}{2};-\tfrac{1}{2}\rangle \\
&= \langle\tfrac{1}{2};\tfrac{1}{2}|\tfrac{1}{2};\tfrac{1}{2}\rangle + \langle\tfrac{1}{2};-\tfrac{1}{2}|2J^3|\tfrac{1}{2};-\tfrac{1}{2}\rangle \qquad ([J^+,J^-] = 2J^3) \\
&= 1 - \langle\tfrac{1}{2};-\tfrac{1}{2}|\tfrac{1}{2};-\tfrac{1}{2}\rangle \\
&= 0.
\end{aligned}$$

The CFTs with $c = 1 - \frac{6(p'-p)^2}{pp'}$ and only Virasoro symmetry are called the <u>minimal models</u> $M(p,p')$. Because the conformal vacua $|\phi_h\rangle$ are parametrised by an energy $E \in \mathbb{R}$, it would seem that all should be allowed. But, this is false because of unphysical states!

Ex: $(p,p')=(2,3) \Rightarrow c = 1 - \frac{6.1^2}{6} = 0$.

In this case, the extra unphysical state beyond $L_{-1}|0\rangle$ is

$$|T\rangle = L_{-2}|0\rangle.$$

Check: $|T\rangle$ is the only energy 2 state, so it's enough to show that $\langle T|T\rangle = 0$. So we do:

$$\langle T|T\rangle = \langle 0|L_2 L_{-2}|0\rangle = \langle 0| L_{-2}\cancel{L_2} + [L_2, L_{-2}]|0\rangle$$

$$= \langle 0|4\cancel{L_0} + \tfrac{1}{2}c|0\rangle = \tfrac{1}{2}c = 0.$$

But, if $|T\rangle = 0$ then $T(z) = 0$ (state-field correspondence) and so $\sum_{n\in\mathbb{Z}} L_n z^{-n-2} = 0 \Rightarrow L_n = 0$ for all n. As $L_0|\phi_h\rangle = h|\phi_h\rangle$ by definition, we can only have $L_0 = 0$ if $h = 0$.

$\therefore$ $|0\rangle$ is the only conformal vacuum in the minimal model $M(2,3)$.

[In fact, $|0\rangle$ is the only state in $M(2,3)$ – the theory is trivial!]

Ex: $(p,p') = (2,5) \Rightarrow c = 1 - \frac{6\cdot 3^2}{10} = -\frac{22}{5}$.

For $M(2,5)$, the extra unphysical state is

$$|\chi\rangle = (L_{-2}^2 - \tfrac{3}{5}L_{-4})|0\rangle.$$

To check this, we check orthogonality to both states of ~~dim~~ energy 4, $L_{-2}^2|0\rangle$ and $L_{-4}|0\rangle$:

$$\begin{aligned}
\langle 0|L_4|\chi\rangle &= \langle 0|L_4 L_{-2}L_{-2} - \tfrac{3}{5}L_4 L_{-4}|0\rangle \\
&= \langle 0|6L_2L_{-2} - \tfrac{3}{5}(8\cancel{L_0} + 5c)|0\rangle \\
&= \langle 0|6(4\cancel{L_0} + \tfrac{1}{2}c) - 3c|0\rangle \\
&= \langle 0|3c - 3c|0\rangle \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle 0|L_2^2|\chi\rangle &= \langle 0|L_2L_2L_{-2}L_{-2} - \tfrac{3}{5}L_2L_2L_{-4}|0\rangle \\
&= \langle 0|L_2L_{-2}L_2L_{-2} + L_2(4L_0 + \tfrac{1}{2}c)L_{-2} - \tfrac{3}{5}L_2\cdot 6L_{-2}|0\rangle \\
&= \langle 0|L_2L_{-2}(4\cancel{L_0} + \tfrac{1}{2}c) + (8 + \tfrac{1}{2}c)(4\cancel{L_0} + \tfrac{1}{2}c) - \tfrac{18}{5}(4\cancel{L_0} + \tfrac{c}{2})|0\rangle \\
&= \langle 0|(4\cancel{L_0} + \tfrac{1}{2}c)\tfrac{1}{2}c + (8 + \tfrac{1}{2}c)\tfrac{1}{2}c - \tfrac{9}{5}c|0\rangle \\
&= \tfrac{1}{2}\left[c^2 + \tfrac{22}{5}c\right] = 0.
\end{aligned}$$

By the state-field correspondance, it follows that $|\chi\rangle = 0$

$$\Rightarrow \quad \chi(z) = \,:T(z)T(z):\, - \frac{3}{10}\partial^2 T(z) = 0 \quad \Rightarrow \quad \sum_{n\in\mathbb{Z}} \chi_n z^{-n-4} = 0,$$

where
$$\chi_n = \sum_{r\in\mathbb{Z}} :L_r L_{n-r}: - \frac{3}{10}(n+2)(n+3)L_n$$
$$= \sum_{r\leq -2} L_r L_{n-r} + \sum_{r\geq -1} L_{n-r}L_r - \frac{3}{10}(n+2)(n+3)L_n \equiv 0.$$

Acting with χ_0 on the conformal vacuum $|\phi_h\rangle$ then gives zero:

$$0 = \chi_0|\phi_h\rangle = \sum_{r\leq -2} L_r \cancel{L_{-r}|\phi_h\rangle} + \sum_{r\geq -1} L_{-r}L_r|\phi_h\rangle - \frac{3}{10}\cdot 2\cdot 3\, L_0|\phi_h\rangle$$
$$= L_1 L_{-1}|\phi_h\rangle + L_0^2|\phi_h\rangle - \frac{9}{5}L_0|\phi_h\rangle = 2L_0|\phi_h\rangle + h^2|\phi_h\rangle - \frac{9}{5}h|\phi_h\rangle$$
$$= (h^2 + \tfrac{1}{5})|\phi_h\rangle = h(h+\tfrac{1}{5})|\phi_h\rangle.$$

$\therefore$ the only conformal vacua in the minimal model $M(2,5)$ are the true vacuum $|0\rangle$ and that with energy $-\frac{1}{5}$: $|\phi_{-1/5}\rangle$.

$M(2,5)$ is identified with the thermodynamic limit of the statistical model known as the Yang-Lee singularity, ~~at the critical~~ at its critical point.
$$\langle \phi_{-1/5}(z,\bar{z})\,\phi_{-1/5}(w,\bar{w})\rangle = |z-w|^{4/5}.$$

Ex: $(p,p') = (3,4) \Rightarrow c = 1 - \dfrac{6.1^2}{12} = \dfrac{1}{2}$.

For $M(3,4)$, the extra unphysical state is
$$|\chi\rangle = \left(L_{-2}^3 + \frac{93}{64}L_{-3}^2 - \frac{33}{8}L_{-4}L_{-2} - \frac{3}{16}L_{-6}\right)|0\rangle.$$

A <u>very</u> tedious computation shows that
$$|\chi\rangle = 0 \quad \Rightarrow \quad |\phi_h\rangle \text{ must have } h = 0, \tfrac{1}{16} \text{ or } \tfrac{1}{2}.$$

$M(3,4)$ is identified with the thermodynamic limit of the Ising model at its critical temperature. $\phi_{1/16} \equiv \sigma$, $\phi_{1/2} \equiv \varepsilon$.

$$\underbrace{\langle \sigma(z,\bar{z})\,\sigma(w,\bar{w})\rangle}_{\text{spin field}} = \frac{1}{|z-w|^{1/4}}, \qquad \underbrace{\langle \varepsilon(z,\bar{z})\,\varepsilon(w,\bar{w})\rangle}_{\text{energy density}} = \frac{1}{|z-w|^2}.$$

Ex: $M(4,5) \sim$ tricritical Ising
$M(5,6) \sim$ 3-state Potts

All minimal models have lattice versions as RSOS models.

The Ising model $M(3,4)$ has $c = \frac{1}{2}$ and conformal vacua of energies 0, $\frac{1}{16}$ (σ) and $\frac{1}{2}$ (ε).

The free fermion has $c = \frac{1}{2}$ and free fermion vacua of energies 0 (NS) and $\frac{1}{2}$ (R).

$\vdots$	$\vdots$	$\vdots$	
$L_{-3}\|0\rangle$	$L_{-3}\|\sigma\rangle$ $L_{-2}L_{-1}\|\sigma\rangle$	$L_{-3}\|\varepsilon\rangle$	
$\|T\rangle = L_{-2}\|0\rangle$	$L_{-2}\|\sigma\rangle$	$L_{-2}\|\varepsilon\rangle$	Ising model
	$L_{-1}\|\sigma\rangle$	$L_{-1}\|\varepsilon\rangle$	
$E=0$ $\|0\rangle$	$E=\frac{1}{16}$ $\|\sigma\rangle$	$E=\frac{1}{2}$ $\|\varepsilon\rangle$	

$\vdots$	$\vdots$	$\vdots$	$\vdots$
$b_{-5/2}b_{-1/2}\|NS\rangle$	$b_{-5/2}\|NS\rangle$	$b_{-3}b_0\|R\rangle$ $b_{-2}b_{-1}\|R\rangle$	$b_{-3}\|R\rangle$ $b_{-2}b_{-1}b_0\|R\rangle$
$b_{-3/2}b_{-1/2}\|NS\rangle$	$b_{-3/2}\|NS\rangle$	$b_{-2}b_0\|R\rangle$	$b_{-2}\|R\rangle$
	$b_{-1/2}\|NS\rangle$ $E=\frac{1}{2}$	$b_{-1}b_0\|R\rangle$	$b_{-1}\|R\rangle$
$E=0$ $\|NS\rangle$		$E=\frac{1}{16}$ $\|R\rangle$	$b_0\|R\rangle$
bos.	ferm.	bos.	ferm.

We can even match up states (up to constants):

$M(3,4)$	$\|0\rangle$	$L_{-2}\|0\rangle$	$\cdots$	$\|\varepsilon\rangle$	$L_{-1}\|\varepsilon\rangle$	$\cdots$	$\|\sigma\rangle$	$L_{-1}\|\sigma\rangle$	$\cdots$	?
FF	$\|NS\rangle$	$b_{-3/2}b_{-1/2}\|NS\rangle$	$\cdots$	$b_{-1/2}\|NS\rangle$	$b_{-3/2}\|NS\rangle$	$\cdots$	$\|R\rangle$	$b_{-1}b_0\|R\rangle$	$\cdots$	$b_0\|R\rangle$

Everything matches except the fermionic states in the Ramond Fock space.

This is accounted for by the disorder operator $|\mu\rangle \longleftrightarrow b_0|R\rangle$. We do not notice ~~see~~ it in $M(3,4)$ as it is indistinguishable from $|\sigma\rangle$ as $M(3,4)$ has no parity.

Conclusion: The minimal model $M(3,4)$ and the free fermion are (nearly) equivalent as CFTs. In fact, this equivalence persists off-criticality which is why Onsager was able to exactly solve the Ising model on the lattice in 1944.