WIPM Lectures on Models in Statistical Mechanics

Lecture 7: Six-Vertex Model and Bethe Ansatz

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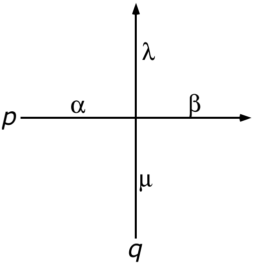
In the previous lectures we discussed the one-dimensional XXX and quantum-lattice-gas models and their solution by the coordinate Bethe Ansatz method.

* We shall now discuss the classical 2-dimensional counter part.
* We shall discuss how the Yang–Baxter equations prove the commutation of a family of transfer matrices and how spin-chain Hamiltonian are obtained as a logarithmic derivative. Therefore, the coordinate Bethe Ansatz solution carries over to the two dimensional model.
* Next we discuss the algebraic Bethe Ansatz method.
* Finally we shall show an example of so-called "analytical Bethe Ansatz," getting results from functional equations.

Vertex Models

The simplest example of a vertex model is defined on a square lattice with periodic boundary conditions. On each bond (edge) lives a state variable that can be discrete or continuous. Let us take the simplest case and choose them all from $\{1,2\}$, i.e., $\alpha,\beta,\lambda,\mu,\cdots \in \{1,2\}$. In the Yang–Baxter integrable vertex model we have also "rapidities" $p,q,r,\cdots$ that live on the lattice lines. To each vertex we assign a Boltzmann weight.



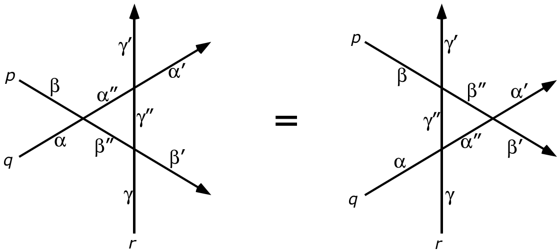
$\omega_{\alpha\mu}^{\lambda\beta}(p,q)$

The Boltzmann weight ω is both graphically and algebraically expressed in the figure. The partition function is the sum over all configurations (i.e., all allowable values of all state variables) of the total Boltzmann weight, which is the product over all vertices of the weight of that vertex:

$$\boxed{Z = \sum_{\text{config.}} \prod_{\text{edges}} \omega_{\alpha\mu}^{\lambda\beta}(p,q)}\,.$$

Yang–Baxter Equation for Vertex Model

In graphical form:

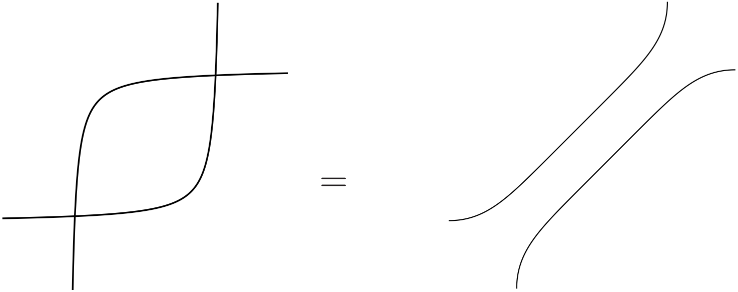


In algebraic form:

$$\sum_{\alpha''}\sum_{\beta''}\sum_{\gamma''} \omega_{\beta\ \alpha}^{\alpha''\beta''}(p,q)\,\omega_{\alpha''\gamma''}^{\gamma'\ \alpha'}(q,r)\,\omega_{\beta''\gamma}^{\gamma''\beta'}(p,r)$$

$$= \sum_{\alpha''}\sum_{\beta''}\sum_{\gamma''} \omega_{\beta''\alpha''}^{\alpha'\ \beta'}(p,q)\,\omega_{\alpha\ \gamma}^{\gamma''\alpha''}(q,r)\,\omega_{\beta\ \gamma''}^{\gamma'\beta''}(p,r)\,.$$

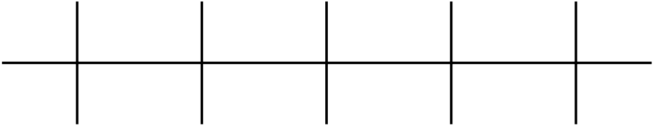
We often also have another relation, namely the inversion relation:



This occurs if for given $\omega_{\alpha\mu}^{\lambda\beta}(p,q)$ there exists an "inverse weight function" $\bar{\omega}_{\lambda\beta}^{\alpha\mu}(p,q)$, so that

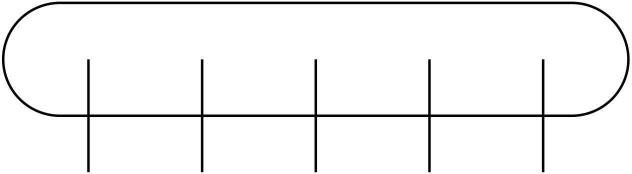
$$\sum_{\lambda}\sum_{\beta}\omega_{\alpha\mu}^{\lambda\beta}(p,q)\bar{\omega}_{\lambda\beta}^{\alpha'\mu'}(p,q) = \delta_{\alpha,\alpha'}\delta_{\mu,\mu'}\,.$$

Next we introduce the “monodromy operator”



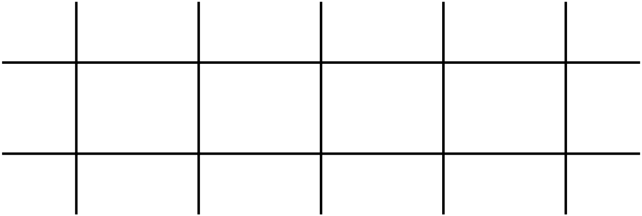
$$L(p,q|\alpha_1,\alpha_{L+1})^{\lambda'_1,\cdots,\lambda'_L}_{\lambda_1,\cdots,\lambda_L} = \sum_{\alpha_2}\cdots\sum_{\alpha_L}\prod_{j=1}^{L}\omega^{\lambda'_j\alpha_{j+1}}_{\alpha_j\lambda_j}(p,q)$$

and transfer matrix

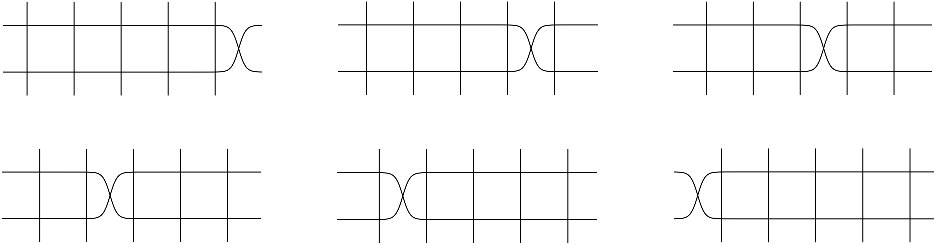


$$T(p,q)^{\lambda'_1,\cdots,\lambda'_L}_{\lambda_1,\cdots,\lambda_L} = \sum_{\alpha_1} L(\alpha_1,\alpha_1)^{\lambda'_1,\cdots,\lambda'_L}_{\lambda_1,\cdots,\lambda_L}\,.$$

The product of two monodromy operators



also satisfies the Yang–Baxter equation:



Then, adding an inverse weight on one side and using the cyclic property of trace,

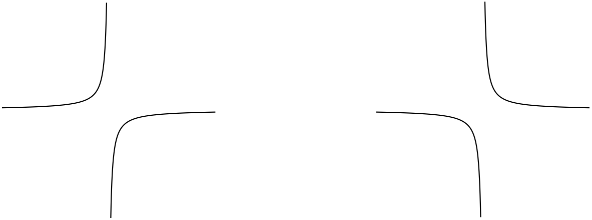
$$\mathsf{T}(p,q)\mathsf{T}(p',q) = \mathsf{T}(p',q)\mathsf{T}(p,q).$$

Hence, transfer matrices with the same vertical rapidities, but with different horizontal rapidities, commute.

If for a special value $p = q_0(q)$,

$$\text{either} \qquad \omega_{\alpha\mu}^{\lambda\beta}(q_0,q) = \delta_{\alpha,\lambda}\delta_{\beta,\mu} \qquad \text{or} \qquad \omega_{\alpha\mu}^{\lambda\beta}(q_0,q) = \delta_{\alpha,\mu}\delta_{\beta,\lambda}\,.$$

In pictures:



Then the transfer matrix becomes either the right-shift or the left-shift operator:

$$T(q_0, q)^{\lambda'_1,\cdots,\lambda'_L}_{\lambda_1,\cdots,\lambda_L} = \prod_{j=1}^{L} \delta_{\lambda_j,\lambda'_{j+1}} \qquad \text{or} \qquad T(q_0, q)^{\lambda'_1,\cdots,\lambda'_L}_{\lambda_1,\cdots,\lambda_L} = \prod_{j=1}^{L} \delta_{\lambda_j,\lambda'_{j-1}} ,$$

where $\lambda'_{j+1} \equiv \lambda'_1$ and $\lambda'_0 \equiv \lambda'_L$. (Exercise: Draw the graphical representation.)

As the $\mathsf{T}(p, q)$ form a commuting family with varying p, we can work out the logarithmic derivative of $\mathsf{T}(p, q)$ at $p = q_0$,

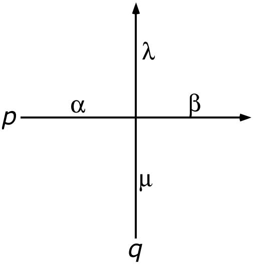
$$\frac{\partial}{\partial p} \ln T(p, q)\bigg|_{p=q_0} = T(q_0, q)^{-1} \; \frac{\partial T(p, q)}{\partial p}\bigg|_{p=q_0} = \mathcal{H},$$

with $\mathcal{H}$ a quantum spin-chain Hamiltonian with nearest-neighbor interactions. (Exercise 2: Draw the graphical representation.)

(Exercise 3: Show that the XXZ Hamiltonian derives from the six-vertex model defined next.)

Six-Vertex Model

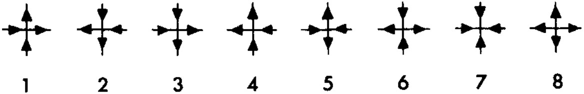
There are two versions, one with $d = 0$, $\alpha = \lambda$, $\beta = \mu$, the other with $c = 0$, $\alpha = \mu$, $\beta = \lambda$:



$\omega^{\lambda\beta}_{\alpha\mu}(p,q)$

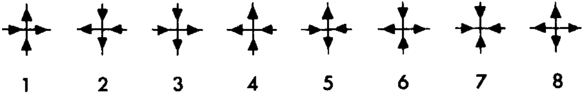
1 arrow up/right

2 arrow down/left



a	a	b	b	c	c	0	0

or



a	a	b	b	0	0	d	d

Arranging the $\omega^{\lambda\beta}_{\alpha\mu}(p,q)$ in an $\hat{\mathbf{R}}$ matrix, we have

$$\hat{\boldsymbol{\omega}}(p,q) \;=\; \begin{array}{r c} & \begin{array}{cccc} \lambda,\beta=1,1 & 1,2 & 2,1 & 2,2 \end{array} \\ \begin{array}{r} \alpha,\mu=1,1 \\ 1,2 \\ 2,1 \\ 2,2 \end{array} & \begin{pmatrix} a & 0 & 0 & d \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ d & 0 & 0 & a \end{pmatrix} \end{array},$$

with

$$a(p,q) = \frac{\sinh\left(\eta-(q-p)\right)}{\sinh(\eta)}, \qquad b(p,q) = \frac{\sinh(q-p)}{\sinh(\eta)},$$

and

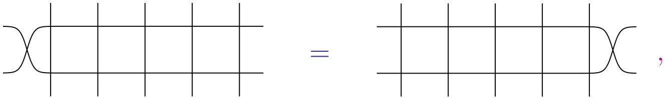
$$\begin{cases} c(p,q)=1, \quad d(p,q)=0, \quad \text{for case 1}, \\ c(p,q)=0, \quad d(p,q)=1, \quad \text{for case 2}. \end{cases}$$

Also, $\eta>0$, (real positive), to have the XXZ quantum chain model in the ordered antiferromagnetic regime.

Algebraic Bethe Ansatz

As the commuting family of transfer matrices $\mathbf{T}(p, q)$ of the six-vertex model also commutes with the XXZ Hamiltonian $\mathcal{H}_{\mathrm{XXZ}}$, one finds that last lecture's coordinate Bethe-Ansatz solution of the quantum chain also diagonalizes the transfer matrix.

Sklyanin, Takhtadzhyan and Faddeev[†] introduced the algebraic Bethe Ansatz method, alias the quantum inverse scattering method (QISM). Let us start with the Yang–Baxter equation for the monodromy matrix:



with all Boltzmann weights being six-vertex model weights to keep it simple. In algebraic form this picture represents:

† E.K. Sklyanin, L.A. Takhtadzhyan and L.D. Faddeev, Theor. Math. Phys. **40**, 688–706 (1979).

$$\sum_{\alpha''=1}^{2}\sum_{\beta''=1}^{2}\sum_{\gamma_1''=1}^{2}\cdots\sum_{\gamma_L''=1}^{2}\omega_{\beta\ \alpha}^{\alpha''\beta''}(p,q)\, L(p,r|\beta'',\beta')_{\gamma_1,\cdots,\gamma_L}^{\gamma_1'',\cdots,\gamma_L''}\, L(q,r|\alpha'',\alpha')_{\gamma_1'',\cdots,\gamma_L''}^{\gamma_1',\cdots,\gamma_L'}$$

$$=\sum_{\alpha''=1}^{2}\sum_{\beta''=1}^{2}\sum_{\gamma_1''=1}^{2}\cdots\sum_{\gamma_L''=1}^{2} L(q,r|\alpha,\alpha'')_{\gamma_1,\cdots,\gamma_L}^{\gamma_1'',\cdots,\gamma_L''}\, L(p,r|\beta,\beta'')_{\gamma_1'',\cdots,\gamma_L''}^{\gamma_1',\cdots,\gamma_L'}\,\omega_{\beta''\alpha''}^{\alpha'\ \beta'}(p,q)\,.$$

We can rearrange the elements of the monodromy matrices to become 2-by-2 matrices with elements that are 2^L-by-2^L matrices:

$$\begin{pmatrix}\mathbf{A}(p,q) & \mathbf{B}(p,q)\\ \mathbf{C}(p,q) & \mathbf{D}(p,q)\end{pmatrix}=\begin{pmatrix}\mathbf{L}(p,q|1,1) & \mathbf{L}(p,q|1,2)\\ \mathbf{L}(p,q|2,1) & \mathbf{L}(p,q|2,2)\end{pmatrix},$$

while

$$\sum_{\alpha''=1}^{2}\sum_{\beta''=1}^{2}\omega_{\beta\ \alpha}^{\alpha''\beta''}(p,q)\,\mathbf{L}(p,r|\beta'',\beta')\cdot\mathbf{L}(q,r|\alpha'',\alpha')$$

$$=\sum_{\alpha''=1}^{2}\sum_{\beta''=1}^{2}\mathbf{L}(q,r|\alpha,\alpha'')\cdot\mathbf{L}(p,r|\beta,\beta'')\,\omega_{\beta''\alpha''}^{\alpha'\ \beta'}(p,q)\,.$$

Let us consider case 1 with

$$\omega^{11}_{11} = \omega^{22}_{22} = a, \quad \omega^{21}_{12} = \omega^{12}_{21} = b, \quad \omega^{12}_{12} = \omega^{21}_{21} = c = 1, \quad \text{all other } 0.$$

Then we first obtain the four commutations

$$\begin{aligned} [\mathbf{A}(p,r), \mathbf{A}(q,r)] &= 0, \qquad & [\mathbf{B}(p,r), \mathbf{B}(q,r)] &= 0, \\ [\mathbf{C}(p,r), \mathbf{C}(q,r)] &= 0, \qquad & [\mathbf{D}(p,r), \mathbf{D}(q,r)] &= 0. \end{aligned}$$

Next,

$$\begin{aligned} b(p,q)[\mathbf{A}(p,r), \mathbf{D}(q,r)] &= \mathbf{C}(q,r)\mathbf{B}(p,r) - \mathbf{C}(p,r)\mathbf{B}(q,r), \\ b(p,q)[\mathbf{B}(p,r), \mathbf{C}(q,r)] &= \mathbf{D}(q,r)\mathbf{A}(p,r) - \mathbf{D}(p,r)\mathbf{A}(q,r), \\ b(p,q)[\mathbf{D}(p,r), \mathbf{A}(q,r)] &= \mathbf{B}(q,r)\mathbf{C}(p,r) - \mathbf{B}(p,r)\mathbf{C}(q,r), \\ b(p,q)[\mathbf{C}(p,r), \mathbf{B}(q,r)] &= \mathbf{A}(q,r)\mathbf{D}(p,r) - \mathbf{A}(p,r)\mathbf{D}(q,r), \end{aligned}$$

with the second pair equivalent to the first pair, as $b(p,q) = -b(q,p)$.

Finally there are eight three-term relations:

$$
\begin{aligned}
a(p,q)\mathbf{A}(p,r)\mathbf{B}(q,r) &= b(p,q)\mathbf{B}(q,r)\mathbf{A}(p,r) + \mathbf{A}(q,r)\mathbf{B}(p,r), \\
a(p,q)\mathbf{B}(p,r)\mathbf{A}(q,r) &= b(p,q)\mathbf{A}(q,r)\mathbf{B}(p,r) + \mathbf{B}(q,r)\mathbf{A}(p,r), \\
a(p,q)\mathbf{C}(p,r)\mathbf{D}(q,r) &= b(p,q)\mathbf{D}(q,r)\mathbf{C}(p,r) + \mathbf{C}(q,r)\mathbf{D}(p,r), \\
a(p,q)\mathbf{D}(p,r)\mathbf{C}(q,r) &= b(p,q)\mathbf{C}(q,r)\mathbf{D}(p,r) + \mathbf{D}(q,r)\mathbf{C}(p,r),
\end{aligned}
$$

$$
\begin{aligned}
a(p,q)\mathbf{A}(q,r)\mathbf{C}(p,r) &= b(p,q)\mathbf{C}(p,r)\mathbf{A}(q,r) + \mathbf{A}(p,r)\mathbf{C}(q,r), \\
a(p,q)\mathbf{C}(q,r)\mathbf{A}(p,r) &= b(p,q)\mathbf{A}(p,r)\mathbf{C}(q,r) + \mathbf{C}(p,r)\mathbf{A}(q,r), \\
a(p,q)\mathbf{B}(q,r)\mathbf{D}(p,r) &= b(p,q)\mathbf{D}(p,r)\mathbf{B}(q,r) + \mathbf{B}(p,r)\mathbf{D}(q,r), \\
a(p,q)\mathbf{D}(q,r)\mathbf{B}(p,r) &= b(p,q)\mathbf{B}(p,r)\mathbf{D}(q,r) + \mathbf{D}(p,r)\mathbf{B}(q,r).
\end{aligned}
$$

Note that the transfer matrix is simply given by

$$
\mathbf{T}(p,r) = \mathbf{A}(p,r) + \mathbf{D}(p,r),
$$

so that these relations can be used to calculate the commutator of $\mathbf{T}(p,r)$ with either $\mathbf{B}(q,r)$ or $\mathbf{C}(q,r)$.

Next we introduce the reference state ("vacuum state with all spins up"):

$$|\,\emptyset\,\rangle = |22\cdots 2\rangle = \bigotimes_{i=1}^{L} |2\rangle.$$

One can then easily check that

$$\mathbf{A}(p,r)|\,\emptyset\,\rangle = a(p,r)^L|\,\emptyset\,\rangle, \quad \mathbf{D}(p,r)|\,\emptyset\,\rangle = b(p,r)^L|\,\emptyset\,\rangle, \quad \mathbf{C}(p,r)|\,\emptyset\,\rangle = 0,$$

as the only nonzero element of the form $\omega_{22}^{\alpha\beta}$ is ω_{22}^{22}, so that once a horizontal link of $\mathbf{L}(\alpha,\beta)$ is in state 2, all states to the right are also 2.

We see that we can consider $\mathbf{C}(p,r)$ as an annihilation operator . Also, as $\mathbf{B}(p,r)$ creates precisely a down spin in $|22\cdots 2\rangle$, we can consider it as a creation operator.

Finally, $|22\cdots 2\rangle$ is an eigenvector of the transfer matrix:

$$\mathbf{T}(p,r)|\,\emptyset\,\rangle = \big(\mathbf{A}(p,r)+\mathbf{D}(p,r)\big)|\,\emptyset\,\rangle = \big(a(p,r)^L + b(p,r)^L\big)|\,\emptyset\,\rangle.$$

The Bethe Ansatz within QISM is that with the vacuum state $|\,\emptyset\,\rangle$ the states

$$|q_1,\cdots,q_n\rangle = \prod_{j=1}^{n} \mathbf{B}(q_j,r)|\,\emptyset\,\rangle \tag{$*$}$$

are also eigenvectors,

$$\mathbf{T}(p,r)|q_1,\cdots,q_n\rangle = \Lambda(p,r,q_1,\cdots,q_n)|q_1,\cdots,q_n\rangle,$$

provided $q_1,\cdots,q_n$ are suitably chosen.

Note that these states are analytic in $q_1,\cdots,q_n$, whereas transfer matrix $\mathbf{T}(p,r)$ is analytic in p. Applying this $\mathbf{T}(p,r)$ on the state $|q_1,\cdots,q_n\rangle$, we want to commute the $\mathbf{A}(p,r)$ and $\mathbf{D}(p,r)$ through the $\mathbf{B}(q_j,r)$, till they hit the vacuum state. To do this we need the following two relations from the list of sixteen:

$$\mathbf{A}(p,r)\mathbf{B}(q,r) = \frac{a(q,p)}{b(q,p)}\mathbf{B}(q,r)\mathbf{A}(p,r) - \frac{1}{b(q,p)}\mathbf{B}(p,r)\mathbf{A}(q,r),$$

$$\mathbf{D}(p,r)\mathbf{B}(q,r) = \frac{a(p,q)}{b(p,q)}\mathbf{B}(q,r)\mathbf{D}(p,r) - \frac{1}{b(p,q)}\mathbf{B}(p,r)\mathbf{D}(q,r).$$

The last terms in these two equations lead to "unwanted terms" that cancel out for the proper choice of the q_j's in $(*)$, so that the eigenvalue $\Lambda(p,r,q_1,\cdots,q_n)$

should be

$$\Lambda(p, r, q_1, \cdots, q_n) = a(p, r)^L \prod_{j=1}^{n} \frac{a(q_j, p)}{b(q_j, p)} + b(p, r)^L \prod_{j=1}^{n} \frac{a(p, q_j)}{b(p, q_j)}.$$

However, this will have poles at $p = q_i$ unless the residues vanish. To work this condition out, we must remember

$$a(p, q) = \frac{\sinh\big(\eta - (q - p)\big)}{\sinh(\eta)}, \qquad b(p, q) = \frac{\sinh(q - p)}{\sinh(\eta)},$$

so that

$$0 = -a(q_i, r)^L \prod_{j=1, j\neq i}^{n} \frac{a(q_j, q_i)}{b(q_j, q_i)} + b(q_i, r)^L \prod_{j=1, j\neq i}^{n} \frac{a(q_i, q_j)}{b(q_i, q_j)},$$

or

$$\left(\frac{a(q_i, r)}{b(q_i, r)}\right)^L = (-1)^{n-1} \prod_{j=1, j\neq i}^{n} \frac{a(q_i, q_j)}{a(q_j, q_i)}, \quad (i = 1, \cdots, n),$$

which is also the XXZ chain Bethe Ansatz equation for $\Delta = -\cosh\eta$.

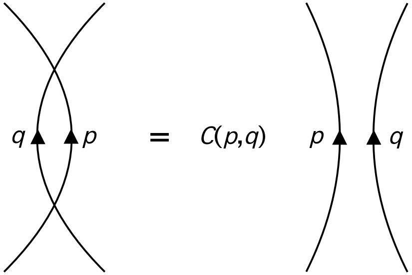
Functional Equations/Analytic Bethe Ansatz

There are other ways of getting properties of the thermodynamic limit without going through the solution of the eigenvectors of the transfer matrix using either coordinate or algebraic Bethe Ansatz methods, which one not always know how to do. One can, for example construct a system of functional equations for the partition function per site z of an $M \times N$ lattice, $M, N \to \infty$,

$$z = \lim_{M,N\to\infty} Z^{1/MN} = \mathrm{e}^{-\beta f}.$$

The oldest method comes from analytic S-matrix theory, which is based on a factorizable S-matrix (satisfying the Yang–Baxter equation). One then uses unitarity and crossing and some analyticity assumptions.

This started with McGuire and Iagolnitzer in the mid 1960s and culminated in the work of Zamolodchikov and Zamolodchikov in the 1970s. In statistical mechanics, unitarity corresponds to inversion of the matrix of Boltzmann weights and crossing to a rotation symmetry of the lattice. This inversion method took off after papers by Stroganov, Schultz and Baxter.



Graphical representation of inversion relation: Here two rapidity lines cross each other twice; this is equivalent to not crossing at all. There may be a scalar factor $C(p,q)$, which can be removed by a normalization of the weights. (Think also of Reidemeister's move II from knot theory.)

We had

$$\hat{\omega}(p,q) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ d & 0 & 0 & a \end{pmatrix}, \quad \begin{cases} c(p,q) = 1, & d(p,q) = 0, \quad \text{for case } 1, \\ c(p,q) = 0, & d(p,q) = 1, \quad \text{for case } 2. \end{cases}$$

and

$$a(p,q) = \frac{\sinh\left(\eta - (q-p)\right)}{\sinh(\eta)}, \qquad b(p,q) = \frac{\sinh(q-p)}{\sinh(\eta)}.$$

To calculate the 'inverse weights' is just inverting a 2-by-2 matrix. For case 1 we find

$$\hat{\boldsymbol{\omega}}^{\text{inv}}(q,p) = \hat{\boldsymbol{\omega}}(q,p)\,\frac{\sinh^2\eta}{\sinh(\eta-u)\sinh(\eta+u)}, \quad u \equiv q-p,$$

and

$$C(p,q) = \frac{\sinh(\eta-u)\sinh(\eta+u)}{\sinh^2\eta}.$$

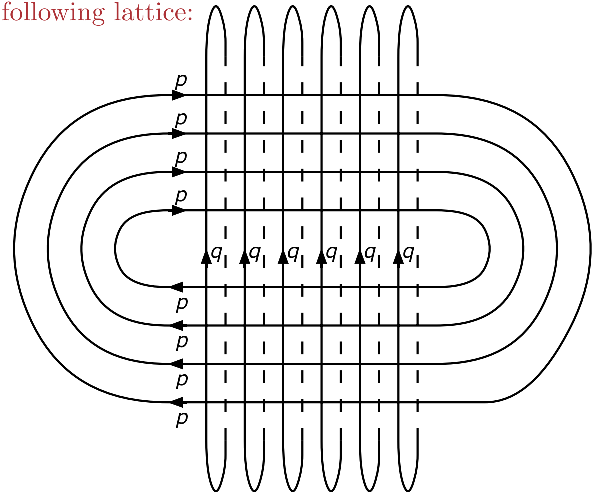
For case 2 we have

$$\hat{\boldsymbol{\omega}}^{\text{inv}}(q,p) = \hat{\boldsymbol{\omega}}(q,p+2\eta)\,\frac{\sinh^2\eta}{\sinh(u)\sinh(2\eta-u)}, \quad u \equiv q-p,$$

and

$$C(p,q) = \frac{\sinh(u)\sinh(2\eta-u)}{\sinh^2\eta}.$$

Consider the following lattice:



Top: $M \times N$ six-vertex model. Bottom: $N \times M$ six-vertex model with inverse weights (model rotated $90°$). Periodic boundary conditions vertically, special boundary conditions as drawn horizontally.

Assume that the thermodynamic limit does not depend macroscopically on these boundary conditions, so that

$$Z_{\text{total}} = z(\text{6-v})^{MN} z(\text{inv 6-v})^{NM} \mathrm{e}^{\mathrm{O}(M)+\mathrm{O}(N)},$$

$$z_{\text{total}} = \lim_{M,N\to\infty} Z^{1/MN} = z(\text{6-v}) z(\text{inv 6-v}),$$

with $z(\text{6-v})$ the partition function per site for the 6-vertex model with periodic boundary conditions and $z(\text{inv 6-v})$ the partition function per site for the inverse 6-vertex model (obtained by analytic continuation).

Now calculate it a second way applying the inversion relation MN times:

$$Z_{\text{total}} = C(p,q)^{MN}\, 2^{M+N},$$

as we get $M+N$ independent loops, each of which can be in two states. Hence,

$$z_{\text{total}} = C(p,q).$$

For case 1 and case 2 we end up with the two relations,

$$\boxed{\begin{aligned} z(u)z(-u) &= \frac{\sinh(\eta-u)\sinh(\eta+u)}{\sinh^2\eta}, \\ z(u)z(2\eta-u) &= \frac{\sinh(u)\sinh(2\eta-u)}{\sinh^2\eta}, \end{aligned}} \tag{1}$$

where $u \equiv q - p$.

In most applications of the inversion method one has only one quadratic inversion relation ('unitarity') and one linear rotational symmetry relation ('crossing'). If the 90° rotated weights and the original weights satisfy $\hat{\boldsymbol{\omega}}^{\mathrm{rot}}(u) = \hat{\boldsymbol{\omega}}(\lambda - u)$ for some constant λ, one has the pair of equations

$$z(u)z(-u) = C(u), \quad z(u) = z(\lambda - u).$$

Now, following Schultz,* we have a second inversion relation in place of 'crossing'.

* C.L. Schultz [Ph.D. Thesis, SUNY, Stony Brook, 1981, Chapter 5] does the sl(n) case.

We have two more relations,

$$\boxed{z(u) = z(u + \pi \mathrm{i}), \qquad z(0) = 1.} \tag{2}$$

The first relation follows from $a(u + \pi\mathrm{i}) = -a(u)$, $b(u + \pi\mathrm{i}) = -b(u)$, $c(u) \equiv 1$, and the fact that on an $M \times N$ lattice, with periodic boundary conditions and MN even, there must be an even number of minus signs, as the number of vertices of types 5 and 6 (and also of type 7 and 8) must be equal.† The other equation follows as $Z(0) < 2^M, 2^N$.

The set of equations (1) and (2) can be solved by iteration for $\eta > 0$ as

$$\boxed{z(u) = \frac{\mathrm{e}^u \sinh(\eta - u)}{\sinh \eta} \prod_{m=0}^{\infty} \frac{(1 - \mathrm{e}^{2u-4(m+1)\eta})(1 - \mathrm{e}^{-2u-2(2m+1)\eta})}{(1 - \mathrm{e}^{-2u-4(m+1)\eta})(1 - \mathrm{e}^{2u-2(2m+1)\eta})}}, \tag{3}$$

as we show next:

† The easiest argument is that the number of sinks of arrows (type 7) must equal the number of sources of arrows (type 8). A similar argument applies for types 5 and 6 weights.

Let us use the notations

$$s \equiv \mathrm{e}^{2u}, \qquad 0 < x \equiv \mathrm{e}^{-2\eta} < 1, \qquad y(s) \equiv \prod_{n=0}^{\infty} \frac{1 - s\,x^{2n+2}}{1 - s\,x^{2n+1}}, \tag{4}$$

so that (3) becomes

$$\boxed{z(s) = \frac{1 - s\,x}{1 - x}\,\frac{y(s)}{y(s^{-1})}}, \tag{3'}$$

which we should obtain from (1) and (2), rewritten as

$$z(s)z(s^{-1}) = \frac{(1 - s\,x)(1 - s^{-1}x)}{(1 - x)^2}, \quad z(s\,x^{-2})z(s^{-1}) = \frac{(1 - s)(1 - s^{-1}x^2)}{(1 - x)^2}, \tag{1'}$$

$$z(s) = z(s\,\mathrm{e}^{2\pi\mathrm{i}}), \qquad z(1) = 1, \tag{2'}$$

from which

$$\frac{z(s)}{z(s\,x^{-2})} = \frac{(1 - s\,x)(1 - s^{-1}x)}{(1 - s)(1 - s^{-1}x^2)}. \tag{5}$$

We split $z(s)$, such that

$$z(s) = z_1(s) z_2(s), \quad \frac{z_1(s)}{z_1(s\,x^{-2})} = \frac{1 - s\,x}{1 - s}, \quad \frac{z_2(s)}{z_2(s\,x^{-2})} = \frac{1 - s^{-1}x}{1 - s^{-1}x^2}. \tag{6}$$

Iterating we get

$$z_1(s) = \frac{1 - s\,x^2}{1 - s\,x^3}\, z_1(s\,x^2) = F_1(s) \prod_{n=0}^{\infty} \frac{1 - s\,x^{2n+2}}{1 - s\,x^{2n+3}} = F_1(s)\,(1 - s\,x)\,y(s),$$

$$z_2(s) = \frac{1 - s^{-1}x}{1 - s^{-1}x^2}\, z_2(s\,x^{-2}) = F_2(s) \prod_{n=0}^{\infty} \frac{1 - s^{-1}x^{2n+1}}{1 - s^{-1}x^{2n+2}} = \frac{F_2(s)}{y(s^{-1})}\,,$$

with

$$F_1(s) = F_1(s\,x^2), \quad F_2(s) = F_2(s\,x^2), \quad F(s) \equiv \frac{F_1(s) F_2(s)}{1 - x} = F(s\,x^2).$$

Hence,

$$z(s) = F(s)\,\frac{1 - s\,x}{1 - x}\,\frac{y(s)}{y(s^{-1})}\,.$$

As one can easily check that (3) satisfies (1) and (2), or, equivalently, that (3′) satisfies (1′) and (2′), one must have

$$F(s) = F(s\,x^2) = F(s\,\mathrm{e}^{2\pi\mathrm{i}}), \quad F(1) = 1,$$

so that $F(s) = F(\mathrm{e}^{2u})$ is doubly periodic in u with periods 2η and $\pi\mathrm{i}$. If we assume that $F(s)$ is analytic, that is that we already captured all zeros and poles in (3′), then by Liouville's theorem $F(s)$ is constant and thus equal 1.

Thus we have established (3) under a few assumptions that are typically used when applying analytic Bethe Ansatz methods.*

* See also, e.g., J.H.H. Perk and F.Y. Wu, Physica A **138**, 100–124 (1986) for some further discussion about such methods.