Introduction to Conformal Field Theory

1. Conformal transformation in two dimensions.

Consider $\mathbb{R}^2$ equipped with metric $g_{\mu\nu}$, so that the length squared of $x = x^\mu$ is

$$x \cdot x = x^\mu g_{\mu\nu} x^\nu$$

Define angle between x and y by $\cos\theta = \dfrac{x^\mu g_{\mu\nu} y^\nu}{(x\cdot x)^{1/2}(y\cdot y)^{1/2}}$ when $x\cdot x,\ y\cdot y \neq 0$

Recall $x_\mu = g_{\mu\nu}x^\nu$ $\quad \partial_\mu = \dfrac{\partial}{\partial x^\mu}$ $\quad \partial^\mu = g^{\mu\nu}\partial_\nu$ $\quad g^{\mu\nu}g_{\nu\rho} = \delta^\nu_\rho$.

Def. A conformal transformation : $x^\mu \longrightarrow x'^\mu$ is one satisfying

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \qquad \text{where } \Lambda(x) > 0 \text{ is a scalar.}$$

Then the angle θ between u and v is preserved.

$$\cos\theta' = \frac{u^\mu g'_{\mu\nu}(x') v^\nu}{(u^\mu g'_{\mu\nu}(x') u^\nu)^{1/2}(v^\mu g'_{\mu\nu}(x') v^\nu)^{1/2}} = \frac{u^\mu \Lambda(x) g_{\mu\nu}(x) v^\nu}{(u^\mu \Lambda(x) g_{\mu\nu}(x) u^\nu)^{1/2}(v^\mu \Lambda(x) g_{\mu\nu}(x) v^\nu)^{1/2}} = \frac{u^\mu g_{\mu\nu}(x) v^\nu}{(u^\mu g_{\mu\nu}(x) u^\nu)^{1/2}(v^\mu g_{\mu\nu}(x) v^\nu)^{1/2}}$$

$$= \cos\theta$$

Consider now a infinitesimal transformation $x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x)$. Expand $\Lambda(x) = 1 - \Omega(x)$, where $\Omega(x)$ is infinitesimal. Since $ds^2 = g'_{\mu\nu}dx'^\mu dx'^\nu = g_{\mu\nu}dx^\mu dx^\nu$. we have

$$\begin{aligned}
g'_{\mu\nu}dx'^\mu dx'^\nu &= g'_{\mu\nu}(x')(dx^\mu + \partial_\rho \varepsilon^\mu dx^\rho)(dx^\nu + \partial_\sigma \varepsilon^\nu dx^\sigma) \\
&= g'_{\mu\nu}(x')(dx^\mu dx^\nu + \partial_\sigma \varepsilon^\nu dx^\mu dx^\sigma + \partial_\rho \varepsilon^\mu dx^\rho dx^\nu) \\
&= g'_{\mu\nu}(x')dx^\mu dx^\nu + [1 - \Omega(x)]g_{\mu\nu}(x)(\partial_\sigma \varepsilon^\nu dx^\mu dx^\sigma + \partial_\rho \varepsilon^\mu dx^\rho dx^\nu) \\
&= g'_{\mu\nu}(x')dx^\mu dx^\nu + \partial_\sigma \varepsilon^\nu g_{\mu\nu} dx^\mu dx^\sigma + \partial_\rho \varepsilon^\mu g_{\mu\nu} dx^\rho dx^\nu \\
&= g'_{\mu\nu}(x')dx^\mu dx^\nu + \partial_\sigma \varepsilon_\mu dx^\mu dx^\sigma + \partial_\rho \varepsilon_\nu dx^\rho dx^\nu \\
&= g'_{\mu\nu}(x')dx^\mu dx^\nu + \partial_\nu \varepsilon_\mu dx^\mu dx^\nu + \partial_\mu \varepsilon_\nu dx^\mu dx^\nu \\
&= (g'_{\mu\nu}(x') + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)dx^\mu dx^\nu
\end{aligned}$$

$$\Rightarrow \quad g_{\mu\nu}(x) = g'_{\mu\nu}(x') + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu.$$

This is for all infinitesimal transformations. For infinitesimal conformal transformations.

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) = g_{\mu\nu} - \Omega g_{\mu\nu}.$$

$$\Rightarrow \quad \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \Omega g_{\mu\nu}.$$

$\times$ by $g^{\nu\mu}$ gives. $\quad \partial_\mu \varepsilon^\mu + \partial_\nu \varepsilon^\nu = \Omega g^\mu_\mu = 2\Omega \qquad (1)$

$$\Rightarrow \quad \Omega = \partial_\mu \varepsilon^\mu$$

Take $\mathbb{R}^2$ with euclidean metric $g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

From equation ①, we have:

$\mu=\nu=1$. $2\partial_1\varepsilon_1 = \partial_1\varepsilon_1 + \partial_2\varepsilon_2$ $\qquad$ $\mu=1, \nu=2$ $\quad \partial_1\varepsilon_2 + \partial_2\varepsilon_1 = 0$

$\mu=\nu=2$ $\quad 2\partial_2\varepsilon_2 = \partial_1\varepsilon_1 + \partial_2\varepsilon_2$ $\qquad$ $\mu=2, \nu=1$ $\quad \partial_2\varepsilon_1 + \partial_1\varepsilon_2 = 0$

$$\Longrightarrow \quad \partial_1\varepsilon_1 = \partial_2\varepsilon_2 \quad \text{AND} \quad \partial_1\varepsilon_2 = -\partial_2\varepsilon_1 \quad ②$$ 复平面上解析函数的条件.

[Cauchy – Riemann equations from complex analysis!]

Change to complex coordinates. ($\mathbb{R}^2 \cong \mathbb{C}$)

$z = x^1 + i x^2 \qquad \varepsilon = \varepsilon^1 + i\varepsilon^2 \qquad \partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$

$\bar{z} = x^1 - i x^2 \qquad \bar{\varepsilon} = \varepsilon^1 - i\varepsilon^2 \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$

So, from equations ②, we have

$$\bar{\partial}\varepsilon = 0 \quad \text{AND} \quad \partial\bar{\varepsilon} = 0$$

$$\therefore \quad \varepsilon = \varepsilon(z) \quad \text{AND.} \quad \bar{\varepsilon} = \bar{\varepsilon}(\bar{z})$$

$$\bar{\partial}\varepsilon = \frac{1}{2}(\partial_1 + i\partial_2)(\varepsilon^1 + i\varepsilon^2)$$
$$= \frac{1}{2}(\partial_1\varepsilon^1 + i\partial_1\varepsilon^2 + i\partial_2\varepsilon^1 - \partial_2\varepsilon^2)$$
$$= \frac{1}{2}[\partial_1\varepsilon^1 - \partial_2\varepsilon^2 + i(\partial_1\varepsilon^2 + \partial_2\varepsilon^1)] = 0$$

So an infinitesimal conformal transformation of euclidean $\mathbb{R}^2 \cong \mathbb{C}$ has form.

$$z' = z + \varepsilon(z) \quad , \quad \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z})$$

Generators of infinitesimal conformal transformation

Let ϕ be a function, under infinitesimal conformal transformation

$$\phi(x') = \phi(x+\varepsilon) = \phi(x) + \varepsilon^\mu \underbrace{\partial_\mu}_{\text{generator!}} \phi(x)$$

A base for ε and $\bar{\varepsilon}$ is $\{-z^{n+1}, -\bar{z}^{n+1}, n \in \mathbb{Z}\}$.

A base for generator is $\{l_n = -z^{n+1}\partial, \ \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}, \ n \in \mathbb{Z}\}$.

These generators form a Lie algebra. eg.

$$[l_m, l_n] = -z^{m+1}\partial(-z^{n+1}\partial) + z^{n+1}\partial(-z^{m+1}\partial)$$
$$= (n+1)z^{m+n+1}\partial + z^{m+n+2}\partial^2 - (m+1)z^{m+n+1}\partial - z^{m+n+2}\partial^2$$
$$= (n-m)z^{m+n+1}\partial = (m-n)l_{m+n}$$

Also. $[l_m, \bar{l}_n] = 0$. $\quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}$

This Lie algebra is two comuting copies of the Witt Algebra. It is the Lie algebra of classical infinitesimal conformal transformations.

The ~~ℓ~~ L_n, $\bar{L}_n$ with $n=-1,0,1$ are global conformal transformations, and form a subalgebra. others are local conformal transformations.

Def: $P_1 = -(L_{-1}+\bar{L}_{-1})$ $\quad$ $D = -(L_0+\bar{L}_0)$ (dilation) $\quad$ $K_1 = -(L_1+\bar{L}_1)$

$P_2 = -i(L_{-1}-\bar{L}_{-1})$ $\quad$ $M = -i(L_0-\bar{L}_0)$ $\quad$ $K_2 = -i(L_1-\bar{L}_1)$

translations $\quad$ rotation $\quad$ special conformal transformations

eg. $P_1 = \partial+\bar{\partial} = \partial_1$ $\quad$ $P_2 = i(\partial-\bar{\partial}) = \partial_2$

$D = z\partial + \bar{z}\bar{\partial}$ $\quad$ $M = x_1\partial_2 - x_2\partial_1$

$= (x_1+ix_2)\frac{1}{2}(\partial_1 - i\partial_2) + (x_1-ix_2)\frac{1}{2}(\partial_1+i\partial_2)$

$= x_1\partial_1 + x_2\partial_2 = x^\mu\partial_\mu$

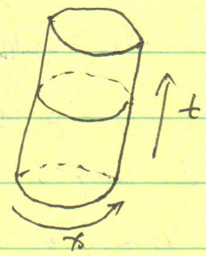
The infintesimal ~~for~~ transformation for P_1 is $x_1' = x_1+\epsilon_1$, $x_2' = x_2$.

$$\phi(x_1', x_2') = \phi(x_1+\epsilon_1, x_2) = \phi(x_1,x_2) + \epsilon_1\partial_1\phi(x_1,x_2)$$
$$= (1+\epsilon_1\partial_1)\phi(x_1,x_2)$$

2. The Free Boson

Let φ be a scalar bosonic field on a cylinder. so

$$\varphi(t,x) = \varphi(t,x+L)$$



The cylinder has Lorentzian metric $g_{\mu\nu} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

Action: $S[\varphi] = \frac{1}{2g}\int_{S^1\times\mathbb{R}} \partial_\mu\varphi\,\partial^\mu\varphi\,dx\,dt = \frac{1}{2g}\int_{S^1\times\mathbb{R}}[-(\partial_t\varphi)^2+(\partial_x\varphi)^2]\,dx\,dt$

coupling constant.

No mass term, no spin, no interaction.

Equation of Motion:

Under infinitesimal transformation $\varphi' = \varphi+\eta$ (η infinitesimal)

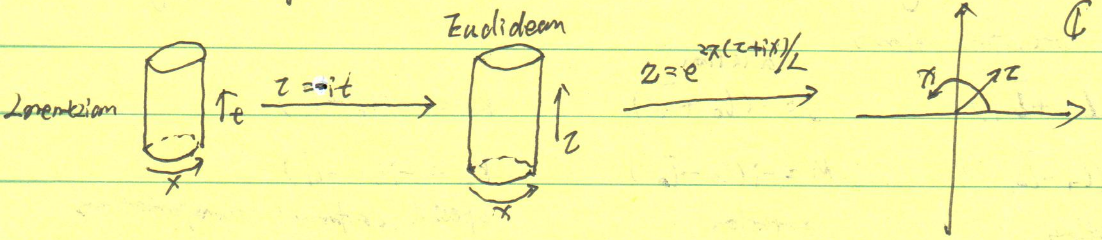
$$S[\varphi'] = \frac{1}{2g}\int_{S^1\times\mathbb{R}}\partial_\mu\varphi'\partial^\mu\varphi'\,dx\,dt = \frac{1}{2g}\int_{S^1\times\mathbb{R}}(\partial_\mu\varphi+\partial_\mu\eta)(\partial^\mu\varphi+\partial^\mu\eta)\,dx\,dt$$
$$= \frac{1}{2g}\int_{S^1\times\mathbb{R}}(\partial_\mu\varphi\partial^\mu\varphi + \partial_\mu\eta\partial^\mu\varphi + \partial^\mu\eta\partial_\mu\varphi)\,dx\,dt.$$

$$\Rightarrow S[\varphi']-S[\varphi] = \frac{1}{2g}\int_{S^1\times\mathbb{R}}(\partial_\mu\eta\partial^\mu\varphi + \partial^\mu\eta\partial_\mu\varphi)\,dx\,dt = \frac{1}{2g}\int_{S^1\times\mathbb{R}}(\partial_\mu\eta\partial^\mu\varphi + \delta^\mu_\nu\partial_\mu\varphi\partial^\nu\eta)\,dx\,dt$$
$$= \frac{1}{2g}\int_{S^1\times\mathbb{R}}(\partial_\mu\eta\partial^\mu\varphi + g^{\mu\sigma}g_{\sigma\nu}\partial_\mu\varphi\partial^\nu\eta)\,dx\,dt = \frac{1}{2g}\int_{S^1\times\mathbb{R}}(\partial_\mu\eta\partial^\mu\varphi + \partial_\sigma\eta\partial^\sigma\varphi)\,dx\,dt$$
$$= \frac{1}{g}\int_{S^1\times\mathbb{R}}\partial_\mu\eta\,\partial^\mu\varphi\,dx\,dt = \frac{1}{g}\int_{S^1\times\mathbb{R}}(-\partial_t\eta\partial_t\varphi + \partial_x\eta\partial_x\varphi)\,dx\,dt$$

分部积分.
$\eta_{t\to\pm\infty}=0$, $\longrightarrow$ $= -\frac{1}{g}\int_{S^1\times\mathbb{R}}\eta(-\partial_t^2\varphi+\partial_x^2\varphi)\,dx\,dt = -\frac{1}{g}\int_{S^1\times\mathbb{R}}\eta\,\partial_\mu\partial^\mu\varphi\,dx\,dt$

$\Rightarrow$ Equation of Motion $\quad \partial_\mu\partial^\mu\varphi = 0 \quad$ ie $\quad \partial_t^2\varphi = \partial_x^2\varphi$.

Switch to complex coordinates & Wick rotation



$$S[\varphi] = \frac{1}{2g}\int_{S\times\mathbb{R}} \partial_\mu\varphi\,\partial^\mu\varphi\, dx\,dt = \frac{1}{2g}\int_{S\times\mathbb{R}}\left[-(\partial_t\varphi)^2 + (\partial_x\varphi)^2\right]dx\,dt$$

Wick rotation: $\tau = it$, $x = x$

$$-(\partial_t\varphi)^2 + (\partial_x\varphi)^2 = (\partial_\tau\varphi)^2 + (\partial_x\varphi)^2$$

$$dx\,dt = \left|\frac{\partial(x,t)}{\partial(x,\tau)}\right| dx\,d\tau = \begin{vmatrix} 1 & 0 \\ 0 & i \end{vmatrix} dx\,d\tau = dx\,d\tau$$

So. $$S[\varphi] = \frac{1}{2g}\int_{S\times\mathbb{R}}\left[(\partial_\tau\varphi)^2 + (\partial_x\varphi)^2\right]dx\,d\tau$$

Switch to complex coordinates:

$$\partial_\tau\varphi = \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\tau} + \frac{\partial\varphi}{\partial\bar z}\frac{\partial\bar z}{\partial\tau} = \frac{2\pi z}{L}\partial\varphi + \frac{2\pi\bar z}{L}\bar\partial\varphi$$

$$\partial_x\varphi = \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial\varphi}{\partial\bar z}\frac{\partial\bar z}{\partial x} = \frac{2\pi i z}{L}\partial\varphi - \frac{2\pi i\bar z}{L}\bar\partial\varphi$$

$$\therefore\quad (\partial_\tau\varphi)^2 + (\partial_x\varphi)^2 = \frac{16\pi^2 z\bar z}{L^2}\partial\varphi\,\bar\partial\varphi$$

$$\cancel{dx\,d\tau}\ dz\,d\bar z = \left|\frac{\partial(z,\bar z)}{\partial(x,\tau)}\right| dx\,d\tau = \begin{vmatrix} \frac{2\pi i z}{L} & \frac{2\pi z}{L} \\ \frac{-2\pi i\bar z}{L} & \frac{2\pi\bar z}{L}\end{vmatrix} dx\,d\tau = \frac{8\pi^2 z\bar z}{L^2}dx\,d\tau$$

So $$S[\varphi] = \frac{1}{2g}\int_{\mathbb{C}} \frac{16\pi^2 z\bar z}{L^2}\partial\varphi\,\bar\partial\varphi\,\frac{L^2}{8\pi^2 z\bar z}dz\,d\bar z$$

$$= \frac{1}{g}\int_{\mathbb{C}}\partial\varphi\,\bar\partial\varphi\,dz\,d\bar z$$

Note: In complex plane z and $\bar z$ should be considered as independent. But when back into physical space we ~~should~~ have to set. $z^* = \bar z$

Equation of motion $\quad \partial\bar\partial\varphi = 0$

$\therefore\quad \partial\varphi = \partial\varphi(z)$ and $\bar\partial\varphi = \bar\partial\varphi(\bar z)$

Classical Conformal Invariance. The action will not change under $z \to z' = z + \varepsilon(z)$, $\bar z \to \bar z' = \bar z + \bar\varepsilon(\bar z)$

Boson is spinless $\quad \varphi'(z',\bar z') = \varphi(z,\bar z)\quad$ [Scalar field]

If: $z' = z + \varepsilon(z)$, $\bar z' = \bar z$ then $\partial = (1+\partial\varepsilon)\partial'$ and $dz' = (1+\partial\varepsilon)dz$.

So $$S'[\varphi'] = \frac{1}{g}\int_{\mathbb{C}}\partial'\varphi'(z',\bar z')\bar\partial'\varphi'(z',\bar z')\,dz'd\bar z' = \frac{1}{g}\int_{\mathbb{C}}\partial'\varphi(z,\bar z)\bar\partial'\varphi(z,\bar z)\,dz'd\bar z'$$

$$= \frac{1}{g}\int_{\mathbb{C}}(1-\partial\varepsilon)\partial\varphi\,\bar\partial\varphi\,(1+\partial\varepsilon)\,dz\,d\bar z$$

$$= \frac{1}{g}\int_{\mathbb{C}}\partial\varphi\,\bar\partial\varphi\,dz\,d\bar z = S[\varphi]$$

Similar. one can proof $S'[\varphi'] = S[\varphi]$, under $z' = z + \varepsilon(z)$, $\bar z' = \bar z + \bar\varepsilon(\bar z)$

Stress - Energy Tensor (Energy — Momentum Tensor)

General coordinate transformations do not preserve the action, but those characterise the stress-Energy Tensor.

under general coordinate transformation $x'^\mu = x^\mu + \eta^\mu$. then

$$S' - S = \int T^{\mu\nu} \partial_\mu \eta_\nu \, dx dt$$

on $\mathbb{C}$, we take $z' = z + \eta(z, \bar z)$, $\bar z' = \bar z + \bar\eta(z, \bar z)$ then

$$\partial = \frac{\partial z'}{\partial z}\partial' + \frac{\partial \bar z'}{\partial z}\bar\partial' \qquad \bar\partial = \frac{\partial z'}{\partial \bar z}\partial' + \frac{\partial \bar z'}{\partial \bar z}\bar\partial'$$

$$= (1+\partial\eta)\partial' + \partial\bar\eta\,\bar\partial' \qquad = \bar\partial\eta\,\partial' + (1+\bar\partial\bar\eta)\bar\partial'$$

$$\partial\varphi\bar\partial\varphi = [(1+\partial\eta)\partial' + \partial\bar\eta\bar\partial']\varphi'[\bar\partial\eta\partial' + (1+\bar\partial\bar\eta)\bar\partial']\varphi' = [(1+\partial\eta)\partial'\varphi' + \partial\bar\eta\bar\partial'\varphi'][\bar\partial\eta\partial'\varphi' + (1+\bar\partial\bar\eta)\bar\partial'\varphi']$$

$$= \bar\partial\eta(\partial'\varphi')^2 + (1+\partial\eta+\bar\partial\bar\eta)\partial'\varphi'\bar\partial'\varphi' + \partial\bar\eta(\bar\partial'\varphi')^2$$

$$\bar\partial\eta(\partial\varphi)^2 = \bar\partial\eta[(1+\partial\eta)\partial'\varphi' + \partial\bar\eta\bar\partial'\varphi']^2 = \bar\partial\eta[(1+2\partial\eta)(\partial'\varphi')^2 + 2\partial\bar\eta\partial'\varphi'\bar\partial'\varphi']$$

$$= \bar\partial\eta(\partial'\varphi')^2$$

同样 $\cancel{\bar\partial\eta(\partial\varphi)^2}$ $\partial\bar\eta(\bar\partial\varphi)^2 = \partial\bar\eta(\bar\partial'\varphi')^2$

$$dz'd\bar z' = \left|\frac{\partial(z', \bar z')}{\partial(z, \bar z)}\right| dz d\bar z = \begin{vmatrix} 1+\partial\eta & \bar\partial\eta \\ \partial\bar\eta & 1+\bar\partial\bar\eta \end{vmatrix} dz d\bar z = (1+\partial\eta+\bar\partial\bar\eta)dz d\bar z$$

So: $$S' - S = \frac{1}{g}\int_{\mathbb{C}} \partial'\varphi'\bar\partial'\varphi' dz' d\bar z' - \frac{1}{g}\int_{\mathbb{C}} \partial\varphi\bar\partial\varphi dz d\bar z$$

$$= -\frac{1}{g}\int_{\mathbb{C}} [\partial\varphi\partial\varphi\bar\partial\eta + \bar\partial\varphi\bar\partial\varphi\partial\bar\eta] dz d\bar z$$

$\therefore$ $T^{zz} = 0 \quad T^{\bar z z} = -\frac{1}{g}\partial\varphi\partial\varphi \quad T^{z\bar z} = -\frac{1}{g}\bar\partial\varphi\bar\partial\varphi \quad T^{\bar z\bar z} = 0$

Renormalise: Define $T(z) = \frac{1}{2}\partial\varphi\partial\varphi$ $= \frac{1}{2}\partial\varphi(z)\partial\varphi(z)$, $\bar T(\bar z) = \frac{1}{2}\bar\partial\varphi(\bar z)\bar\partial\varphi(\bar z)$

Canonical Quantisation:

(equal-time)

1) Determine "degrees of freedom" 2) Compute conjugate momenta 3) Impose canonical commutation rules

1. Fourier decomposition (since $\varphi(t, x) = \varphi(t, x+L)$)

$$\varphi(t, x) = \sum_{n\in\mathbb{Z}} \underbrace{\varphi_n(t)}_{\text{degrees of freedom.}} e^{2\pi i n x/L}$$

$$S = \frac{1}{2g}\int_{-\infty}^{\infty}\int_0^L [-(\partial_t\varphi)^2 + (\partial_x\varphi)^2] dx dt = -\frac{L}{2g}\int_{-\infty}^{\infty}\sum_{m\in\mathbb{Z}}[\dot\varphi_m(t)\dot\varphi_{-m}(t) - \frac{4\pi^2 m^2}{L^2}\varphi_m(t)\varphi_{-m}(t)]dt.$$

2. Conjugate momentum to $\varphi_n(t)$ is $\Pi_n(t) = \frac{\partial L}{\partial\dot\varphi_n(t)}$, ie.

$$\Pi_n(t) = -\frac{L}{g}\dot\varphi_{-n}(t)$$

3. We promote both $\varphi_n(t)$ and $\Pi_n(t)$ to operators with commutation relations ($\hbar = 1$)

$$[\varphi_m(t), \varphi_n(t)] = [\Pi_m(t), \Pi_n(t)] = 0, \quad [\varphi_m(t), \Pi_n(t)] = i\delta_{m,n}$$

Thus, $[\varphi_m(t), \dot\varphi_n(t)] = \frac{-ig}{L}\delta_{m+n,0}$

Now change back to complex coordinates:

$$\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \qquad \bar\partial\varphi(\bar z) = \sum_{n\in\mathbb{Z}} \bar a_n \bar z^{-n-1}$$

The a_n and $\bar a_n$ are the degrees of freedom in these coordinates. Thus, they become operators in the quantum field.

Note: Cauchy's theorem tells:

$$a_n = \oint_0 \partial\varphi(z) z^n \frac{dz}{2\pi i} \qquad \left(\begin{array}{c}\text{any anticlockwise} \\ \text{contour around } 0\end{array}\right)$$

Cauchy's Theorem:

If: $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}$ then $\oint_0 f(z) z^n \frac{dz}{2\pi i} = f_n$

Generalisation:

If: $f(z) = \sum_{n\in\mathbb{Z}} f_n (z-w)^{-n-1}$ the $\oint_w f(z)(z-w)^n \frac{dz}{2\pi i} = f_n$

Alternatively:

$$\oint_w \frac{f(z)}{(z-w)^{n+1}} \frac{dz}{2\pi i} = \frac{1}{n!}\partial^n f(w)$$

Take the contour to be a circle of fixed radius. Since $z = e^{2\pi(\tau+ix)/L}$, this circle is constant in time! Thus,

$$a_n = \oint_0 \partial\varphi(z) z^n \frac{dz}{2\pi i} = \frac{1}{4\pi i}\int_0^L (\partial_x\varphi - \partial_t\varphi) e^{2\pi i n(x-t)/L} dx \quad \left(\text{using } \partial = \frac{L}{4\pi i} z^{-1}(\partial_x - \partial_t) \text{ and } dz = \frac{2\pi i}{L} z\, dx\right)$$

$$= -\left(\frac{n}{2}\varphi_{-n}(t) + \frac{L}{4\pi i}\dot\varphi_{-n}(t)\right)e^{-2\pi i n t/L} \quad \left(\text{using } \varphi(t,x) = \sum_n \varphi_n(t) e^{2\pi i n x/L}\right)$$

The commutators of the a_n are thus

$$[a_m, a_n] = m\delta_{m+n,0}\frac{g}{4\pi^2} \quad [\bar a_m, \bar a_n] = m\delta_{m+n,0}\frac{g}{4\pi^2} \quad [a_m, \bar a_n] = 0$$

At this point, we set $g = 4\pi^2$ to simplify above commutators.

The Lie algebra generated by a_n is called the Heisenberg algebra.

Exercise: $\dot a_n = 0$ Pf: $a_n = -\left(\frac{n}{2}\varphi_{-n}(t) + \frac{L}{4\pi i}\dot\varphi_{-n}(t)\right)e^{-2\pi i n t/L} = \left(-\frac{n}{2}\varphi_{-n}(t) + \frac{g}{4\pi i}\Pi_n(t)\right)e^{-2\pi i n t/L}$

From steps 1 and 2, the Hamiltonian, $H = -\frac{g}{2L}\sum_{m\in\mathbb{Z}}\left[\Pi_m(t)\Pi_{-m}(t) + \frac{4\pi^2 m^2}{g^2}\varphi_m(t)\varphi_{-m}(t)\right]$

$$[\varphi_{-n}, H] = -\frac{g}{2L}\sum_m [\varphi_{-n}, \Pi_m\Pi_{-m}] = -\frac{ig}{L}\Pi_n \qquad ①$$

$$[\Pi_n, H] = -\frac{g}{2L}\frac{4\pi^2}{g^2}\sum_m [\Pi_n, m^2\varphi_m\varphi_{-m}] = \frac{4\pi^2 i}{2g} n^2\varphi_{-n} \qquad ②$$

$$①\times\left(-\frac{n}{2}\right) + ②\times\frac{g}{4\pi i} = \left[\left(-\frac{n}{2}\varphi_{-n} + \frac{g}{4\pi i}\pi_n\right), H\right] = \frac{ign}{2L}\pi_n + \frac{\pi n^2}{L}\varphi_{-n} = -\frac{2\pi n}{L}\left(-\frac{n}{2}\varphi_{-n} + \frac{g}{4\pi i}\pi_n\right)$$

from $i\frac{d}{dt}\hat{O}_H(t) = [\hat{O}_H(t), \hat{H}]$, we can get.

$$\left[-\frac{n}{2}\varphi_{-n}(t) + \frac{g}{4\pi i}\pi_n(t)\right] = \left[-\frac{n}{2}\varphi_{-n}(0) + \frac{g}{4\pi i}\pi_n(0)\right] e^{2\pi i n t/L}$$

So $$a_n = \left[-\frac{n}{2}\varphi_{-n}(t) + \frac{g}{4\pi i}\pi_n(t)\right] e^{-2\pi i n t/L} = \left[-\frac{n}{2}\varphi_{-n}(t=0) + \frac{g}{4\pi i}\pi_n(t=0)\right]$$

$$\therefore \quad \dot{a}_n = 0$$

Fock Spaces:

Given the symmetry algebra of the a_n and $\bar{a}_n$, we can consider those with $n>0$ to be annihilation operators, those with $n<0$ to be creation operators. and those with $n=0$ to be zero modes.

Let $a_n^+ = a_{-n}$, and $\bar{a}_n^+ = \bar{a}_{-n}$. So the zero-modes are self-adjoint.

$$a_0^+ = a_0 \qquad \bar{a}_0^+ = \bar{a}_0$$

Their eigenvalues are real — they are physical observables! In fact. the eigenvalues are the momenta in the "z and $\bar{z}$" directions.

So, consider a vacuum (ground state) of momentum $p, \bar{p}$, call $|p, \bar{p}\rangle$ $p, \bar{p} \in \mathbb{R}$.

It is annihilated by $a_n, \bar{a}_n$ with $n>0$

$$a_n|p,\bar{p}\rangle = \bar{a}_n|p,\bar{p}\rangle = 0. \quad \text{for } n>0$$

It also satisfies $a_0|p,\bar{p}\rangle = p|p,\bar{p}\rangle$, $\bar{a}_0|p,\bar{p}\rangle = \bar{p}|p,\bar{p}\rangle$

Acting on it with creation operators $a_{-n}, \bar{a}_{-n}$ $(n>0)$ give excited states. eg.

$$a_{-1}|p,\bar{p}\rangle, \quad a_{-1}^3|p,\bar{p}\rangle \quad \bar{a}_{-2}|p,\bar{p}\rangle \cdot \quad a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}|p,\bar{p}\rangle \quad \cdots$$

The span of all excited states of a vacuum $|p\bar{p}\rangle$ is called the Fock space $F_{p,\bar{p}}$.

Note that these excited states have the same momentum, because ~~that~~.

$$[a_0, a_n] = 0 \quad [\bar{a}_0, \bar{a}_n] = 0 \quad [a_0, \bar{a}_n] = 0$$

eg: $$a_0 a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}|p,\bar{p}\rangle = a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}a_0|p,\bar{p}\rangle = p(a_{-3}^2\bar{a}_{-7}\bar{a}_{-4}|p,\bar{p}\rangle$$

Why are they excited? They have more energy! So we turn to the stress-energy tensor $T(z), \bar{T}(\bar{z})$

$$T(z) = \frac{1}{2}\partial\varphi(z)\partial\varphi(z) = \frac{1}{2}\sum_{r,s\in\mathbb{Z}} a_r a_s z^{-r-s-2} \qquad (\text{forget about } \bar{z} \text{ stuff})$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\underbrace{\left[\sum_{r\in\mathbb{Z}} a_r a_{n-r}\right]}_{2L_n} z^{-n-2}$$

problem: The Fourier mode $L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}$ is a infinite sum. it might diverge when we act on some quantum states.

eg. $L_0|p\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r}|p\rangle = \frac{1}{2}\left[\sum_{r=-\infty}^{-1} a_r a_{-r}|p\rangle + a_0^2|p\rangle + \sum_{r=1}^{\infty} a_r a_{-r}|p\rangle\right]$

$= p^2|p\rangle + \frac{1}{2}\sum_{r=1}^{\infty}[a_r, a_{-r}]|p\rangle$

$= p^2|p\rangle + \frac{1}{2}\sum_{r=1}^{\infty} r|p\rangle \leftarrow$ diverges!

We regularise the above divergence by introducing "normal — ordering" in which annihilators are moved to the right of creators.

More precisely, $:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \le -1 \\ a_n a_m & \text{if } m \ge 0 \end{cases}$

Justification: Classically, the a_n are numbers, so they commute. When we quantise, there is an ambiguity in the ordering chosen. The naive ordering gives divergence. So we replace it by normal — ordering

Redefine: $T(z) = \frac{1}{2}:\partial\phi(z)\partial\phi(z): = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}: z^{-n-2}$

$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{r\le -1} a_r a_{n-r} + \sum_{r\ge 0} a_{n-r} a_r\right] z^{-n-2}$

$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}:$

These L_n act on excited states without divergence, eg.

$L_0|p\rangle = \frac{1}{2}\sum_{r\le -1} a_r a_{-r}|p\rangle + \frac{1}{2}\sum_{r\ge 0} a_{-r} a_r|p\rangle = \frac{1}{2}a_0^2|p\rangle = \frac{1}{2}p^2|p\rangle$

The eigenvalue of L_0 is called the energy (actually, it is the sum of the eigenvalues of L_0 and $\bar{L}_0$ which is the total energy, their difference is the angular momentum or spin).

Exercise. check that ① $L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}$ $(n\neq 0)$

Pf: If $n\neq 0$, then $[a_r, a_{n-r}] = 0$.

② $L_0 = \frac{1}{2}a_0^2 + \sum_{r\ge 1} a_{-r} a_r$

Pf: $L_0 = \frac{1}{2}\sum_{r\in\mathbb{Z}} :a_r a_{-r}: = \frac{1}{2}\left[\sum_{r\le -1} a_r a_{-r} + a_0^2 + \sum_{r\ge 1} a_{-r} a_{r}\right]$

$= \frac{1}{2}\left[\sum_{s\ge 1} a_{-s} a_{s} + a_0^2 + \sum_{r\ge 1} a_{-r} a_{r}\right]$

$\cancel{= \frac{1}{2}a_0^2 + \sum_{r\ge 1} a_{-r} a_r}$

$= \frac{1}{2}a_0^2 + \sum_{r\ge 1}^{\infty} a_{-r} a_r$

③ $[L_m, a_n] = -n a_{m+n}$.

Pf: When $m\neq 0$, $[L_m, a_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}}[a_r a_{m-r}, a_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}}\left(a_r[a_{m-r}, a_n] + [a_r, a_n]a_{m-r}\right)$

$= \frac{1}{2}\sum_{r\in\mathbb{Z}}\left[(m-r)\delta_{r,m+n} a_r + r\delta_{r,-n} a_{m-r}\right] = \frac{1}{2}(-n a_{m+n} - n a_{m+n})$

$= -n a_{m+n}$.

when $m=0$. $[L_0, a_n] = [\frac{1}{2}a_0^2, a_n] + \sum_{r=1}^{\infty}[a_{-r}a_r, a_n] = \sum_{r=1}^{\infty}(a_{-r}[a_r, a_n] + [a_{-r}, a_n]a_r)$

$$= \sum_{r=1}^{\infty}[a_{-r}\, r\, \delta_{r+n,0} + a_r(-r)\,\delta_{n-r,0}] = -n a_n$$

④ $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\, c$. where $c=1$.

See below!

These commutation relations are similar to those of the classical infinitesimal conformal generators. l_n : $[l_m, l_n] = (m-n)l_{m+n}$

The extra term with $c=1$ is due to the quantisation. c is call the central charge or conformal anomaly.

Note that $[L_0, a_{-n}] = n a_{-n}$ means that acting with a creator a_{-n} increase the energy by n units.

e.g. if $|\psi\rangle$ has energy E: $L_0|\psi\rangle = E|\psi\rangle$

the $a_{-n}|\psi\rangle$ has energy $E+n$: $L_0 a_{-n}|\psi\rangle = (a_{-n}L_0 + [L_0, a_{-n}])|\psi\rangle = (a_{-n}E + n a_{-n})|\psi\rangle$

$$= (E+n)a_{-n}|\psi\rangle$$

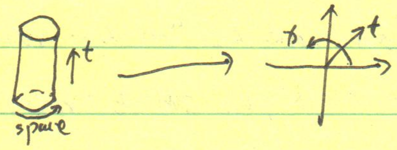
So. excited states do have more energy, as claimed!

State – Field Correspondence

Since for vacuum $|p\rangle$, $E_p = \frac{1}{2}p^2$, the vacuum $|0\rangle$ has minimal energy. It's the "true vacuum"

Given a field $\varphi(z)$, (e.g. $\partial\varphi(z)$, $T(z)$, etc.) there is a corresponding quantum state $|\varphi\rangle$ define by

$$|\varphi\rangle = \lim_{z\to 0}\varphi(z)|0\rangle \qquad \text{(AND. vice-versa!)}$$

Recall 

$z \to 0$ means $t \to -\infty$

In scattering language. $|\varphi\rangle$ is an "asymptotic in state".

Example: ① The identity field. $\Omega(z) = 1 = \sum_{n\in\mathbb{Z}}\delta_{n,0}z^{-n}$

$\lim_{z\to 0}\Omega(z)|0\rangle = |0\rangle$ Thus the corresponding state is $|\Omega\rangle = |0\rangle$

② The holomorphic derivative of the boson field $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$

$$\lim_{z\to 0}\partial\varphi(z)|0\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}} a_n z^{-n-1}|0\rangle = \lim_{z\to 0}\sum_{n\le -1} a_n z^{-n-1}|0\rangle = \lim_{z\to 0}[a_{-1}|0\rangle + a_{-2}z|0\rangle + a_{-3}z^2|0\rangle + \cdots]$$

$$= a_{-1}|0\rangle$$

i.e. $|\partial\varphi\rangle$ is the excited state $a_{-1}|0\rangle$

③ $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ corresponds to $L_{-2}|0\rangle$.

$$L_{-2}|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r-2}|0\rangle = \frac{1}{2}a_{-1}^2|0\rangle \qquad \begin{bmatrix} a_r|0\rangle = 0 \ \forall r\geq 0 \\ a_{-r-2}|0\rangle = 0 \ \forall r\leq -2 \end{bmatrix}$$

$$\lim_{z\to 0} T(z)|0\rangle \overset{\lim_{z\to0}}{=} \sum_{n<-2} L_n z^{-n-2}|0\rangle + L_{-2}|0\rangle + \lim_{z\to 0}\sum_{n>-2} z^{-n-2} L_n|0\rangle$$

$$L_{-1}|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r-1}|0\rangle = 0 \qquad \begin{bmatrix} a_r|0\rangle = 0 \ \forall r\geq 0 \\ a_{-r-1}|0\rangle = 0 \ \forall r\leq -1 \end{bmatrix} \quad [a_r, a_{-r-1}] = 0$$

$$L_0|0\rangle = 0$$

$$(n>0) \quad L_n|0\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{-r+n}|0\rangle = 0 \qquad \begin{bmatrix} a_r|0\rangle = 0 \ \forall r\geq 0 \\ a_{-r-n}|0\rangle = 0 \ \forall r\neq 0 \end{bmatrix} \quad [a_r, a_{n-r}] = 0$$

So $\lim_{z\to 0} T(z)|0\rangle = L_{-2}|0\rangle$

Exercise: show that $\partial^2\varphi(z) = \partial(\partial\varphi(z)) = -\sum_{n\in\mathbb{Z}}(n+1)a_n z^{-n-2}$ corresponds to $a_{-2}|0\rangle$

$$\lim_{z\to 0}\partial^2\varphi(z)|0\rangle = \lim_{z\to 0} -\sum_{n\in\mathbb{Z}} a_n z^{-n-2}|0\rangle = \lim_{z\to 0} -\sum_{n<-2} a_n z^{-n-2}|0\rangle(n+1) + a_{-2}|0\rangle + \lim_{z\to 0} -\sum_{n\geq 0} z^{-n-2} a_n|0\rangle$$
$$= a_{-2}|0\rangle$$

show that $\partial^j\varphi(z)$ corresponds to $(j-1)!\,a_{-j}|0\rangle$

$$\partial^j\varphi(z) = \partial^{j-1}(\partial\varphi(z)) = \partial^{j-1}\sum_{n\in\mathbb{Z}} a_n z^{-n-1} = \sum_{n\in\mathbb{Z}}(-n-1)(-n-2)\cdots(-n-j+1)\,a_n z^{-n-j}$$

$$\lim_{z\to 0}\partial^j\varphi(z) = \lim_{z\to 0}\sum_{n<-j}(-n-1)(-n-2)\cdots(-n-j+1)\,a_n z^{-n-j}|0\rangle + (j-1)(j-2)\cdots 1\cdot a_{-j}|0\rangle$$
$$+ \lim_{z\to 0}\sum_{n\geq 0}(-n-1)(-n-2)\cdots(-n-j+1)\,z^{-n-j}a_n|0\rangle$$
$$= (j-1)!\,a_{-j}|0\rangle$$

Operator product Expansion.

Recall: $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $[a_m, a_n] = m\,\delta_{m+n,0}$.

$$\Rightarrow [a_m, \partial\varphi(w)] = \sum_{n\in\mathbb{Z}}[a_m, a_n]w^{-n-1} = \sum_{n\in\mathbb{Z}} m\,\delta_{m+n,0}w^{-n-1} = m w^{m-1}$$

But: $[\partial\varphi(z), \partial\varphi(w)] = \sum_{m\in\mathbb{Z}}[a_m, \partial\varphi(w)]z^{-m-1} = \sum_{m\in\mathbb{Z}} m w^{m-1} z^{-m-1} = \frac{1}{zw}\underbrace{\sum_{m\in\mathbb{Z}} m\left(\frac{w}{z}\right)^m}_{\text{Divergent!}}$

We again have to remove such divergences. This is done by "time-ordering".
Because time on cylinder is the radial direction on $\mathbb{C}$, we call it "radial-ordering"

$$R\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z|>|w| \\ B(w)A(z) & \text{if } |z|<|w| \end{cases}$$

why is this reasonable?

These fields act on quantum states $|C\rangle$ which are field at $z=0$ (i.e. $t=-\infty$). Thus

$$A(z)B(w)|C\rangle$$

is interpreted as:

① $|C\rangle$ is a state at time $-\infty$
② Act with $B(w)$ at time $|w|$
③ Act with $A(z)$ at time $|z|$

This only makes sense if $|z| > |w|$ *! If $|z| < |w|$, then $A(z)$ act before $B(w)$ and we must write $B(w)A(z)|c\rangle$

In all cases. physics requires that the fields are radially-ordered.

The radial-ordering removes the divergence in $[\partial\varphi(z), \partial\varphi(w)]$

$$R\{[\partial\varphi(z), \partial\varphi(w)]\} = R\{\partial\varphi(z)\partial\varphi(w)\} - R\{\partial\varphi(w)\partial\varphi(z)\} = 0$$

A interesting computation:

Assume $|z| > |w|$ for now.

$$R\{\partial\varphi(z)\partial\varphi(w)\} = \partial\varphi(z)\partial\varphi(w) = \sum_{r,s} a_r a_s z^{-r-1} w^{-s-1} = \sum_{n,r} a_r a_{n-r} z^{-r-1} w^{-n-1+r}$$

For $n \neq 0$. $a_r a_{n-r} = :a_r a_{n-r}:$ $\quad ([a_r, a_{n-r}] = 0.\ n \neq 0)$

For $n = 0$, $a_r a_{-r} = :a_r a_{-r}:$ if $r \leq -1$

$a_r a_{-r} = a_{-r} a_r + [a_r, a_{-r}] = :a_r a_{-r}: + r$ if $r \geq 0$

ie
$$R\{\partial\varphi(z)\partial\varphi(w)\} = \sum_{n,r} :a_r a_{n-r}: z^{-r-1} w^{-n-1+r} + \sum_{r=0}^{\infty} r z^{-r-1} w^{r-1}$$
$$= :\partial\varphi(z)\partial\varphi(w): + \frac{1}{z^2}\sum_{r=0}^{\infty} r \left(\frac{w}{z}\right)^{r-1}$$
$$= :\partial\varphi(z)\partial\varphi(w): + \frac{1}{z^2}\frac{1}{(1-\frac{w}{z})^2} \quad (\text{since } |z| > |w|)$$
$$= :\partial\varphi(z)\partial\varphi(w): + \frac{1}{(z-w)^2}$$

If $|z| < |w|$ then $R\{\partial\varphi(z)\partial\varphi(w)\} = :\partial\varphi(w)\partial\varphi(z): + \frac{1}{(w-z)^2}$

Claim: $:a_r a_s: = :a_s a_r:$ so $:\partial\varphi(z)\partial\varphi(w): = :\partial\varphi(w)\partial\varphi(z):$

pf: ① $r+s \neq 0 \Rightarrow :a_r a_s: = a_r a_s = a_s a_r = :a_s a_r:$ as $[a_r, a_s] = 0$

② $r+s = 0$ 1) $r \leq -1 \Rightarrow :a_r a_{-r}: = a_r a_{-r} = :a_{-r} a_r:$

2) $r \geq 1 \Rightarrow :a_r a_{-r}: = a_{-r} a_r = :a_{-r} a_r:$

3) $r = 0 \Rightarrow :a_0 a_0: = :a_0 a_0:$

So:

$$\boxed{R\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w):}$$

We can Taylor-expand the RHS of above equation about $z = w$.

$$R\{\partial\varphi(z)\partial\varphi(w)\} = \frac{1}{(z-w)^2} + \underbrace{:\partial\varphi(w)\partial\varphi(w):}_{2T(w)} + :\partial^2\varphi(w)\partial\varphi(w):(z-w) + \cdots$$

Note that: $:\partial\varphi(w)\partial\varphi(w): = \oint_w \frac{R\{\partial\varphi(z)\partial\varphi(w)\}}{z-w}\frac{dz}{2\pi i}$

There is a pole for $z-w$, so $R\{\partial\varphi(z)\partial\varphi(z)\}$ diverges.

This Laurent series expansion is called an Operator product Expansion (OPE)

In general, an OPE always has the form.

$$R\{A(z)B(w)\} = [\text{singular terms as } z\to w] + :A(z)B(w):$$

[In fact, the general definition of normal-ordering is

$$:A(w)B(w): = \oint_w \frac{R\{A(z)B(w)\}}{z-w}\frac{dz}{2\pi i}$$]

Because of this, lazy physicists just write the singular terms:

$$R\{\partial\varphi(z)\partial\varphi(w)\} \sim \frac{1}{(z-w)^2}$$

Usually they leave out the radial-ordering symbol too:

$$\partial\varphi(z)\partial\varphi(w) \sim \frac{1}{(z-w)^2}$$

The OPE contains all information about commutators in its singular terms.

Recall that $\quad a_n = \oint_0 \partial\varphi(z)\, z^n \frac{dz}{2\pi i}$

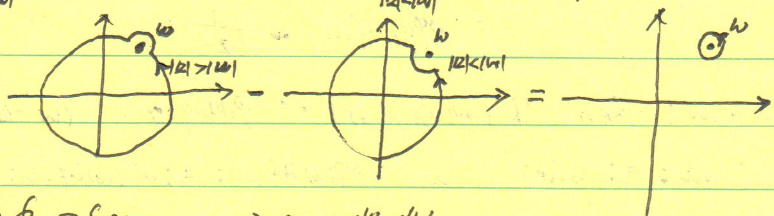
$$\therefore [a_m, a_n] = a_m a_n - a_n a_m$$

$$= \oint_0\oint_0 \partial\varphi(z)\partial\varphi(w)\, z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0\oint_0 \partial\varphi(w)\partial\varphi(z)\, z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

(these products must be radially-ordered to make sense!

$$= \oint_0\oint_0{}_{|z|>|w|} R\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0\oint_0{}_{|z|<|w|} R\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

z-Contours:



$$= \oint_0\oint_w R\{\partial\varphi(z)\partial\varphi(w)\} z^m w^n \frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0\oint_w \Big[\frac{z^m w^n}{(z-w)^2} + \underbrace{:\partial\varphi(z)\partial\varphi(w): z^m w^n}_{\text{no pole at } z=w}\Big]\frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0 m\, w^{m+n-1}\frac{dw}{2\pi i}$$

$$= m\,\delta_{m+n,0}$$

Equivalently, the commutation rules determine the OPE using the state-field correspondence

Assume $R\{\partial\varphi(z)\partial\varphi(w)\} = \sum_n \varphi_n(w)(z-w)^{-n-1}$, for some unknown $\varphi_n(w)$

Then apply both sides to $|0\rangle$ and let $w\to 0$:

LHS: $\lim_{w\to 0} R\{\partial\varphi(z)\partial\varphi(w)\}|0\rangle = \partial\varphi(z)\lim_{w\to 0}\partial\varphi(w)|0\rangle = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}|\partial\varphi\rangle$

$$= \sum_{n\in\mathbb{Z}} a_n a_{-1}|0\rangle z^{-n-1} \qquad (\text{recall } |\partial\varphi\rangle = a_{-1}|0\rangle)$$

RHS: $\lim_{w\to 0} \sum_n \varphi_n(w)(z-w)^{-n-1}|0\rangle = \sum_n \left[\lim_{w\to 0}\varphi_n(w)|0\rangle\right] z^{-n-1} = \sum_n |\varphi_n\rangle z^{-n-1}$

Comparing gives $\quad |\varphi_n\rangle = a_n a_{-1}|0\rangle \quad$ for all $n \in \mathbb{Z}$

For $n \geq 2$. $a_n a_{-1}|0\rangle = a_{-1}a_n|0\rangle = 0 \Rightarrow |\varphi_n\rangle = 0 \Rightarrow \varphi_n(w) = 0$

$n=1$ $\quad a_1 a_{-1}|0\rangle = (a_{-1}a_1 + [a_1, a_{-1}])|0\rangle = |0\rangle \Rightarrow |\varphi_1\rangle = |0\rangle \Rightarrow \varphi_1(w) = 1$

$n=0$. $\quad a_0 a_{-1}|0\rangle = a_{-1}a_0|0\rangle = 0 \Rightarrow |\varphi_0\rangle = 0 \Rightarrow \varphi_0(w) = 0$

$n=-1$ $\quad a_{-1}^2|0\rangle$ is $|\varphi_{-1}\rangle$ so $\varphi_{-1}(w)$ is $:\partial\varphi(w)\partial\varphi(w):$

etc ...

$$\therefore R\{\partial\varphi(z)\partial\varphi(w)\} = \frac{\varphi_1(w)}{(z-w)^2} + \varphi_{-1}(w) + \varphi_{-2}(w)(z-w) + \cdots$$
$$= \frac{1}{(z-w)^2} + :\partial\varphi(w)\partial\varphi(w): + \cdots.$$

Exercise: Use $[L_m, a_n] = -n a_{m+n}$ to compute the OPE
$$R\{T(z)\partial\varphi(w)\}$$

Assume. $\quad R\{T(z)\partial\varphi(w)\} = \sum_n \varphi_n(w)(z-w)^{-n-2}$

LHS $= \lim_{w\to 0} R\{T(z)\partial\varphi(w)\}|0\rangle = T(z)|\partial\varphi\rangle = \sum_n L_n z^{-n-2} a_{-1}|0\rangle \qquad (|\partial\varphi\rangle = a_{-1}|0\rangle)$

RHS $= \sum_n |\varphi_n\rangle z^{-n-2}$

Comparing gives $\qquad |\varphi_n\rangle = L_n a_{-1}|0\rangle \qquad$ for all n.

For $n \leq -2$: $\quad \varphi_n(w)(z-w)^{-n-2}$ is regular term.

$n=-1$ $\quad |\varphi_{-1}\rangle = L_{-1}a_{-1}|0\rangle = a_{-2}|0\rangle \qquad (L_{-1}|0\rangle = 0)$

$\therefore \quad \varphi_{-1}(w) = \partial^2\varphi(w)$

$n=0$ $\quad |\varphi_0\rangle = L_0 a_{-1}|0\rangle = a_{-1}|0\rangle \qquad L_0|0\rangle = 0$

$\therefore \quad \varphi_0(w) = \partial\varphi(w)$

$n>0$. $\quad |\varphi_n\rangle = L_n a_{-1}|0\rangle = a_{n-1}|0\rangle = 0 \qquad L_n|0\rangle = 0.$ for $n>0$.

$$\therefore \quad R\{T(z)\partial\varphi(w)\} = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{(z-w)} + \text{regular terms.}$$

Wick's Theorem For OPEs.

Wick's theorem lets us compute the OPE of normally-ordered product of $\partial\varphi(z)$.

eg. we can compute $R\{T(z)\partial\varphi(w)\} = \frac{1}{2}R\{:\partial\varphi(z)\partial\varphi(z):\partial\varphi(w)\}$

To do this, we introduce the contraction
$$\overbrace{\partial\varphi(z)\partial\varphi(w)} = \frac{1}{(z-w)^2} \quad \text{(singular term)}$$

the singular terms of the OPE are computed by taking all possible contractions.

and ~~to~~ normal-ordering what remains:

$$R\{T(z)\partial\varphi(w)\} = \frac{1}{2}R\{:\partial\varphi(z)\partial\varphi(z):\partial\varphi(w)\}$$

$$\sim \frac{1}{2}\left[:\partial\varphi(z)\overbrace{\partial\varphi(z):\partial\varphi(w)} + :\overbrace{\partial\varphi(z)\partial\varphi(z):\partial\varphi(w)}\right]$$

$$= \frac{1}{2}\left[\frac{\partial\varphi(z)}{(z-w)^2} + \frac{\partial\varphi(z)}{(z-w)^2}\right]$$

$$= \frac{\partial\varphi(z)}{(z-w)^2}$$

$$\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{(z-w)} \qquad \text{Laurent series expansion}$$

Another example:

$$R\{T(z)T(w)\} = \frac{1}{4}R\{:\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)\partial\varphi(w):\}$$

$$\sim \frac{1}{4}\Big[:\overbrace{\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)}\partial\varphi(w): + :\overbrace{\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)\partial\varphi(w)}:$$

$$+ :\partial\varphi(z)\overbrace{\partial\varphi(z)::\partial\varphi(w)}\partial\varphi(w): + :\partial\varphi(z)\overbrace{\partial\varphi(z)::\partial\varphi(w)\partial\varphi(w)}:$$

$$+ :\overbrace{\partial\varphi(z)\underbrace{\partial\varphi(z)::\partial\varphi(w)}\partial\varphi(w)}: + :\overbrace{\partial\varphi(z)\underbrace{\partial\varphi(z)::\partial\varphi(w)}}\partial\varphi(w): \Big]$$

$$= \frac{:\partial\varphi(z)\partial\varphi(w):}{(z-w)^2} + \frac{1/2}{(z-w)^4}$$

$$\sim \frac{1/2}{(z-w)^4} + \frac{:\partial\varphi(w)\partial\varphi(w):}{(z-w)^2} + \frac{:\partial^2\varphi(w)\partial\varphi(w):}{z-w} + \cdots$$

$$= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

Here $\partial T(w) = \frac{1}{2}\partial:\partial\varphi(w)\partial\varphi(w): = \frac{1}{2}:\partial^2\varphi(w)\partial\varphi(w): + \frac{1}{2}:\partial\varphi(w)\partial^2\varphi(w):$

$$= :\partial^2\varphi(w)\partial\varphi(w): \qquad (:A_1A_2: = :A_2A_1:)$$

Exercise: Use $T(z) = \sum_n L_n z^{-n-2}$ and so $L_n = \oint_0 T(z) z^{n+1}\frac{dz}{2\pi i}$ to compute $[L_m, L_n]$

$$[L_m, L_n] = \oint_0\oint_0 T(z) z^{m+1} T(w) w^{n+1}\frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0\oint_0 T(w)T(z) w^{n+1} z^{m+1}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0\oint_0{}_{|z|>|w|} R\{T(z)T(w)\} z^{m+1} w^{n+1}\frac{dz}{2\pi i}\frac{dw}{2\pi i} - \oint_0\oint_0{}_{|z|<|w|} R\{T(z)T(w)\} z^{m+1} w^{n+1}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0\oint_w R\{T(z)T(w)\} z^{m+1} w^{n+1}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

$$= \oint_0\oint_w \left[\frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] z^{m+1}\frac{dz}{2\pi i}\, w^{n+1}\frac{dw}{2\pi i} \qquad \left(\oint_w \frac{f(z)}{(z-w)^{n+1}}\frac{dz}{2\pi i} = \frac{1}{n!}\partial^n f(w)\right)$$

$$= \oint_0 \left[\frac{m^3-m}{12} w^{m-2} + 2T(w)(m+1)w^m + \partial T(w) w^{m+1}\right] w^{n+1}\frac{dw}{2\pi i}$$

$$= \oint \left[\frac{m^3-m}{12} w^{m+n-1} + 2T(w)(m+1) w^{m+n+1} + \partial T(w) w^{m+n+2}\right]\frac{dw}{2\pi i}$$

分部积分 $= \oint \left[\frac{m^3-m}{12} w^{m+n-1} + T(w)(2m+2) w^{m+n+1} - T(w)(m+n+2) w^{m+n+1}\right]\frac{dw}{2\pi i}$

$$= \oint \left[\frac{m^3-m}{12} w^{m+n-1} + (m-n)T(w) w^{m+n+1}\right]\frac{dw}{2\pi i}$$

$$= \frac{m^3-m}{12}\delta_{m+n,0} + (m-n)L_{m+n}$$

Primary field:

Define a primary field to be any field that corresponds to a vacuum under the state-field correspondence. All other fields are said to be secondary.

The primary field corresponding to the momentum vacuum $|p\rangle$ is denoted by

$$V_p(z)$$

we must have $\lim_{z\to 0} V_p(z)|0\rangle = |p\rangle$. Aside from $V_0(z)=1$, we haven't found any primary field yet.

Being primary, or being a vacuum, depends upon the algebra.

- A free boson vacuum has $a_0|p\rangle = p|p\rangle$ and $a_n|p\rangle = 0$. $\forall n>0$
- A conformal vacuum has $L_0|h\rangle = h|h\rangle$ and $L_n|h\rangle = 0$. $\forall n>0$

A free boson primary field $V_p(z)$ is automatically a conformal primary field. because $|p\rangle$ satisfies $L_0|p\rangle = \frac{1}{2}p^2|p\rangle$ and $L_n|p\rangle = 0$ $\forall n>0$. But, conformal primary need not be free boson primaries.

Proof that $L_n|p\rangle = 0$. $\forall n>0$:

If $L_n|p\rangle \neq 0$. then $L_0L_n|p\rangle = (L_nL_0 + [L_0, L_n])|p\rangle = (L_n\frac{1}{2}p^2 - nL_n)|p\rangle$

$$= (\tfrac{1}{2}p^2 - n)L_n|p\rangle$$

So the state $L_n|p\rangle$ would have energy $\frac{1}{2}p^2 - n$. But the states of the Fock space have energy at least $\frac{1}{2}p^2$, a contradiction. So $L_n|p\rangle = 0$

Examples:

① The field $\partial\varphi(z)$ is a conformal primary becase. the corresponding state $|\partial\varphi\rangle = a_{-1}|0\rangle$ is a conformal vaccum:

$$L_0|\partial\varphi\rangle = L_0 a_{-1}|0\rangle = (a_{-1}L_0 + [L_0, a_{-1}])|0\rangle = a_{-1}|0\rangle = |\partial\varphi\rangle \quad (E=1)$$

$$L_1|\partial\varphi\rangle = L_1 a_{-1}|0\rangle = (a_{-1}L_1 + [L_1, a_{-1}])|0\rangle = a_0|0\rangle = 0$$

$n\geq 2$ $L_n|\partial\varphi\rangle = 0$. because $L_n|\partial\varphi\rangle$ woud have energy $1-n<0$

$\partial\varphi(z)$ is not a free boson primary field. because

$$a_1|\partial\varphi\rangle = a_1a_{-1}|0\rangle = (a_{-1}a_1 + [a_1, a_{-1}])|0\rangle = |0\rangle \neq 0$$

ie. $|\partial\varphi\rangle$ is not a free boson vacuum

② $T(z)$ is a conformed primary if and only if the central charge is $c=0$.

$L_0|T\rangle = L_0L_{-2}|0\rangle = (L_{-2}L_0 + [L_0, L_{-2}])|0\rangle = 2L_{-2}|0\rangle = 2|T\rangle$ ($|T\rangle = L_{-2}|0\rangle$)

$L_1|T\rangle = L_1L_{-2}|0\rangle = (L_{-2}L_1 + [L_1, L_{-2}])|0\rangle = 3L_{-1}|0\rangle = 0$ (⊗ $L_{-1}|0\rangle = 0$ in any conformal field theory so that $\lim_{z\to 0} T(z)|0\rangle$ exists, and is $L_{-2}|0\rangle$)

$L_2|T\rangle = L_2L_{-2}|0\rangle = (L_{-2}L_2 + [L_2, L_{-2}])|0\rangle = (4L_0 + \frac{1}{2}C)|0\rangle = \frac{1}{2}C|0\rangle$

$n \geq 3$, $L_n|T\rangle = 0$ because. $L_n|T\rangle$ would have energy $2-n<0$

How to characterise a primary field using OPEs?

- Let $V_p(w)$ be a free boson primary, so $a_0|p\rangle = p|p\rangle$, $a_n|p\rangle = 0$. $\forall n>0$

  Assume $R\{\partial\varphi(z)V_p(w)\} = \sum_n \varphi_n(w)(z-w)^{-n-1}$ ← unknown field.

  apply to $|0\rangle$ and let $w\to 0$, we get:

  $LHS = \partial\varphi(z)|p\rangle = \sum_n a_n|p\rangle z^{-n-1}$ $\quad RHS = \sum_n |\varphi_n\rangle z^{-n-1}$

  So. $|\varphi_n\rangle = a_n|p\rangle$

  So. $\forall n\geq 1$. $|\varphi_n\rangle = a_n|p\rangle = 0$. $\Rightarrow \varphi_n(w) = 0$

  $n=0$, $|\varphi_0\rangle = a_0|p\rangle = p|p\rangle \Rightarrow \varphi_0(w) = pV_p(w)$

  $n\leq 1$ $(z-w)^{-n-1}$ is regular

  $\Rightarrow R\{\partial\varphi(z)V_p(w)\} = \dfrac{pV_p(w)}{z-w} +$ regular terms as $z\to w$.

- Let $\phi_h(w)$ be a conformal primary. so $L_0|h\rangle = h|h\rangle$. $L_n|h\rangle = 0$. $\forall n>0$.

  Assume $R\{T(z)\phi_h(w)\} = \sum_n \varphi_n(w)(z-w)^{-n-2}$, apply to $|0\rangle$ and let $w\to 0$, we get:

  $LHS = T(z)|h\rangle = \sum_n L_n|h\rangle z^{-n-2}$ $\quad RHS = \sum_n |\varphi_n\rangle z^{-n-2}$

  So $|\varphi_n\rangle = L_n|h\rangle$

  For $n\geq 1$. $|\varphi_n\rangle = L_n|h\rangle = 0 \Rightarrow \varphi_n(w) = 0$

  $n=0$, $|\varphi_0\rangle = L_0|h\rangle = h|h\rangle \Rightarrow \varphi_0(w) = h\phi_h(w)$

  $n=-1$, $|\varphi_{-1}\rangle = L_{-1}|h\rangle \Rightarrow \varphi_{-1}(w) = \partial\phi_h(w)$ ⊗ This requires the following fact: In any CFT. if $A(z)$ corresponds to $|A\rangle$, the $\partial A(z)$ corresponds to $L_{-1}|A\rangle$

  $n\leq -2$ $(z-w)^{-n-2}$ is regular.

  $\Rightarrow R\{T(z)\phi_h(w)\} = \dfrac{h\phi_h(w)}{(z-w)^2} + \dfrac{\partial\phi_h(w)}{(z-w)} +$ regular terms.

Correlation Functions:

In quantum mechanics. all physical quantities may be expressed in terms of amplitudes $\langle\phi|\psi\rangle$. eg. $\langle\phi|H|\phi\rangle$ is the energy of the state ϕ.

In quantum field theory, this is generalised so that the physical quantities are related to

$$\langle \phi | A(z) B(w) \cdots C(x) | \varphi \rangle .$$

By the state-field correspondence, state $|\varphi\rangle$ can be replaced by $\varphi(z)|0\rangle$, if the limit $z \to 0$ is taken. Therefore, we consider the correlation functions:

$$\langle 0 | \underbrace{A(z) B(w) \cdots C(x)}_{\text{radially-ordered}} | 0 \rangle \equiv \langle A(z) B(w) \cdots C(x) \rangle$$

We interpret $|0\rangle$ as being at $t = -\infty$, $C(x)$ as acting at time $|x|$, ..., $A(z)$ as acting at time $|z| > |w| > \cdots > |x|$. $\langle 0|$ then projects the result onto the true vacuum — so we interpret bras $\langle \phi |$ as being at $t = +\infty$.

Recall that operators act on bras using the adjoint:

$$\langle 0 | a_n = (a_n^\dagger |0\rangle)^\dagger = (a_{-n}|0\rangle)^\dagger$$

Thus, $\langle 0 | a_n = 0$ for $n \leq 0$.

Examples:

① Correlators involving only the identity field are always 1.

$$\langle 0 | 1 \cdots 1 | 0 \rangle = \langle 0 | 0 \rangle = 1$$

② $\langle 0 | \partial\varphi(z) | 0 \rangle = \sum_n \langle 0 | a_n | 0 \rangle z^{-n-1} = 0$ $\quad \left[\begin{array}{l} a_n|0\rangle = 0 \ \forall n \geq 0 \\ \langle 0 | a_n = 0 \ \forall n \leq 0 \end{array} \right]$

③ $\langle 0 | :\partial\varphi(z)\partial\varphi(w): | 0 \rangle = \sum_{r,s} \langle 0 | :a_r a_s: | 0 \rangle z^{-r-1} w^{-s-1}$

$$= \sum_{r\leq -1} \sum_s \underbrace{\langle 0 | a_r a_s | 0 \rangle}_{0} z^{-r-1} w^{-s-1} + \sum_{r\geq 0} \sum_s \underbrace{\langle 0 | a_s a_r | 0 \rangle}_{0} z^{-r-1} w^{-s-1}$$

$$= 0$$

normally-ordered correlators always vanish!

④ $\langle 0 | T(z) | 0 \rangle = \frac{1}{2} \sum_{r,s} :a_r a_s: z^{-r-s-2} = 0$

⑤ $\langle 0 | R\{\partial\varphi(z)\partial\varphi(w)\} | 0 \rangle = \langle 0 | \frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w): | 0 \rangle = \frac{1}{(z-w)^2}$

Correlators of n field are called n-point functions.

⑥ $\langle 0 | R\{T(z)T(w)\} | 0 \rangle = \langle 0 | \frac{c/2}{(z-w)^4} + \cancel{\frac{2T(w)}{(z-w)^2}} + \frac{:\partial\varphi(z)\partial\varphi(w):}{(z-w)^2} | 0 \rangle$

$$= \frac{c/2}{(z-w)^4}$$

Correlators involving only $\partial\varphi(z)$ and derivatives can be computed using Wick's theorem:

$$\langle 0 | R\{\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\} | 0 \rangle$$

$$= \langle 0| :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3): |0\rangle + \langle 0| :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3): |0\rangle$$

$$+ \langle 0| :\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3): |0\rangle$$

$$= \frac{\langle 0|\partial\varphi(z_2)|0\rangle}{(z_1-z_3)^2} + \frac{\langle 0|\partial\varphi(z_3)|0\rangle}{(z_1-z_2)^2} + \frac{\langle 0|\partial\varphi(z_1)\rangle}{(z_2-z_3)^2}$$

$$= 0$$

$$\langle 0| R\{\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4)\}|0\rangle$$

$$= \langle 0| z_1 z_2 z_3 z_4 |0\rangle + \langle 0| z_1 z_2 z_3 z_4 |0\rangle + \langle 0| z_1 z_2 z_3 z_4 |0\rangle$$

$$= \frac{1}{(z_1-z_2)^2(z_3-z_4)^2} + \frac{1}{(z_1-z_3)^2(z_2-z_4)^2} + \frac{1}{(z_1-z_4)^2(z_2-z_3)^2}$$

Constraints on correlators:

Let $V_{p_1}(z_1), \ldots, V_{p_n}(z_n)$ be free boson primaries.

$$[a_m, V_p(w)] = \oint_{0,\, |z|>|w|} \partial\varphi(z) V_p(w) z^m \frac{dz}{2\pi i} - \oint_{0,\, |z|<|w|} V_p(w)\partial\varphi(z) z^m \frac{dz}{2\pi i}$$

$$= \oint_w R\{\partial\varphi(z) V_p(w)\} z^m \frac{dz}{2\pi i}$$

$$= \oint_w \frac{p V_p(w)}{z-w} z^m \frac{dz}{2\pi i}$$

$$= p w^m V_p(w)$$

In particular. $[a_0, V_p(w)] = p V_p(w)$

Since $a_0|0\rangle = 0$ and $\langle 0|a_0 = 0$, we compute

$$0 = \langle 0| a_0 V_{p_1}(z_1) \cdots V_{p_n}(z_n)|0\rangle$$

$$= \sum_{j=1}^{n} \langle 0| V_{p_1}(z_1) \cdots [a_0, V_{p_j}(z_j)] \cdots V_{p_n}(z_n)|0\rangle + \langle 0| V_{p_1}(z_1) \cdots V_{p_n}(z_n) a_0|0\rangle$$

$$= \sum_{j=1}^{n} p_j \cdot \langle 0| V_{p_1}(z_1) \cdots V_{p_n}(z_n)|0\rangle$$

Conclusion: A correlator of free boson primaries is 0 unless their momenta sum to zero.

Interpretation: Momentum is conserved: $|0\rangle$ has $p=0$ and we want to compare with $\langle 0|$, which likewise has $p=0$.

Let $\phi_{h_1}(z_1), \ldots, \phi_{h_n}(z_n)$ be conformal primaries.

$$[L_m, \phi_h(w)] = \oint_{0,\, |z|>|w|} T(z)\phi_h(w) z^{m+1} \frac{dz}{2\pi i} - \oint_{0,\, |z|<|w|} \phi_h(w) T(z) z^{m+1} \frac{dz}{2\pi i}$$

$$= \oint_w \mathcal{R}\{T(z)\phi_h(w)\} z^{m+1}\frac{dz}{2\pi i} = \oint_w \left[\frac{h\phi_h(w)z^{m+1}}{(z-w)^2} + \frac{\partial\phi_h(w)z^{m+1}}{z-w}\right]\frac{dz}{2\pi i}$$

$$= h(m+1)w^m\phi_h(w) + z^{m+1}\partial\phi_h(w)$$

We consider L_{-1}, L_0, L_1 as they each annihilate $|0\rangle$ and $\langle 0|$.

$$\Rightarrow\ 0 = \langle 0|L_m\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle = \sum_{j=1}^{n}\langle 0|\phi_{h_1}(z_1)\cdots[L_m, \phi_{h_j}(z_j)]\cdots\phi_{h_n}(z_n)|0\rangle$$

$$\sum_{j=1}^{n}\langle 0|\phi_{h_1}(z_1)\cdots[h_j(m+1)z_j^m\phi_{h_j}(z_j) + z_j^{m+1}\partial_j\phi_{h_j}(z_j)]\cdots\phi_{h_n}(z_n)|0\rangle \qquad \partial_j = \partial_{z_j}$$

$$= \sum_{j=1}^{n}[h_j(m+1)z_j^m + z_j^{m+1}\partial_j]\cdot\langle 0|\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle$$

These are constraint equations for the n point function! Explicity:

$$m=-1 \Rightarrow \sum_{j=1}^{n}\partial_j\langle 0|\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle = 0 \qquad (-1)$$

$$m=0 \Rightarrow \sum_{j=1}^{n}(z_j\partial_j + h_j)\langle 0|\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle = 0 \qquad (0)$$

$$m=1 \Rightarrow \sum_{j=1}^{n}(z_j^2\partial_j + 2h_jz_j)\langle 0|\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)|0\rangle = 0 \qquad (1)$$

Let's solve them:

$n=1$: $(-1) \Rightarrow \partial_1\langle 0|\phi_{h_1}(z_1)|0\rangle = 0 \Rightarrow \langle 0|\phi_{h_1}(z_1)|0\rangle$ is constant

$(0) \Rightarrow (z_1\partial_1 + h_1)\langle 0|\phi_{h_1}(z_1)|0\rangle = 0 \Rightarrow \langle 0|\phi_{h_1}(z_1)|0\rangle = 0$ or $h_1 = 0$.

Note: $\langle 0|\mathbb{1}|0\rangle = 1$. and indeed. $\mathbb{1} = \phi_0(z_1)$ has $h=0$

$n=2$: $(-1) \Rightarrow (\partial_1 + \partial_2)\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)|0\rangle = 0$

change coordinates to $z = z_1 + z_2$ and $z_{12} = z_1 - z_2$

$\Rightarrow \partial\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)|0\rangle = 0 \Rightarrow \langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)|0\rangle = f(z_{12})$

$(0) \Rightarrow (z_{12}\partial_{12} + h_1 + h_2)f(z_{12}) = 0 \Rightarrow f(z_{12}) = \frac{C_{12}}{z_{12}^{h_1+h_2}}$

$(1) \Rightarrow [2z_{12}\partial_{12} + (h_1+h_2)z + (h_1-h_2)z_{12}]f(z_{12}) = 0$

Substitute for f to get $(h_1-h_2)\frac{C_{12}}{z_{12}^{h_1+h_2-1}} = 0 \Rightarrow C_{12} = 0$ or $h_1 = h_2$

$$\therefore\ \langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)|0\rangle = \frac{C_{12}\,\delta_{h_1,h_2}}{(z_1-z_2)^{2h_1}}$$

$n=3$: $(-1) \Rightarrow (\partial_1 + \partial_2 + \partial_3)\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)|0\rangle = 0$

$(0) \Rightarrow (z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + h_1 + h_2 + h_3)\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)|0\rangle = 0$

$(1) \Rightarrow (z_1^2\partial_1 + z_2^2\partial_2 + z_3^2\partial_3 + h_1z_1 + h_2z_2 + h_3z_3)\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)|0\rangle = 0$

change coordinates to $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

$\langle 0|\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)|0\rangle$
$= f(z_1, z_2, z_3)$
$= f(a, b, c)$

So. $\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix} \qquad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

After a long boring calculation, we get:

$(-1) \Rightarrow \quad \partial_c f(a,b,c) = 0$

$(0) \Rightarrow \quad (a\partial_a + b\partial_b + h_1 + h_2 + h_3) f(a,b,c) = 0$

$(1) \Rightarrow \quad \left[\frac{1}{3}a^2\partial_a - \frac{1}{3}b^2\partial_b + \frac{2}{3}ab\partial_a + \frac{2}{3}ac\partial_a - \frac{2}{3}ab\partial_b + \frac{2}{3}bc\partial_b + a\left(\frac{4h_1}{3} - \frac{2h_2}{3} - \frac{2h_3}{3}\right) + b\left(\frac{2h_1}{3} + \frac{2h_2}{3} - \frac{4h_3}{3}\right) + c\left(\frac{2h_1}{3} + \frac{2h_2}{3} + \frac{2h_3}{3}\right)\right] f(a,b,c) = 0$

$\Rightarrow \left[\left(\frac{a}{3} + \frac{2b}{3} + \frac{2c}{3}\right) a\partial_a + \left(-\frac{2a}{3} - \frac{b}{3} + \frac{2c}{3}\right) b\partial_b + a\left(\frac{4h_1}{3} - \frac{2h_2}{3} - \frac{2h_3}{3}\right) + b\left(\frac{2h_1}{3} + \frac{2h_2}{3} - \frac{4h_3}{3}\right) + c\left(\frac{2h_1}{3} + \frac{2h_2}{3} + \frac{2h_3}{3}\right)\right] f(a,b,c) = 0$

$\Rightarrow \left[\left(\frac{a}{3} + \frac{2b}{3} + \frac{2c}{3}\right)(a\partial_a + b\partial_b) - (a+b)b\partial_b + \cdots \cdots\right.$

$\Rightarrow \left[-(a+b)b\partial_b + a(h_1 - h_2 - h_3) - 2bh_3\right] f(a,b,c) = 0$

$\Rightarrow \quad \frac{\partial_b f}{f} = -\frac{h_2 + h_3 - h_1}{b} - \frac{h_3 + h_1 - h_2}{a+b}$

$\Rightarrow \quad f = \frac{g(a)}{b^{h_2+h_3-h_1}(a+b)^{h_3+h_1-h_2}}$ $\quad g(a)$ is a function of a.

Put above equation into (0), one can easily find that

$$g(a) = C_{123} / a^{h_2+h_1-h_3} \qquad C_{123} \text{ is a constant.}$$

$$\text{So. } f(z_1, z_2, z_3) = \frac{C_{123}}{a^{h_2+h_1-h_3}\, b^{h_2+h_3-h_1}\, (a+b)^{h_3+h_2-h_1}}$$

$$= \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_1-z_3)^{h_1+h_3-h_2}}$$

If we include the antiholomorphic contributions, then we could get, eg.

$$\langle 0|\phi_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\phi_{h_2,\bar{h}_2}(z_2,\bar{z}_2)|0\rangle = \frac{C_{12}\,\delta_{h_1=h_2}\,\delta_{\bar{h}_1=\bar{h}_2}}{(z_1-z_2)^{2h_1}(\bar{z}_1-\bar{z}_2)^{2\bar{h}_1}}$$

$$= \frac{C_{12}\, e^{2i(h-\bar{h})\arg z_{12}}}{|z_1-z_2|^{2(h_1+\bar{h}_1)}}\,\delta_{h_1,h_2}\,\delta_{\bar{h}_1,\bar{h}_2}$$

This power law decay is the basis for CFT predictions of universal scaling laws and critical exponents!

The Free Fermion.

Just as the free boson describes a massless spinless bosonic string. The free fermion describes a massless spin $-\frac{1}{2}$ fermion string. Thus, the fermion field has a $j=\frac{1}{2}$ and $j=-\frac{1}{2}$ component:

$$\Psi(t,x)=\begin{pmatrix}\psi(t,x)\\ \bar\psi(t,x)\end{pmatrix}.$$

Being fermionic, we may impose periodic ~~and~~ or antiperiodic boundary conditions on the cylinder, e.g.

$$\psi(t,x+L)=\begin{cases}+\psi(t,x) & \text{periodic (Ramond sector)}\\ -\psi(t,x) & \text{antiperiodic (Neveu-Schwarz sector)}\end{cases}$$

With $\bar\psi$, there are four sectors in total!

The action $S[\Psi]=\frac{1}{4\pi}\int_{cyl}\Psi^+(t,x)\gamma^t\gamma^\mu\partial_\mu\Psi(t,x)\,dx\,dt$ $\quad\mu=t,x.$

$$=\frac{1}{4\pi}\int_{cyl}(\psi\bar\partial\psi+\bar\psi\partial\bar\psi)\,dz\,d\bar z$$

where $z=t+ix$ and the gamma matrices are $\gamma^t=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\gamma^x=i\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ in Euclidian metric with

$$\gamma^t(\gamma^t\partial_t+\gamma^x\partial_x)=2\begin{pmatrix}\bar\partial&0\\0&\partial\end{pmatrix}$$

the equations of motion are $\partial\bar\psi=\bar\partial\psi=0$. ie. $\psi=\psi(z)$ and $\bar\psi=\bar\psi(\bar z)$

infinitesimal

Pf: $S[\Psi']=\frac{1}{4\pi}\int[(\psi+\eta)\bar\partial(\psi+\eta)+(\bar\psi+\xi)\partial(\bar\psi+\xi)]\,dz\,d\bar z$ $\quad(\psi'=\psi+\eta,\ \bar\psi'=\bar\psi+\xi)$

$$=\frac{1}{4\pi}\int_{cyl}[\psi\bar\partial\psi+\psi\bar\partial\eta+\eta\bar\partial\psi+\bar\psi\partial\bar\psi+\bar\psi\partial\xi+\xi\partial\bar\psi]\,dz\,d\bar z$$

$$=S[\Psi]+\frac{1}{4\pi}\int_{cyl}[\psi\bar\partial\eta+\eta\bar\partial\psi+\bar\psi\partial\xi+\xi\partial\bar\psi]\,dz\,d\bar z$$

分部积分 $=S[\Psi]+\frac{1}{4\pi}\int_{cyl}[-(\bar\partial\psi)\eta+\eta\bar\partial\psi-(\partial\bar\psi)\xi+\xi\partial\bar\psi]\,dz\,d\bar z$

$$=S[\Psi]+\frac{1}{2\pi}\int_{cyl}[\eta\bar\partial\psi+\xi\partial\bar\psi]\,dz\,d\bar z$$

Transforming to complex coordinates. $w=e^{2\pi z/L}$ (Conformal transformation)

$\partial=\partial_w\frac{2\pi w}{L}$, $\bar\partial=\bar\partial_{\bar w}\frac{2\pi\bar w}{L}$ $\qquad dw=\frac{2\pi w}{L}dz$, $d\bar w=\frac{2\pi\bar w}{L}d\bar z$

$$S[\Psi]=\frac{1}{4\pi}\int_{cyl}[\psi(z)\bar\partial\psi(z)+\bar\psi(\bar z)\partial\bar\psi(\bar z)]\,dz\,d\bar z$$

$$=\frac{1}{4\pi}\int_{cyl}\left[\psi(z)\frac{2\pi\bar w}{L}\bar\partial_{\bar w}\psi(z)+\bar\psi(\bar z)\frac{2\pi w}{L}\partial_w\bar\psi(\bar z)\right]\frac{L^2}{4\pi^2 w\bar w}\,dw\,d\bar w$$

$$=\frac{1}{4\pi}\int_{cyl}\left[\psi(z)\frac{L}{2\pi w}\bar\partial_{\bar w}\psi(z)+\bar\psi(\bar z)\frac{L}{2\pi\bar w}\partial_w\bar\psi(\bar z)\right]dw\,d\bar w$$

Set: $\psi(w)=\left(\frac{L}{2\pi w}\right)^{1/2}\psi(z)$ and $\bar\psi(\bar w)=\left(\frac{1}{2\pi\bar w}\right)^{1/2}\bar\psi(\bar z)$

ie. $\psi(t,x)=\psi(z)=\left(\frac{2\pi w}{L}\right)^{1/2}\psi(w)$

$$S[\Psi]=\frac{1}{4\pi}\int_{\mathbb{C}}[\psi\bar\partial\psi+\bar\psi\partial\bar\psi]\,dw\,d\bar w$$

Since $x \to x+L$ amounts to $w \to e^{2\pi i} w$ on the plane, the nature of the boundary conditions swaps as we transform:

$$\psi(t, x+L) = e^{i\pi}\sqrt{\frac{2\pi w}{L}}\,\psi(e^{2\pi i} w) = \begin{cases} \psi(t,x) = \sqrt{\frac{2\pi w}{L}}\,\psi(w) & \text{(Ramond Sector)} \\ -\psi(t,x) = -\sqrt{\frac{2\pi w}{L}}\,\psi(w) & \text{(Neveu-Schwarz Sector)} \end{cases}$$

$$\Rightarrow \quad \psi(e^{2\pi i} w) = \begin{cases} -\psi(w) & \text{antiperiodic (Ramond Sector)} \\ \psi(w) & \text{periodic (Neveu-Schwarz Sector)} \end{cases}$$

As the string is spin $\frac{1}{2}$, the angular momentum operator $L_0 - \bar{L}_0$ should have eigenvalue $+\frac{1}{2}$ on $\psi(w)$ and $-\frac{1}{2}$ on $\bar{\psi}(\bar{w})$. Thus, the eigenvalues of $L_0 + \bar{L}_0$ must be $+\frac{1}{2}$ and $+\frac{1}{2}$, respectively. So we propose the decompositions:

$$\psi(w) = \begin{cases} \sum_{n\in\mathbb{Z}} b_n w^{-n-\frac{1}{2}} & \text{(Ramond)} \\ \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n w^{-n-\frac{1}{2}} & \text{(Neveu-Schwarz)} \end{cases} \qquad b_n = \oint_0 \psi(w)\, w^{n-\frac{1}{2}} \frac{dw}{2\pi i}$$

Canonical quantisation gives anticommutation relations:

$$b_m b_n + b_n b_m = \{b_m, b_n\} = \delta_{m+n,0}$$

Pf: The action defined in page 21 is in Euclidian form. So t actually repersents τ, and the wick rotation is $t = -i\tau \Rightarrow \tau = it$, $\partial_\tau = -i\partial_t$, $d\tau = i\,dt$

So: $$S[\Psi] = \frac{1}{4\pi}\int_{cyl} \psi(\partial_\tau + i\partial_x)\,dx\,d\tau \qquad \text{ignore the } \bar{\psi} \text{ part}$$
$$= \frac{1}{4\pi}\int_{cyl} \psi(-i\partial_t + i\partial_x)\,dx\,dt$$

$$\psi(t,x) = \sum_n \psi_n(t) e^{2\pi i n x/L} \Rightarrow S[\psi] = \frac{1}{4\pi}\int_{cyl} \sum_m \psi_m(t) e^{2\pi i m x/L}\left(-i\sum_n \dot{\psi}_n(t) e^{2\pi i n x/L} - \sum_n \psi_n(t)\frac{2\pi n}{L} e^{2\pi i n x/L}\right) dx\,dt$$
$$= \frac{L}{4\pi}\int_{-\infty}^{\infty} \sum_n \left[-i\psi_{-n}(t)\dot{\psi}_n(t) - \psi_{-n}(t)\psi_n(t)\frac{2\pi n}{L}\right] dt$$

$$\Pi_n(t) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_n(t)} = \frac{iL}{2\pi}\psi_{-n}(t) \qquad \left(\text{note: } \psi \text{ is grassmann field and } \psi_{-n}(t)\dot{\psi}_n(t)\,dt \xrightarrow{\text{分部积分}} -\dot{\psi}_{-n}(t)\psi_n(t)\,dt\right)$$

$$\therefore \{\psi_m(t), \Pi_n(t)\} = i\delta_{m,n} \longrightarrow \{\psi_m(t), \psi_{-n}(t)\} = \frac{2\pi}{L}\delta_{m,n} \longrightarrow \{\psi_{-m}(t), \psi_{-n}(t)\} = \frac{2\pi}{L}\delta_{m+n,0}$$

$$b_n = \oint_0 \psi(w)\, w^{n-\frac{1}{2}}\frac{dw}{2\pi i} \xrightarrow{\text{固定t的围路}} \int_0^L \psi(w)\, w^{n-\frac{1}{2}}\frac{1}{2\pi i}\frac{2\pi i w}{L}\,dx \qquad \left(w = e^{2\pi(\tau + ix)/L}\right)$$
$$= \frac{1}{\sqrt{2\pi L}}\int_0^L \psi(w)\left(\frac{2\pi w}{L}\right)^{1/2} w^n\,dx$$
$$= \frac{1}{\sqrt{2\pi L}}\int_0^L \psi(t,x)\, e^{2\pi n(\tau + ix)/L}\,dx$$
$$= \frac{1}{\sqrt{2\pi L}}\int_0^L \sum_m \psi_m(t)\, e^{2\pi i m x/L} e^{2\pi n(\tau + ix)/L}\,dx$$
$$= \sqrt{\frac{L}{2\pi}}\,\psi_{-n}(t)\, e^{2\pi n\tau/L}$$

$$\therefore \{b_m, b_n\} = \frac{L}{2\pi}\{\psi_{-m}(t), \psi_{-n}(t)\}\, e^{2\pi(m+n)\tau/L}$$
$$= \delta_{m+n,0}$$

Fock spaces:

In each sector, let the b_n with $n>0$ be annihilators and the b_n with $n<0$ be creators.

In the Ramond sector, we have b_0. It is not a zero-mode as $b_0|\lambda\rangle=\lambda|\lambda\rangle$, which would require that $|\lambda\rangle$ be both bosonic and fermionic! It is not an annihilator as $b_0|\lambda\rangle=0$ contradicts $\frac{1}{2}|\lambda\rangle=b_0^2|\lambda\rangle=b_0(b_0|\lambda\rangle)$ $\therefore$ b_0 is a creator.

Without any zero-modes, we cannot distinguish vacua. $\therefore$ there is a single vaccum $|NS\rangle$ in the Neveu-Schwarz and a single vaccum $|R\rangle$ in the Ramond sector.

$$\begin{array}{lllll}
 & \vdots & & \vdots & \vdots \\
 & b_{-\frac{3}{2}}b_{-\frac{1}{2}}|NS\rangle & & & \\
(b_{-\frac{1}{2}}^2=0) & b_{-\frac{3}{2}}|NS\rangle & b_{-2}|R\rangle & b_{-2}b_0|R\rangle & (b_{-1}^2=0) \\
 & b_{-\frac{1}{2}}|NS\rangle & b_{-1}|R\rangle & b_{-1}b_0|R\rangle & \\
 & |NS\rangle & |R\rangle & b_0|R\rangle & \longleftarrow \text{degenerate!}
\end{array}$$

Which one is the true vacuum?

It can't be $|R\rangle$ because

$$\lim_{z\to 0}\psi(z)|R\rangle=\lim_{z\to 0}\sum_{n\in\mathbb{Z}}b_n|R\rangle z^{-n-\frac{1}{2}}=\lim_{z\to 0}\left[b_0|R\rangle z^{-\frac{1}{2}}+b_{-1}|R\rangle z^{\frac{1}{2}}+\cdots\right],$$

which doesn't exist! However,

$$\lim_{z\to 0}\psi(z)|NS\rangle=\lim_{z\to 0}\sum_{n\in\mathbb{Z}+\frac{1}{2}}b_n|NS\rangle z^{-n-\frac{1}{2}}=\lim_{z\to 0}\left[b_{-\frac{1}{2}}|NS\rangle+b_{-\frac{3}{2}}|NS\rangle z+\cdots\right]=b_{-\frac{1}{2}}|NS\rangle$$

does. We write $|0\rangle\equiv|NS\rangle$ and $|\psi\rangle=b_{-\frac{1}{2}}|0\rangle$.

Radical-Ordering:

Swapping fermionic field gives a sign. Let the parity of the field $A(z)$ be

$$\bar{A}=\begin{cases}0 & \text{if } A(z) \text{ is bosonic}\\ 1 & \text{if } A(z) \text{ is fermionic}\end{cases}.$$

Define

$$R\{A(z)B(w)\}=\begin{cases}A(z)B(w) & \text{if } |z|>|w|\\ (-1)^{\bar{A}\bar{B}}B(w)A(z) & \text{if } |z|<|w|\end{cases}$$

Operator Product Expansion

$$R\{\psi(z)\psi(w)\} \overset{\text{assume}}{=\!=} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \phi_n(w)(z-w)^{-n-\frac{1}{2}}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n|\psi\rangle z^{-n-\frac{1}{2}} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} |\phi_n\rangle z^{-n-\frac{1}{2}} \qquad (\text{Note } |0\rangle \equiv |NS\rangle)$$

$$\Rightarrow |\phi_n\rangle = b_n|\psi\rangle = b_n b_{-\frac{1}{2}}|0\rangle$$

$n>\frac{1}{2} \Rightarrow |\phi_n\rangle = b_n b_{-\frac{1}{2}}|0\rangle = -b_{-\frac{1}{2}} b_n|0\rangle = 0 \Rightarrow \phi_n(w)=0$

$n=\frac{1}{2} \Rightarrow |\phi_{1/2}\rangle = b_{\frac{1}{2}} b_{-\frac{1}{2}}|0\rangle = (-b_{-\frac{1}{2}} b_{\frac{1}{2}}+1)|0\rangle = |0\rangle \Rightarrow \phi_{1/2}(w) = 1$

$n<\frac{1}{2} \Rightarrow \phi_n(w)(z-w)^{-n-\frac{1}{2}}$ is regular.

$$\therefore\ \underline{R\{\psi(z)\psi(w)\} \sim \frac{1}{z-w}}$$

Note that. $R\{\psi(z)\psi(w)\} \sim \frac{1}{z-w} = -\frac{1}{w-z} = -R\{\psi(w)\psi(z)\}$

Stress-Energy Tenser:

For the free boson, $T(z)$ was a dimension 2 field:

$$|T\rangle = L_{-2}|0\rangle = \tfrac{1}{2}a_{-1}^2|0\rangle. \quad L_0|T\rangle = 2|T\rangle$$

We therefore guess that.

$$|T\rangle = \alpha\, b_{-\frac{3}{2}} b_{-\frac{1}{2}}|0\rangle = -\alpha\, b_{-\frac{1}{2}} b_{-\frac{3}{2}}|0\rangle$$

$$\Rightarrow \quad T(z) = -\alpha :\psi(z)\partial\psi(z):$$

where the normal-ordering is defined, IN THE NS SECTOR, by.

$$:b_m b_n: = \begin{cases} b_m b_n & \text{if } m\le -\frac{1}{2} \\ -b_n b_m & \text{if } m \ge \frac{1}{2} \end{cases}.$$

Since $\psi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{-n-\frac{1}{2}}$. $\partial\psi(z) = -\sum_{n\in\mathbb{Z}+\frac{1}{2}} (n+\frac{1}{2}) b_n z^{-n-3/2}$

$$\Rightarrow \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = T(z) = +\alpha \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} (n-r+\tfrac{1}{2}) :b_r b_{n-r}: z^{-n-2}$$

$$\Rightarrow L_n = \alpha \sum_{r\in\mathbb{Z}+\frac{1}{2}} (n-r+\tfrac{1}{2}) :b_r b_{n-r}:$$

$$= \alpha \sum_{r\le -\frac{1}{2}} (n-r+\tfrac{1}{2}) b_r b_{n-r} - \alpha \sum_{r\ge\frac{1}{2}} (n-r+\tfrac{1}{2}) b_{n-r} b_r.$$

Proofment of $T(z) = -\alpha :\psi(z)\partial\psi(z):$

$$\lim_{z\to 0} T(z)|0\rangle = -\alpha \lim_{z\to 0} :\psi(z)\partial\psi(z):|0\rangle = \sum_{n\in\mathbb{Z}} \lim_{z\to 0} L_n|0\rangle z^{-n-2}$$

From above definition of L_n, we can easily get.

for $n\ge 0 \qquad L_n|0\rangle = 0$

$n=-2 \qquad L_{-2}|0\rangle = \alpha \sum_{r\le -\frac{1}{2}} (-2-r+\frac{1}{2}) b_r b_{-2-r}|0\rangle - \alpha \sum_{r\ge\frac{1}{2}} (-2-r+\frac{1}{2}) b_{-2-r} b_r|0\rangle$

$$= \alpha \sum_{r\le -\frac{5}{2}} (-2-r+\tfrac{1}{2}) b_r b_{-2-r}|0\rangle + \alpha(-1) b_{-\frac{3}{2}} b_{-\frac{1}{2}}|0\rangle + \alpha(-2) b_{-\frac{1}{2}} b_{-\frac{3}{2}}|0\rangle$$

$$= \alpha\, b_{-\frac{3}{2}} b_{-\frac{1}{2}}|0\rangle$$

Recall that $L_{-1}|0\rangle = 0$ in any conformal field theory. (See below)

$\therefore \lim_{z\to 0} T(z)|0\rangle = \alpha b_{-\frac{3}{2}} b_{-\frac{1}{2}}|0\rangle = |T\rangle$.

Wick's theorem works for OPEs of $\psi(z)$ and its derivatives, but we need to introduce a sign every time we exchange two fermions to contract:

e.g. $\psi(z)\partial\psi(x)\psi(w) = \frac{\psi(w)}{(z-x)^2}$, $\psi(z)\partial\psi(x)\psi(w) = -\partial\psi(x)\psi(z)\psi(w) = \frac{-\partial\psi(x)}{z-w}$

since $R\{\psi(z)\partial\psi(x)\} = \partial_x R\{\psi(z)\psi(x)\} \sim \partial_x \frac{1}{z-x} = \frac{1}{(z-x)^2}$

$$\therefore R\{T(z)\psi(w)\} = -\alpha R\{:\psi(z)\partial\psi(z):\psi(w)\}$$
$$\sim -\alpha:\psi(z)\partial\psi(z):\psi(w) - \alpha:\psi(z)\partial\psi(z):\psi(w)$$
$$= \frac{\alpha\partial\psi(z)}{z-w} + \frac{\alpha\psi(z)}{(z-w)^2}$$
$$\sim \frac{\alpha\psi(w)}{(z-w)^2} + \frac{2\alpha\partial\psi(w)}{z-w}$$

since $R\{\partial\psi(z)\psi(w)\} = \partial_z R\{\psi(z)\psi(w)\} \sim \partial_z \frac{1}{z-w} = \frac{-1}{(z-w)^2}$

Actually $\alpha = \frac{1}{2}$. and $T(z) = -\frac{1}{2}:\psi(z)\partial\psi(z):$

Exercise: $R\{T(z)T(w)\} = \frac{1}{4}R\{:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w):\}$

$$= \frac{1}{4}\Big[:\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): + :\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w):$$
$$+ :\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): + :\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w):$$
$$+ :\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w): + :\psi(z)\partial\psi(z)::\psi(w)\partial\psi(w):\Big]$$

$R\{\partial\psi(z)\partial\psi(w)\} \sim \partial_z \frac{1}{(z-w)^2} = \frac{-2}{(z-w)^3}$

$$= \frac{1}{4}\Big[\frac{-:\partial\psi(z)\partial\psi(w):}{z-w} + \frac{:\partial\psi(z)\psi(w):}{(z-w)^2} + \frac{-:\psi(z)\partial\psi(w):}{(z-w)^2}$$
$$+ \frac{2:\psi(z)\psi(w):}{(z-w)^3} + \frac{1}{z-w}\frac{+2}{(z-w)^3} + \frac{1}{(z-w)^2}\frac{-1}{(z-w)^2}\Big]$$

$R\{\partial\psi(z)\psi(w)\} \sim \partial_z \frac{1}{z-w} = \frac{-1}{(z-w)^2}$

$\partial T(w) = -\frac{1}{2}:\partial\psi(w)\partial\psi(w): -\frac{1}{2}:\psi(w)\partial^2\psi(w):$

$$\sim \frac{1}{4}\Big[\frac{-:\partial\psi(w)\partial\psi(w):}{z-w} + \frac{:\partial\psi(w)\psi(w):}{(z-w)^2} + \frac{:\partial^2\psi(w)\psi(w):}{z-w} - \frac{:\psi(w)\partial\psi(w):}{(z-w)^2}$$
$$- \frac{:\partial\psi(w)\partial\psi(w):}{z-w} + \underbrace{\frac{2:\psi(w)\psi(w):}{(z-w)^3}}_{=0} + \frac{2:\partial\psi(w)\psi(w)}{(z-w)^2} + \frac{:\partial^2\psi(w)\psi(w):}{(z-w)} + \frac{1}{(z-w)^4}\Big]$$

$$= \frac{1/4}{(z-w)^4} + \frac{3:\partial\psi(w)\psi(w): - :\psi(w)\partial\psi(w):}{4(z-w)^2} + \frac{2:\partial^2\psi(w)\psi(w): - 2:\partial\psi(w)\partial\psi(w):}{4(z-w)}$$

$$= \frac{1/4}{(z-w)^4} + \frac{-:\psi(w)\partial\psi(w):}{(z-w)^2} + \frac{\frac{1}{2}:\psi(w)\partial^2\psi(w): - \frac{1}{2}:\partial\psi(w)\partial\psi(w):}{(z-w)}$$

$$= \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$\therefore c = \frac{1}{2}$ (It can be found by calculating $[L_m, L_n]$ from $R\{T(z)T(w)\}$)

This demonstrate that the free fermion is a CFT.

We can now compute the energy of the vacua $|0\rangle = |NS\rangle$ and $|R\rangle$.

For $|NS\rangle$, this is easy and the answer is as expected:

$$L_0|NS\rangle = -\frac{1}{2}\sum_{r\le -\frac{1}{2}}(r-\frac{1}{2})b_r b_{-r}|NS\rangle + \frac{1}{2}\sum_{r\ge\frac{1}{2}}(r-\frac{1}{2})b_{-r}b_r|NS\rangle = 0$$

But, we cannot use this expression for L_0 in the Ramond sector.

(the normal-ordering is only valid in the NS sector!)

To compute the energy of $|R\rangle$, we ~~can~~ derive a generalised commutation relation:

$$R\{\psi(z)\psi(w)\} = \frac{1}{z-w} + :\psi(z)\psi(w):$$

$$= \frac{1}{z-w} + \underbrace{:\psi(w)\psi(w):}_{0} + \underbrace{:\partial\psi(w)\psi(w):}_{2T(w)}(z-w) + \cdots$$

$$= \frac{1}{z-w} + 2T(w)(z-w) + \cdots$$

We therefore compute $\oint_0\oint_w R\{\psi(z)\psi(w)\} z^{m+3/2} w^{n-\frac{1}{2}} (z-w)^{-2} \frac{dz}{2\pi i}\frac{dw}{2\pi i}$ in two ways.

① $= \oint_0\oint_w \left[\frac{z^{m+3/2}w^{n-1/2}}{(z-w)^3} + \frac{2T(w)z^{m+1/2}w^{n-1/2}}{z-w}\right]\frac{dz}{2\pi i}\frac{dw}{2\pi i} = \oint_0\left[\frac{1}{2}(m+\frac{1}{2})(m+\frac{3}{2})w^{m+n-1} + 2T(w)w^{m+n+1}\right]\frac{dw}{2\pi i}$

$= \frac{1}{2}(m+\frac{1}{2})(m+\frac{3}{2})\delta_{m+n,0} + 2L_{m+n}$

② $= \oint_0\oint_0 \psi(z)\psi(w) z^{m-1/2} w^{n-1/2}(1-w/z)^{-2}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$ $\quad(|z|>|w|)$

$\quad - \oint_0\oint_0 -\psi(w)\psi(z) w^{n-5/2} z^{m+\frac{3}{2}}(1-\frac{z}{w})^{-2}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$ $\quad(|z|<|w|)$

$$\frac{1}{(1-x)^2} = \partial_x\frac{1}{1-x} = \sum_{r=1}^{\infty} r x^{r-1} = \sum_{r=0}^{\infty}(r+1)x^r \quad (|x|<1)$$

$= \sum_{r=0}^{\infty}(r+1)\left[\oint_0 \psi(z) z^{m-r-\frac{1}{2}}\frac{dz}{2\pi i}\oint_0 \psi(w) w^{n+r-\frac{1}{2}}\frac{dw}{2\pi i} + \oint_0 \psi(w) w^{n-r-\frac{5}{2}}\frac{dw}{2\pi i}\oint_0 \psi(z) z^{m+r+\frac{3}{2}}\frac{dz}{2\pi i}\right]$

$= \sum_{r=0}^{\infty}(r+1)\left[\psi_{m-r}\psi_{n+r} + \psi_{n-r-2}\psi_{m+r+2}\right]$

Thus:

$$\sum_{r=0}^{\infty}(r+1)\left[\psi_{m-r}\psi_{n+r} + \psi_{n-r-2}\psi_{m+r+2}\right] = \frac{(2m+1)(2m+3)}{8}\delta_{m+n,0} + 2L_{m+n}.$$

We apply this, with $m = -\frac{1}{2}$, $n = \frac{1}{2}$, to $|NS\rangle$

$$\sum_{r=0}^{\infty}(r+1)\left[\psi_{-r-\frac{1}{2}}\psi_{r+\frac{1}{2}} + \psi_{-r-\frac{3}{2}}\psi_{r+\frac{3}{2}}\right]|NS\rangle = 2L_0|NS\rangle \Rightarrow L_0|NS\rangle = 0$$

With $m=-1$ and $n=1$, we can apply it to $|R\rangle$

$$\sum_{r=0}^{\infty}(r+1)\left[\psi_{-r-1}\psi_{r+1} + \psi_{-r-1}\psi_{r+1}\right]|R\rangle = -\frac{1}{8} + 2L_0|R\rangle \Rightarrow L_0|R\rangle = \frac{1}{16}|R\rangle$$

The Ramond vacuum has energy $1/16$

With above equation, we can get that $|R\rangle$ and $b_0|R\rangle$ are degenerated!

Today, we will discuss CFTs that only have the conformal symmetry:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3-m}{12} \delta_{m+n,0}\, C$$

Here, the Fock spaces are built by exciting a conformal vacuum $|\phi_h\rangle$ statisfying

$$L_0|\phi_h\rangle = h|\phi_h\rangle \quad \text{and} \quad L_n|\phi_h\rangle = 0 \quad \text{for } n>0$$

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
L_{-1}^3|\phi_h\rangle & L_{-2}L_{-1}|\phi_h\rangle & L_{-3}|\phi_h\rangle & h+3 \\
L_{-1}^2|\phi_h\rangle & L_{-2}|\phi_h\rangle & & h+2 \\
L_{-1}|\phi_h\rangle & & & E = h+1 \\
|\phi_h\rangle & & & E = h
\end{array}$$

Something interesting happens when $h=0$, because the true vacuum $|\phi_0\rangle = |0\rangle$ must be annihilated by L_{-1}:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
L_{-2}^2|0\rangle & L_{-4}|0\rangle & 4 \\
L_{-3}|0\rangle & & 3 \\
L_{-2}|0\rangle = |T\rangle & & E=2 \\
|0\rangle & & E=0
\end{array}$$

- $L_{-1}|0\rangle = 0$ for the state-field correspondence to work.
  $\lim_{z\to 0} T(z)|0\rangle$ must exist!
- $L_{-1}|0\rangle$ corresponds to the derivative ($L_{-1} \leftrightarrow \partial$) of the identity field ($|0\rangle \leftrightarrow 1$)
- The true vacuum is maximally symmetric:

$$L_n|0\rangle = 0 \quad \text{for all } n \geq -1$$

Here is a fourth way of thinking about this:

- The state $L_{-1}|0\rangle$ is unphysical, meaning that it vanishes in any amplitude:
  $\langle\psi|L_{-1}|0\rangle = 0$, hence any physical measurement involving this state gives zero!

Physically, $L_{-1}|0\rangle$ is indistinguishable from the zero state!

Why is $\langle\psi|L_{-1}|0\rangle = 0$ for all $|\psi\rangle$?

1) L_0 is hermitian ($L_0^\dagger = L_0$). So its eigenvectors are orthogonal.
   $\therefore\ \langle\psi|L_{-1}|0\rangle = 0$, unless $|\psi\rangle$ has energy $E=1$.
2) Different Fock spaces are orthogonal (by definition)
   $\therefore\ \langle\psi|L_{-1}|0\rangle = 0$, unless $|\psi\rangle$ is created from $|0\rangle$ with $E=1$.

There is therefor only one possibility for $|\psi\rangle = L_{-1}|0\rangle$ itself. But then

$$\langle\psi|L_{-1}|0\rangle = \langle 0|L_1 L_{-1}|0\rangle = \langle 0|L_{-1}L_1 + [L_1, L_{-1}]|0\rangle \qquad [L_n^+ = L_{-n}]$$

$$= \langle 0|2L_0|0\rangle = 0$$

~~There~~ .

There are no unphysical states in the Fock spaces of the free boson of the free fermion.

Clarifying Example:

In quantum mechanics, a spin $\frac{1}{2}$ particle is described by $SU(2)$, where we have two states $|\frac{1}{2};\frac{1}{2}\rangle$ and $|\frac{1}{2};-\frac{1}{2}\rangle$ upon which the spin operator $S = J^3 = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and the ladder operators $J^+ = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $J^- = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ act. Here, J^3 is a zero mode (it measure the spin) and, if $|\frac{1}{2};\frac{1}{2}\rangle$ is taken as a vacuum, then J^+ is an annihilation operator while J^- is a creation operator. But, the Fock space created from $|\frac{1}{2};\frac{1}{2}\rangle$ should look like this:

$$\begin{array}{ccc} & \vdots & \\ J^- & |\tfrac{1}{2};-\tfrac{3}{2}\rangle & \\ J^- & |\tfrac{1}{2};-\tfrac{1}{2}\rangle & J^+ \\ & |\tfrac{1}{2};\tfrac{1}{2}\rangle & \end{array}$$

However, there are no states with spin $< -\frac{1}{2}$, because $|\frac{1}{2};-\frac{3}{2}\rangle$ is unphysical:

1) J^3 is hermitian, so it is orthogonal to all states with spin $\sigma \neq -\frac{3}{2}$

2) $\langle\frac{1}{2};-\frac{3}{2}|\frac{1}{2};-\frac{3}{2}\rangle = \langle\frac{1}{2};-\frac{1}{2}|J^+J^-|\frac{1}{2};-\frac{1}{2}\rangle \qquad ((J^-)^+ = J^+)$

$$= \langle\tfrac{1}{2};-\tfrac{1}{2}|J^-J^+ + [J^+, J^-]|\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

$$= \langle\tfrac{1}{2};-\tfrac{1}{2}|\tfrac{1}{2};-\tfrac{1}{2}\rangle + \langle\tfrac{1}{2};-\tfrac{1}{2}|2J^3|\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

$$= 0$$

FACT:

The Fock space built from the true vacuum by acting with conformal creators $L_n\,(n<0)$ has an unphysical state unrelated to $L_{-1}|0\rangle$ if and only if the central charge has the form:

$$C = 1 - \frac{6(p'-p)^2}{pp'}$$

where $p, p' \in \{2, 3, 4, \cdots\}$ and $\gcd\{p, p'\} = 1$.

The CFTs with $C = 1 - \frac{6(p'-p)^2}{pp'}$ and only Virasoro ~~symmetry~~ symmetry are called the Minimal Models $M(p, p')$. Because the conformal vacua $|\phi_h\rangle$ are parameterised by an energy $E \in \mathbb{R}$, it would seem that all should be allowed. But, this is false because of unphysical states!

Example: $(p, p') = (2, 3) \Rightarrow C = 0$

In this case, the extra unphysical state beyond $L_{-1}|0\rangle$ is

$$|T\rangle = L_{-2}|0\rangle$$

Check: $|T\rangle$ is the only state with energy 2. So it's enough to show that $\langle T|T\rangle = 0$

$$\langle T|T\rangle = \langle 0|L_2 L_{-2}|0\rangle = \langle 0|L_{-2}L_2 + [L_2, L_{-2}]|0\rangle$$

$$= \langle 0|4L_0 + \frac{c}{2}|0\rangle = 0$$

But, if $|T\rangle = 0$, then $T(z) = 0$ (state-field correspondence) and so $\sum_{n\in\mathbb{Z}} L_n z^{-n-2} = 0 \Rightarrow$ $L_n = 0$ for all n. As $L_0|\phi_h\rangle = h|\phi_h\rangle$ by definition, we can only have $L_0 = 0$ if $h = 0$.

$\therefore$ $|0\rangle$ is the only conformal vacuum in the minimal model $M(2, 3)$.

[In fact, $|0\rangle$ is the only state in $M(2,3)$ — the theory is trivial!]

Example: $(p, p') = (2, 5) \Rightarrow C = -\frac{22}{5}$

For $M(2,5)$, the extra unphysical state is

$$|\chi\rangle = (L_{-2}^2 - \frac{3}{5}L_{-4})|0\rangle$$

To check this, we check orthogonality to both states of energy 4: $L_{-2}^2|0\rangle$ and $L_{-4}|0\rangle$

$$\langle 0|L_4|\chi\rangle = \langle 0|L_4 L_{-2}L_{-2} - \frac{3}{5}L_4 L_{-4}|0\rangle$$

$$= \langle 0|6L_2L_{-2} - \frac{3}{5}(8L_0 + 5c)|0\rangle$$

$$= \langle 0|6(4L_0 + \frac{c}{2}) - 3c|0\rangle$$

$$= 0$$

$$\langle 0|L_2^2|\chi\rangle = \langle 0|L_2L_2L_{-2}L_{-2} - \frac{3}{5}L_2L_2L_{-4}|0\rangle$$

$$= \langle 0|L_2L_{-2}(4L_0 + \frac{c}{2}) + L_2(4L_0 + \frac{c}{2})L_{-2} - \frac{3}{5}L_2 \cdot 6L_{-2}|0\rangle$$

$$= \langle 0|\frac{c}{2}L_2L_{-2} + \frac{c}{2}L_2L_{-2} - \frac{18}{5}(4L_0 + \frac{c}{2})|0\rangle$$

$$= \langle 0|(4L_0 + \frac{c}{2})\frac{c}{2} + \frac{c}{2}(8 + \frac{c}{2}) - \frac{9c}{5}|0\rangle$$

$$= \langle 0|\frac{c^2}{4} + \frac{c}{2}(8 + \frac{c}{2}) - \frac{9c}{5}|0\rangle = 0$$

By the state-field correspondence, it follows that

$$|\chi\rangle = 0 \Rightarrow \chi(z) = :T(z)T(z): - \frac{3}{10}\partial^2 T(z) = 0 \Rightarrow \sum_{n\in\mathbb{Z}} \chi_n z^{-n-4} = 0$$

where $\chi_n = \sum_{r\in\mathbb{Z}} :L_r L_{n-r}: - \frac{3}{10}(n+2)(n+3)L_n$

$$= \sum_{r\le -2} L_r L_{n-r} + \sum_{r\ge -1} L_{n-r} L_r - \frac{3}{10}(n+2)(n+3) L_n \equiv 0$$

Acting with X_0 on the conformal vacuum $|\phi_h\rangle$ then gives zero:

$$0 = X_0|\phi_h\rangle = \sum_{r\le -2} L_r L_{-r}|\phi_h\rangle + \sum_{r\ge -1} L_{-r} L_r|\phi_h\rangle - \frac{3}{10}\cdot 2\cdot 3 L_0|\phi_h\rangle$$

$$= L_1 L_{-1}|\phi_h\rangle + L_0^2|\phi_h\rangle - \frac{9}{5} L_0|\phi_h\rangle$$

$$= 2L_0|\phi_h\rangle + h^2|\phi_h\rangle - \frac{9}{5} h|\phi_h\rangle$$

$$= (h^2 + \frac{h}{5})|\phi_h\rangle$$

$$= h(h+\frac{1}{5})|\phi_h\rangle.$$

$\therefore$ the only conformal vacua in the minimal model $M(2,5)$ are the true vacuum $|0\rangle$ and that with energy $-\frac{1}{5}$: $|\phi_{-\frac{1}{5}}\rangle$.

$M(2,5)$ is identified with the thermodynamic limit of the ~~stati~~ statistical model known as the Yang-Lee singularity, at its critical point $\langle \phi_{-1/5}(z,\bar{z})\phi_{-1/5}(w,\bar{w})\rangle = |z-w|^{4/5}$

Example:

$M(4,5)$ ~ tricritical Ising Model
$M(5,6)$ ~ 3-states potts model.

All minimal models have lattice versions as RSOS models.

Example: $(p,p') = (3,4) \Rightarrow c = \frac{1}{2}$

For $M(3,4)$, the extra unphysical state is

$$|\chi\rangle = (L_{-2}^3 + \frac{93}{64} L_{-3}^2 - \frac{33}{8} L_{-4}L_{-2} - \frac{3}{16} L_{-6})|0\rangle$$

A very tedious computions shows that

$$|\chi\rangle = 0 \Rightarrow |\phi_h\rangle \text{ must have } h = 0,\ \frac{1}{16} \text{ or } \frac{1}{2}$$

$M(3,4)$ is identified with the thermodynamic limit of the Ising model at its critical temperature.

$\phi_{1/16} \equiv \sigma$ (spin field) $\quad \phi_{1/2} \equiv \varepsilon$ (energy density)

$$\langle \sigma(z,\bar{z})\sigma(w,\bar{w})\rangle = \frac{1}{|z-w|^{1/4}} \qquad \langle \varepsilon(z,\bar{z})\varepsilon(w,\bar{w})\rangle = \frac{1}{|z-w|^2}$$

The Ising model $M(3,4)$ has $c=\frac{1}{2}$ and conformal vacua of energy 0, $\frac{1}{16}(\sigma)$ and $\frac{1}{2}(\varepsilon)$.

The free fermion has $c=\frac{1}{2}$ and free fermion vacua of energy 0 (NS) and $\frac{1}{2}$ (R)

Ising Model

⋮		⋮		⋮	
$L_{-3}\|0\rangle$	3	$L_{-3}\|\sigma\rangle,\ L_{-2}L_{-1}\|\sigma\rangle$		$L_{-3}\|\varepsilon\rangle$	$\frac{7}{2}$
$\|T\rangle = L_{-2}\|0\rangle$	$E=2$	$L_{-2}\|\sigma\rangle$	$\frac{1}{16}+2$	$L_{-2}\|\varepsilon\rangle$	$\frac{5}{2}$
		$L_{-1}\|\sigma\rangle$	$E=\frac{1}{16}+1$	$L_{-1}\|\varepsilon\rangle$	$E=\frac{3}{2}$
$\|0\rangle$	$E=0$	$\|\sigma\rangle$	$E=\frac{1}{16}$	$\|\varepsilon\rangle$	$E=\frac{1}{2}$

Free Fermion

	⋮	⋮		⋮	⋮
$E=3$	$b_{-\frac{5}{2}}b_{-\frac{1}{2}}\|NS\rangle$				
		$b_{-\frac{5}{2}}\|NS\rangle$	$\frac{5}{2}$		
$E=2$	$b_{-\frac{3}{2}}b_{-\frac{1}{2}}\|NS\rangle$			$b_{-3}b_0\|R\rangle\ b_{-2}b_{-1}\|R\rangle$	$b_{-3}\|R\rangle,\ b_{-2}b_{-1}b_0\|R\rangle$ $\quad \frac{1}{16}+3$
				$b_{-2}b_0\|R\rangle$	$b_{-2}\|R\rangle$ $\quad \frac{1}{16}+2$
		$b_{-\frac{3}{2}}\|NS\rangle$	$E=\frac{3}{2}$		
		$b_{-\frac{1}{2}}\|NS\rangle$	$E=\frac{1}{2}$	$b_{-1}b_0\|R\rangle$	$b_{-1}\|R\rangle$ $\quad E=\frac{1}{16}+1$
$E=0$	$\|NS\rangle$			$\|R\rangle$	$b_0\|R\rangle$ $\quad E=\frac{1}{16}$
	bosonic	fermionic		bosonic	fermionic

We can even match up states (up to constants):

$M(3,4)$	$\|0\rangle$	$L_{-2}\|0\rangle$	-----	$\|\varepsilon\rangle$	$L_{-1}\|\varepsilon\rangle$	----	$\|\sigma\rangle$	$L_{-1}\|\sigma\rangle$	---	?	
FF	$\|NS\rangle$	$b_{-3/2}b_{-\frac{1}{2}}\|0\rangle$	-----	$b_{-\frac{1}{2}}\|NS\rangle$	$b_{-\frac{3}{2}}\|NS\rangle$	----	$\|R\rangle$	$b_{-1}b_0\|R\rangle$	---	$b_0\|R\rangle$	$b_{-1}\|R\rangle$ -----

Everything matches except the fermionic states in the Ramond Fock space.

This is accounted for by the disorder operator $|\mu\rangle \longleftrightarrow b_0|R\rangle$.

We do not notice it in $M(3,4)$ as it is indistinguishable from $|\sigma\rangle$, as $M(3,4)$ has no parity.

Conclusion:

The minimal model $M(3,4)$ and the free fermion are (nearly) equivalent as CFTs. In fact, this equivalence persists off-criticality which is why Onsager was able to exactly solve the Ising model on the lattice in 1944.