WIPM Lectures on Models in Statistical Mechanics

Lecture 6: Quantum Lattice Gas and Bethe Ansatz

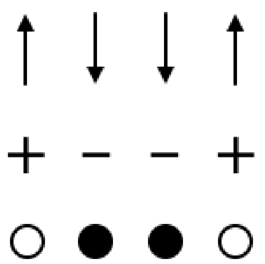
Jacques H. H. Perk, Oklahoma State University

In the previous lectures we discussed the 2D Ising model and the 1D quantum Ising chain in transverse field. Even though much more can be said, we leave it with that. Today we start a series on Bethe Ansatz and Yang–Baxter equations.

- We first introduce the one-dimensional quantum lattice gas (QLG) with nearest-neighbor interactions.
- Next we shall show how this QLG model relates to the one-dimensional Heisenberg–Ising model (also called XXZ chain).
- After this we shall discuss the coordinate Bethe Ansatz method.
- Finally we shall show how these models in the limit $\Delta \to 1$ reproduce the Lieb-Liniger 1D Bose gas with delta interaction.

One-Dimensional Quantum Lattice Model

Yang and Yang[*] mention the paper[†] of Matsubara and Matsuda on the quantum lattice gas model as a principal motivation to study the Heisenberg–Ising model. The connection with the Bose gas with delta interaction and the Bethe Ansatz solution come out more naturally if we treat the 1D QLG first:[‡]



As indicated in the figure, we have a direct map between a spin model and a hard-core particle model:

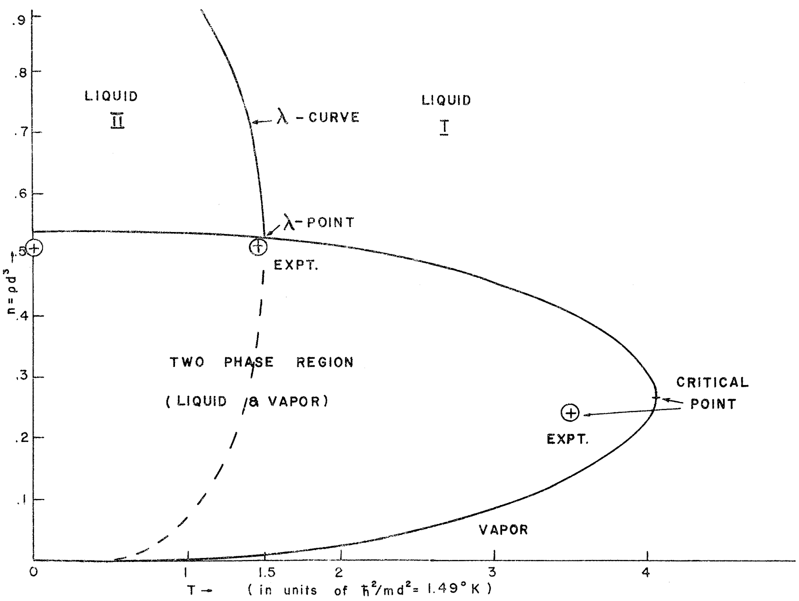
Spin up or + corresponds with a hole (or 0), while spin down or − corresponds with a particle present (or 1).

We can represent the creation of a particle at site j by σ_j^- and its annihilation by σ_j^+.

[*] C.N. Yang and C.P. Yang, Phys. Lett. **20**, 9–10; Phys. Rev. **147**, 303–306, **150** 321–327 (1966).

[†] T. Matsubara and H. Matsuda, Progr. Theor. Phys. **16**, 569–582 (1956).

[‡] J.H.H. Perk's Scriptie (University of Amsterdam, June 1974) contains unpublished details.



Interestingly, P.R. Zilsel [Phys. Rev. Lett. **15**, 476–479 (1965)] got this phase diagram in the mean-field approximation applied to the Quantum Lattice Gas. It semiquantitatively represents the phase diagram of Helium-4.

For a one-dimensional system of spinless bosons interacting via pair potential $V(x)$ we can write the Hamiltonian in second quantized form as

$$\begin{aligned}\mathcal{H} &= \frac{\hbar^2}{2m}\int_0^L \mathrm{d}x\,\frac{\mathrm{d}\boldsymbol{\psi}^\dagger}{\mathrm{d}x}\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}x} + \frac{1}{2}\int_0^L \mathrm{d}x\int_0^L \mathrm{d}y\, V(|x-y|)\boldsymbol{\psi}^\dagger(x)\boldsymbol{\psi}^\dagger(y)\boldsymbol{\psi}(y)\boldsymbol{\psi}(x)\\ &= -\frac{\hbar^2}{2m}\int_0^L \mathrm{d}x\,\boldsymbol{\psi}^\dagger\frac{\mathrm{d}^2\boldsymbol{\psi}}{\mathrm{d}x^2} + \frac{1}{2}\int_0^L \mathrm{d}x\int_0^L \mathrm{d}y\, V(|x-y|)\boldsymbol{\psi}^\dagger(x)\boldsymbol{\psi}^\dagger(y)\boldsymbol{\psi}(y)\boldsymbol{\psi}(x),\end{aligned}$$

where $\boldsymbol{\psi}^\dagger(x)$ creates a particle at position x and $\boldsymbol{\psi}(x)$ annihilates it.

Following Matsubara and Matsuda, we can discretize it assuming a periodic lattice of N sites and lattice spacing a, so that $L = Na$. We also assume pair potential $V(x_i, x_j) = V(x_j, x_i)$ given by

$$\left.\begin{aligned} V(i,i) &= +\infty\\ V(i,i\pm 1) &= -2\Delta\\ V(i,i\pm t) &= 0,\ \text{if } 2\leqslant t\leqslant N-2\end{aligned}\right\}\quad i = 1, 2, \cdots, N.$$

We then have to replace the integration $\int_0^L \mathrm{d}x$ by the summation $\sum_{i=1}^N$ and the differentiations by differences according to

$$\frac{\mathrm{d}f}{\mathrm{d}x} \;\to\; \frac{\Delta f}{\Delta x} = \frac{f_{i+1}-f_i}{a}, \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \;\to\; \frac{\Delta^2 f}{\Delta x^2} = \frac{f_{i+1}+f_{i-1}-2f_i}{a^2}.$$

Since we have infinite on-site repulsion, we can have only no or one particle on each site. Considering the vacuum as all spins up and a particle as a down spin, we thus also have to replace

$$\boldsymbol{\psi}^\dagger \;\to\; \sigma_i^-, \qquad \boldsymbol{\psi} \;\to\; \sigma_i^+,$$

so that we arrive at the quantum-lattice-gas Hamiltonian

$$\begin{aligned}\mathcal{H}_{\mathrm{QLG}} &= \frac{\hbar^2}{2ma^2}\sum_{i=1}^{N}(\sigma_{i+1}^- - \sigma_i^-)(\sigma_{i+1}^+ - \sigma_i^+) + \frac{1}{2}\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N} V(i,j)\sigma_i^-\sigma_i^+\sigma_j^-\sigma_j^+ \\ &= -\frac{\hbar^2}{2ma^2}\sum_{i=1}^{N}(\sigma_{i+1}^- + \sigma_{i-1}^- - 2\sigma_i^-)\sigma_i^+ - 2\Delta\sum_{i=1}^{N}\sigma_i^-\sigma_i^+\sigma_{i+1}^-\sigma_{i+1}^+,\end{aligned}$$

with the periodic boundary condition $\boldsymbol{\sigma}_{i+N} \equiv \boldsymbol{\sigma}_i$.

Next we follow the convention of choosing units such that $\frac{\hbar^2}{2ma^2} = 1$ and we may also replace

$$\sigma_j^\pm = \tfrac{1}{2}(\sigma_j^x \pm \mathrm{i}\sigma_j^y), \qquad \sigma_j^-\sigma_j^+ = \tfrac{1}{2}(1-\sigma_j^z),$$

and find

$$\begin{aligned}\mathcal{H}_{\mathrm{QLG}} &= -\sum_{i=1}^{N}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + 2\Delta\,\sigma_i^-\sigma_i^+\sigma_{i+1}^-\sigma_{i+1}^+ - 2\sigma_i^-\sigma_i^+\right) \\ &= -\frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\,\sigma_i^z\sigma_{i+1}^z\right) + \sum_{i=1}^{N}\left[(\Delta-1)\sigma_i^z + 1 - \tfrac{1}{2}\Delta\right],\end{aligned}$$

or

$$\boxed{\begin{aligned}\mathcal{H}_{\mathrm{QLG}} &= \mathcal{H}_{\mathrm{XXZ}} + (\Delta-1)M^z + N - \tfrac{1}{2}N\Delta \\ &= \mathcal{H}_{\mathrm{XXZ}} - 2(\Delta-1)N^- + \tfrac{1}{2}N\Delta\end{aligned}}$$

with

$$\mathcal{H}_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\,\sigma_i^z\sigma_{i+1}^z\right), \quad M^z = \sum_{i=1}^{N}\sigma_i^z, \quad N^- = \sum_{i=1}^{N}\sigma_i^-\sigma_i^+.$$

Here N^- counts the number of particles (or down spins) in a given state and $M^z = N - 2N^-$ is the magnetization (in some special unit system).

One can easily check that the operators $\mathcal{H}_{\mathrm{QLG}}$, $\mathcal{H}_{\mathrm{XXZ}}$, M^z, and N^- form a set of mutually commuting Hermitian operators, e.g.,

$$\mathcal{H}_{\mathrm{XXZ}}^\dagger = \mathcal{H}_{\mathrm{XXZ}}, \qquad [\mathcal{H}_{\mathrm{XXZ}}, M^z] = 0, \quad \text{etc.}$$

Thus they have common eigenvectors $|\Psi\rangle$

$$\mathcal{H}_{\mathrm{QLG}}|\Psi\rangle = E_{\mathrm{QLG}}|\Psi\rangle, \quad N^-|\Psi\rangle = n|\Psi\rangle,$$

$$\mathcal{H}_{\mathrm{XXZ}}|\Psi\rangle = E_{\mathrm{XXZ}}|\Psi\rangle, \quad M^z|\Psi\rangle = Nm|\Psi\rangle, \quad Nm = N - 2n,$$

$$E_{\mathrm{QLG}} - E_{\mathrm{XXZ}} = (\Delta - 1)Nm + N - \tfrac{1}{2}N\Delta = -2(\Delta - 1)n + \tfrac{1}{2}N\Delta,$$

where $m = \langle\Psi|M^z|\Psi\rangle/N$ is the average magnetization per site and $n = \langle\Psi|N^-|\Psi\rangle$ the number of particles (down spins) in state $|\Psi\rangle$.

XXZ Hamiltonian

Remarks:

- For the XYZ Hamiltonian,

$$\mathcal{H}_{\mathrm{XYZ}}(J_x, J_y, J_z) = -\frac{1}{2}\sum_{i=1}^{N}\left(J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z\right),$$

  to have the same particle picture we need $[\mathcal{H}_{\mathrm{XYZ}}, M^z] = 0$, or $J_x = J_y$.
- If N is even, we can choose J_x to be positive, as

$$\mathcal{H}_{\mathrm{XYZ}}(J_x, J_y, J_z) = A\,\mathcal{H}_{\mathrm{XYZ}}(-J_x, -J_y, J_z)A^{-1}, \quad A \equiv \prod_{j=1}^{N/2}\sigma_{2j}^z.$$

- Therefore, if N is even and $J_x = J_y \neq 0$, it is sufficient to study

$$\mathcal{H}_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\,\sigma_i^z\sigma_{i+1}^z\right), \qquad \Delta = \frac{J_z}{|J_x|},$$

  as we can always choose $J_x > 0$ and $\mathcal{H}_{\mathrm{XYZ}}(|J_x|, |J_x|, J_z) = |J_x|\mathcal{H}_{\mathrm{XXZ}}$.

- $\mathcal{H}_{\rm XXZ}(-\Delta)$ is unitarily equivalent to $-\mathcal{H}_{\rm XXZ}(\Delta)$, as
$$A\,\mathcal{H}_{\rm XYZ}(1,1,\Delta)A^{-1} = \mathcal{H}_{\rm XYZ}(-1,-1,\Delta) = -\mathcal{H}_{\rm XYZ}(1,1,-\Delta).$$
- We have the following special cases of $\mathcal{H}_{\rm XXZ}(-\Delta)$:
$$\begin{cases} \Delta = 0: & \text{Isotropic XY Hamiltonian,} \\ \Delta = +1: & \text{Isotropic ferromagnet,} \\ \Delta = -1: & \text{Isotropic antiferromagnet,} \\ \Delta = \pm\infty: & \text{One-dimensional Ising model.} \end{cases}$$
- For fixed magnetization m and finite N:
$$\begin{cases} 1^\circ & \text{The ground state of } \mathcal{H}_{\rm XXZ} \text{ is nondegenerate.} \\ 2^\circ & \text{Its energy } E_N(m,\Delta) \text{ is analytic in } \Delta. \\ 3^\circ & \text{Corresponding eigenvector } |\Psi_0(m,\Delta)\rangle \text{ has only positive elements.} \end{cases}$$
The proof uses the Perron–Frobenius theorem applied to $(\lambda\mathbf{1}-\mathcal{H}_{\rm XXZ})^k$, which is strictly positive for λ and k large enough. (In the given sector any basis state is connected with any other basis state by a string of nonzero off-diagonal elements of the non-negative matrix $\lambda\mathbf{1}-\mathcal{H}_{\rm XXZ}$.)

- Rewriting $\mathcal{H}_{\rm XXZ}$ as
$$\mathcal{H}_{\rm XXZ} = -\frac{1}{2}N\Delta\,\mathbf{1} - \frac{1}{2}\sum_{i=1}^{N}\big(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\,\sigma_i^z\sigma_{i+1}^z - \Delta\,\mathbf{1}\big),$$
we see that it is the sum of $-\frac{1}{2}N\Delta\,\mathbf{1}$ and N matrices based on
$$-\frac{1}{2}(\sigma^x\otimes\sigma^x + \sigma^y\otimes\sigma^y + \Delta\sigma^z\otimes\sigma^z - \Delta\mathbf{1}_{4\times 4}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Delta & -1 & 0 \\ 0 & -1 & \Delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
having eigenvalues 0 and $\Delta\pm 1$. Therefore, for all allowed m or n values
$$E_{\rm XXZ} \geqslant -\frac{1}{2}N\Delta + N\min\{0,\Delta-1\} \geqslant -\frac{1}{2}N\Delta, \quad \text{if } \Delta\geqslant 1.$$
- Taking trial state $|\Phi\rangle = \big|\underbrace{\downarrow\downarrow\cdots\downarrow}_{n}\underbrace{\uparrow\uparrow\cdots\uparrow}_{N-n}\big\rangle = \big|\underbrace{\bullet\bullet\cdots\bullet}_{n}\underbrace{\circ\circ\cdots\circ}_{N-n}\big\rangle$, we find then, for any $n=\frac{1}{2}N(1-m)$,
$$-\frac{1}{2}N\Delta \leqslant E_{\rm XXZ}^{\rm min} \leqslant \langle\Phi|\mathcal{H}_{\rm XXZ}|\Phi\rangle = -\frac{1}{2}N\Delta + 2\Delta, \quad \text{if } \Delta\geqslant 1.$$

- For the XXZ model we can define the ground-state energy per site in the thermodynamic limit as

$$e(m,\Delta) = \lim_{N\to\infty} \frac{1}{N}\, E^{\min}_{\mathrm{XXZ}}(N, m_N, \Delta), \qquad m_N \equiv \frac{\lfloor Nm \rfloor}{N},$$

where $\lfloor Nm \rfloor$ is the integer part of Nm. We know $e(m,\Delta) = -\frac{1}{2}\Delta$, if $\Delta \geqslant 1$.
- The existence of the limit follows from the variational principle (or the "Bogolyubov Inequality at $T = 0$"):
Let $|\Phi_i\rangle$ be a ground state of $\mathcal{H}_i$, with $i = 1$ or 2. Then,

$$\langle \Phi_i|\mathcal{H}_i|\Phi_i\rangle \leqslant \langle \Phi_j|\mathcal{H}_i|\Phi_j\rangle = \langle \Phi_j|\mathcal{H}_j|\Phi_j\rangle + \langle \Phi_j|\mathcal{H}_i - \mathcal{H}_j|\Phi_j\rangle,$$

$$\Longrightarrow \qquad \boxed{\left|E_1^{\min} - E_2^{\min}\right| \leqslant \|\mathcal{H}_1 - \mathcal{H}_2\|}\,.$$

- Comparing the case with N_1N_2 sites with the case with N_1 copies of the XXZ model with N_2 sites, replacing N_1 bonds with other bonds, we find

$$\left|E^{\min}_{\mathrm{XXZ}}(N_1N_2, m_{N_2}, \Delta) - N_1 E^{\min}_{\mathrm{XXZ}}(N_2, m_{N_2}, \Delta)\right| \leqslant N_1 C, \quad C \equiv 2 + |\Delta|.$$

Similarly, comparing with N_2 copies of the XXZ model with N_1 sites,

$$\left|E^{\min}_{\mathrm{XXZ}}(N_1N_2, m_{N_1}, \Delta) - N_2 E^{\min}_{\mathrm{XXZ}}(N_1, m_{N_1}, \Delta)\right| \leqslant N_2 C.$$

Hence, if we define

$$e_N(m,\Delta) \equiv \frac{1}{N}\, E^{\min}_{\mathrm{XXZ}}(N, m, \Delta), \qquad e(m,\Delta) = \lim_{N\to\infty} e_N(m_N, \Delta)$$

we can rewrite the last two inequalities as

$$\left|e_{N_1N_2}(m_{N_i}, \Delta) - e_{N_i}(m_{N_i}, \Delta)\right| \leqslant \frac{C}{N_i}, \quad (i = 1, 2),$$

so that

$$\left|e_{N_1}(m_{N_1}, \Delta) - e_{N_2}(m_{N_2}, \Delta)\right| \leqslant \frac{C}{N_1} + \frac{C}{N_2} + \left|e_{N_1N_2}(m_{N_1}, \Delta) - e_{N_1N_2}(m_{N_2}, \Delta)\right|.$$

For the last step in the proof we need Lemma 3 from C.N. Yang and C.P. Yang, Phys. Rev. **147**, 303–306 (1966):

Suppose we have ground state wave vector $|\Psi_n\rangle$ and ground-state energy $E^{\min}_{\mathrm{XXZ}}(N,m,\Delta)$ for the XXZ chain on N sites in the sector with n down spins, $m = 1 - \frac{2n}{N}$. Then we can use $\sigma_j^{\pm}|\Psi_n\rangle$ as a variational trial wave vector for $|\Psi_{n\mp 1}\rangle$ with $m \to m \pm \frac{2}{N}$:

$$
\begin{aligned}
E^{\min}_{\mathrm{XXZ}}(N, m \pm \tfrac{2}{N}, \Delta) &\leqslant \frac{\langle\Psi_n|\sigma_j^{\mp}\mathcal{H}_{\mathrm{XXZ}}\sigma_j^{\pm}|\Psi_n\rangle}{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle} \\
&= \frac{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}\mathcal{H}_{\mathrm{XXZ}}|\Psi_n\rangle + \langle\Psi_n|\sigma_j^{\mp}[\mathcal{H}_{\mathrm{XXZ}},\sigma_j^{\pm}]|\Psi_n\rangle}{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle} \\
&= \frac{E^{\min}_{\mathrm{XXZ}}(N,m,\Delta)\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle + \langle\Psi_n|\sigma_j^{\mp}[\mathcal{H}_{\mathrm{XXZ}},\sigma_j^{\pm}]|\Psi_n\rangle}{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle} \\
&\leqslant E^{\min}_{\mathrm{XXZ}}(N,m,\Delta) + \left|\frac{\langle\Psi_n|\sigma_j^{\mp}[\mathcal{H}_{\mathrm{XXZ}},\sigma_j^{\pm}]|\Psi_n\rangle}{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle}\right| \\
&\leqslant E^{\min}_{\mathrm{XXZ}}(N,m,\Delta) + \frac{\|[\mathcal{H}_{\mathrm{XXZ}},\sigma_j^{\pm}]\|\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle^{1/2}\langle\Psi_n|\Psi_n\rangle^{1/2}}{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle},
\end{aligned}
$$

having used the Cauchy–Schwarz inequality in the last step. One can easily check that $\|[\mathcal{H}_{\mathrm{XXZ}},\sigma_j^{\pm}]\| \leqslant C' \equiv 2 + 2|\Delta|$. Also, $\sigma_j^{\mp}\sigma_j^{\pm} = \frac{1}{2}(1 \mp \sigma_j^z)$, while from the Perron–Frobenius theorem $|\Psi_n\rangle$ is the unique ground state in the sector with $n = \frac{1}{2}N(1-m)$ down spins and, therefore, translationally invariant. Hence,

$$
\begin{aligned}
E^{\min}_{\mathrm{XXZ}}(N, m \pm \tfrac{2}{N}, \Delta) &\leqslant E^{\min}_{\mathrm{XXZ}}(N,m,\Delta) + \frac{\|[\mathcal{H}_{\mathrm{XXZ}},\sigma_j^{\pm}]\|}{\langle\Psi_n|\sigma_j^{\mp}\sigma_j^{\pm}|\Psi_n\rangle^{1/2}} \\
&\leqslant E^{\min}_{\mathrm{XXZ}}(N,m,\Delta) + \frac{C'}{\sqrt{\frac{1}{2}(1 \mp m)}}, \quad C' = 2 + 2|\Delta|.
\end{aligned}
$$

If m not too close to ± 1 or, more precisely, both numbers of up spins and down spins must be at least $N\varepsilon$ for all m considered, then $\frac{1}{2}(1 \mp m) > \varepsilon > 0$. If we now apply the inequality repeatedly, we find

$$
|E^{\min}_{\mathrm{XXZ}}(N,m,\Delta) - E^{\min}_{\mathrm{XXZ}}(N,m',\Delta)| \leqslant \frac{NC'}{2\sqrt{\varepsilon}}\,|m - m'|,
$$

$$
\Longrightarrow \quad \boxed{\left|e_{N_1N_2}(m_{N_1},\Delta) - e_{N_1N_2}(m_{N_2},\Delta)\right| \leqslant \frac{C'}{2\sqrt{\varepsilon}}\,|m_{N_1} - m_{N_2}|}.
$$

Putting it all together,

$$\left|e_{N_1}(m_{N_1},\Delta) - e_{N_2}(m_{N_2},\Delta)\right| \leqslant \frac{C}{N_1} + \frac{C}{N_2} + \frac{C'}{2\sqrt{\varepsilon}}\left|\frac{\lfloor N_1 m\rfloor}{N_1} - \frac{\lfloor N_2 m\rfloor}{N_2}\right|$$
$$\leqslant \left(C + \frac{C'}{2\sqrt{\varepsilon}}\right)\left(\frac{1}{N_1} + \frac{1}{N_2}\right).$$

The thermodynamic limit exists as we have a converging Cauchy sequence when $|m| < 1$, while $e_N(\pm 1,\Delta) = -\frac{1}{2}\Delta$ for all N.

- Yang and Yang give several more properties in their Phys. Rev. **147** paper. Some are required by thermodynamics or by symmetry. As $\mathcal{H}_{\mathrm{XXZ}}$ commutes with $\prod_j \sigma_j^x$, while $\sum_j \sigma_j^z$ anticommutes with it, we have

$$E_{\mathrm{XXZ}}^{\min}(N,m,\Delta) = E_{\mathrm{XXZ}}^{\min}(N,-m,\Delta), \quad e(m,\Delta) = e(-m,\Delta).$$

Also, $e(m,\Delta)$ is convex in m and concave in Δ. (Yang and Yang use the now less common 'concave upward' and 'convex upward'.) We have,

$$e(m,\Delta) \equiv -\frac{1}{2}\Delta, \quad \text{for } \Delta \geqslant 1, \qquad e(0,\Delta) < e(m,\Delta), \quad \text{for } \Delta < 1,\ m \neq 0.$$

Coordinate Bethe Ansatz for Quantum Lattice Gas

We can express the wave vector in terms of basis vectors, with $|x_1, x_2, \cdots, x_n\rangle$ denoting the state with particles on sites $x_1, x_2, \cdots, x_n$. Then each n-particle state can be expressed as

$$|\Psi\rangle = \sum_{1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant N}\sum\cdots\sum f(x_1, x_2, \cdots, x_n)\,|x_1, x_2, \cdots, x_n\rangle,$$

with wave function $f(x_1, x_2, \cdots, x_n)$. We may also write

$$|\Psi\rangle = \sum_{1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant N}\sum\cdots\sum f(x_1, x_2, \cdots, x_n)\,|x_1, x_2, \cdots, x_n\rangle,$$

provided we set $f(x_1, x_2, \cdots, x_n) = 0$ when to coordinates are equal, $(x_j = x_{j+1})$, as we allowed no two particles to occupy the same site.

(Note that we only say *if* a particle is at x or not, *not* which particle.)

Remember we had

$$\mathcal{H}_{\text{QLG}} = -\frac{\hbar^2}{2ma^2}\sum_{i=1}^{N}(\sigma_{i+1}^- + \sigma_{i-1}^- - 2\sigma_i^-)\sigma_i^+ - 2\Delta\sum_{i=1}^{N}\sigma_i^-\sigma_i^+\sigma_{i+1}^-\sigma_{i+1}^+,$$

with $\frac{\hbar^2}{2ma^2} = 1$. As σ_i^+ gives zero unless there is a particle at x_i, this translates into

$$\hat{\mathcal{H}} = -\sum_{j=1}^{n}\frac{\Delta^2}{\Delta x_j^2} - 2\Delta\sum_{j=1}^{n}\delta_{x_j,x_{j+1}}, \quad \hat{\mathcal{H}}f = E_{\text{QLG}}f,$$

with $x_{n+1} \equiv x_1$. Therefore,

$$\boxed{-\sum_{j=1}^{n}\frac{\Delta^2 f}{\Delta x_j^2} - 2\Delta\, N_{\text{nn}}f = E_{\text{QLG}}f},$$

where, for any function $\phi(x_j)$,

$$\frac{\Delta^2\phi}{\Delta x_j^2} \equiv \phi(x_j+1)+\phi(x_j-1)-2\phi(x_j) \quad \text{and} \quad N_{\text{nn}}(x_1,x_2,\cdots,x_n) = \sum_{j=1}^{n}\delta_{x_j,x_{j+1}}$$

counts the number of nearest-neighbor pairs among $x_1, x_2, \cdots, x_n$.

Therefore, we have the *non-interacting* second-order linear partial difference equation

$$\boxed{-\sum_{j=1}^{n}\frac{\Delta^2 f}{\Delta x_j^2} = E_{\text{QLG}}f}, \tag{1}$$

with two kinds of boundary conditions:

1° Equal coordinate/nearest-neighbor conditions,

$$\boxed{\begin{aligned} f(x_1,\cdots,x_j{+}1,x_{j+1},\cdots,x_n) + f(x_1,\cdots,x_j,x_{j+1}{-}1,\cdots,x_n) \\ = 2\Delta\, f(x_1,\cdots,x_j,x_{j+1},\cdots,x_n), \quad \text{whenever } x_{j+1} = x_j. \end{aligned}} \tag{2}$$

2° Periodic boundary conditions,

$$\boxed{\begin{aligned} &f(0,x_2,x_3,\cdots,x_n) \equiv f(x_2,x_3,\cdots,x_n,N), \\ &f(x_1,x_2,\cdots,x_{n-1},N{+}1) \equiv f(1,x_1,x_2,\cdots,x_{n-1}). \end{aligned}} \tag{3}$$

Condition 2° is self-evident. Let us explain condition 1°:

Let us start with the three terms of $\hat{\mathcal{H}}$ are relevant if $x_{j+1} = x_j + 1$:

$$
\begin{aligned}
&-\frac{\Delta^2 f}{\Delta x_j^2} - \frac{\Delta^2 f}{\Delta x_{j+1}^2} - 2\Delta\, f \\
&\qquad = -(f(x_1, \cdots, x_j{+}1, x_{j+1}, \cdots, x_n) + f(x_1, \cdots, x_j{-}1, x_{j+1}, \cdots, x_n) \\
&\qquad\qquad\qquad - 2f(x_1, \cdots, x_j, x_{j+1}, \cdots, x_n)) \\
&\qquad\quad - (f(x_1, \cdots, x_j, x_{j+1}{+}1, \cdots, x_n) + f(x_1, \cdots, x_j, x_{j+1}{-}1, \cdots, x_n) \\
&\qquad\qquad\qquad - 2f(x_1, \cdots, x_j, x_{j+1}, \cdots, x_n)) \\
&\qquad\quad - 2\Delta\, f(x_1, \cdots, x_j, x_{j+1}, \cdots, x_n) \\
&\qquad = -(f(x_1, \cdots, x_j{+}1, x_j{+}1, \cdots, x_n) + f(x_1, \cdots, x_j{-}1, x_j{+}1, \cdots, x_n) \\
&\qquad\qquad\qquad - 2f(x_1, \cdots, x_j, x_j{+}1, \cdots, x_n)) \\
&\qquad\quad - (f(x_1, \cdots, x_j, x_j{+}2, \cdots, x_n) + f(x_1, \cdots, x_j, x_j, \cdots, x_n) \\
&\qquad\qquad\qquad - 2f(x_1, \cdots, x_j, x_j{+}1, \cdots, x_n)) \\
&\qquad\quad - 2\Delta\, f(x_1, \cdots, x_j, x_j{+}1, \cdots, x_n).
\end{aligned}
$$

The term $2\Delta\, f$ is absent in the noninteracting equation (1), while the two terms $f(x_1, \cdots, x_j{+}1, x_j{+}1, \cdots, x_n)$ and $f(x_1, \cdots, x_j, x_j, \cdots, x_n)$ are zero there.

Since we only need $f(x_1, \cdots, x_n)$ for $x_1 < \cdots < x_j < x_{j+1} < \cdots < x_n$, we can alter the value of f when two coordinates are the same. Making the choice (2) incorporates the nearest-neighbor interaction -2Δ into the equation

$$
\boxed{-\sum_{j=1}^{n} \frac{\Delta^2 f}{\Delta x_j^2} = E_{\mathrm{QLG}} f}, \tag{1}
$$

for the non-interacting QLG with infinite on-site repulsion, just setting

$$
\boxed{\begin{aligned} &f(x_1, \cdots, x_j{+}1, x_{j+1}, \cdots, x_n) + f(x_1, \cdots, x_j, x_{j+1}{-}1, \cdots, x_n) \\ &\qquad = 2\Delta f(x_1, \cdots, x_j, x_{j+1}, \cdots, x_n), \quad \text{whenever } x_{j+1} = x_j. \end{aligned}} \tag{2}
$$

The solution of the Schrödinger equation for our QLG is fully determined by these two equations and

$$
\boxed{\begin{aligned} &f(0, x_2, x_3, \cdots, x_n) \equiv f(x_2, x_3, \cdots, x_n, N), \\ &f(x_1, x_2, \cdots, x_{n-1}, N{+}1) \equiv f(1, x_1, x_2, \cdots, x_{n-1}). \end{aligned}} \tag{3}
$$

By the separation-of-variables method, the general solution of equation (1) is a linear combination of plane waves of the form

$$f(x_1, \cdots, x_n) = \frac{1}{\mathcal{N}} \exp\left(\mathrm{i}\sum_{j=1}^{n} k_j x_j\right),$$

such that

$$E_{\mathrm{QLG}} = \sum_{j=1}^{n}(2 - 2\cos k_j) = \sum_{j=1}^{n} 4\sin^2\frac{k_j}{2}.$$

The wave vectors k_j could be complex, as there could be bound states.

The two cases $n = 0$ or 1 are easy:

If $n = 0$ we have the vacuum state $|\,\emptyset\,\rangle$ and $f(\emptyset) = 1$, $E_{\mathrm{QLG}} = 0$.

If $n = 1$ we have $f(0) = f(N)$ from (3), so that

$$f(x_1) = \frac{1}{\sqrt{N}}\,\mathrm{e}^{\mathrm{i}k_1x_1}, \quad E_{\mathrm{QLG}} = 4\sin^2\frac{k_1}{2}, \quad k_1 = \frac{2\pi\ell_1}{N}, \quad (\ell_1 = 0, 1, \cdots, N-1).$$

If $n > 1$, we find from (2) that with the ordered set of wave vectors $[k_1, \cdots, k_n]$ we also need the set with k_j and k_{j+1} interchanged. From (3) we see that we need all cyclic permutations of $k_1, \cdots, k_n$. At minimum we shall need all $n!$ permutations P, i.e., all $n!$ choices $[k_{\mathrm{P}(1)}, \cdots, k_{\mathrm{P}(n)}]$, so that we end up with the Bethe Ansatz:

$$\boxed{f(x_1, \cdots, x_n) = \frac{1}{\mathcal{N}} \sum_{\mathrm{P}} a(\mathrm{P}) \exp\left(\mathrm{i}\sum_{j=1}^{n} k_{\mathrm{P}(j)} x_j\right)}, \tag{4}$$

with n wave vectors k_j and $n!$ coefficients $a(\mathrm{P})$ to be determined. [Here $\mathcal{N}$ is a normalization that could be absorbed into the $a(\mathrm{P})$.]

Substituting (4) into (2), we have

$$\sum_{\mathrm{P}} a(\mathrm{P})\, \mathrm{e}^{\sum_{\ell\neq j,j+1} \mathrm{i}k_{\mathrm{P}(\ell)}x_\ell} \Big\{ \mathrm{e}^{\mathrm{i}k_{\mathrm{P}(j)}(x_j+1) + \mathrm{i}k_{\mathrm{P}(j+1)}(x_j+1)} + \mathrm{e}^{\mathrm{i}k_{\mathrm{P}(j)}x_j + \mathrm{i}k_{\mathrm{P}(j+1)}x_j} - 2\Delta\, \mathrm{e}^{\mathrm{i}k_{\mathrm{P}(j)}x_j + \mathrm{i}k_{\mathrm{P}(j+1)}(x_j+1)} \Big\} = 0.$$

or

$$\sum_{\mathrm{P}} a(\mathrm{P})\, \mathrm{e}^{\sum_{\ell\neq j,j+1} \mathrm{i}k_{\mathrm{P}(\ell)}x_\ell}\, \mathrm{e}^{\mathrm{i}(k_{\mathrm{P}(j)}+k_{\mathrm{P}(j+1)})x_j} \left\{\mathrm{e}^{\mathrm{i}k_{\mathrm{P}(j)}+\mathrm{i}k_{\mathrm{P}(j+1)}}+1-2\Delta\, \mathrm{e}^{\mathrm{i}k_{\mathrm{P}(j+1)}}\right\} = 0.$$

Equating the coefficients of similar exponentials in the x_j we find

$$\sum_{\mathrm{P}'=\mathrm{P},\mathrm{P}^*} a(\mathrm{P}')\left\{\mathrm{e}^{\mathrm{i}k_{\mathrm{P}'(j)}+\mathrm{i}k_{\mathrm{P}'(j+1)}}+1-2\Delta\, \mathrm{e}^{\mathrm{i}k_{\mathrm{P}'(j+1)}}\right\} = 0,$$

where now P and P^* are any fixed pair of permutations that only differs in the action on j and $j+1$:

$$\left.\begin{aligned} &\mathrm{P}(j) = \mathrm{P}^*(j+1),\\ &\mathrm{P}(j+1) = \mathrm{P}^*(j),\\ &\mathrm{P}(\ell) = \mathrm{P}^*(\ell),\ \ell\neq j, j+1, \end{aligned}\right\} \qquad \begin{aligned} k_{\mathrm{P}(j)} &= k_{\mathrm{P}^*(j+1)} \equiv p,\\ k_{\mathrm{P}(j+1)} &= k_{\mathrm{P}^*(j)} \equiv q. \end{aligned}$$

Putting this in the equation relating $a(\mathrm{P})$ and $a(\mathrm{P}^*)$, we find

$$a(\mathrm{P})(\mathrm{e}^{\mathrm{i}p+\mathrm{i}q}+1-2\Delta\mathrm{e}^{\mathrm{i}q}) + a(\mathrm{P}^*)(\mathrm{e}^{\mathrm{i}p+\mathrm{i}q}+1-2\Delta\mathrm{e}^{\mathrm{i}p}) = 0,$$

or

$$\frac{a(\mathrm{P})}{a(\mathrm{P}^*)} = -\frac{2\Delta\mathrm{e}^{\mathrm{i}p}-1-\mathrm{e}^{\mathrm{i}p+\mathrm{i}q}}{2\Delta\mathrm{e}^{\mathrm{i}q}-1-\mathrm{e}^{\mathrm{i}p+\mathrm{i}q}} \equiv -\mathrm{e}^{-\mathrm{i}\Theta(p,q)}.$$

where $\Theta(p,q)$ is also given by (exercise)

$$\boxed{\tan\frac{\Theta(p,q)}{2} = \frac{\tan\frac{q}{2}-\tan\frac{p}{2}}{(1-\frac{1}{\Delta})+(1+\frac{1}{\Delta})\tan\frac{q}{2}\tan\frac{p}{2}}}. \tag{5}$$

We can solve $a(\mathrm{P})$ as

$$\boxed{a(\mathrm{P}) = (-1)^{\mathrm{P}}\prod_{\ell<j}\exp\left[-\frac{\mathrm{i}}{2}\Theta(k_{\mathrm{P}(\ell)},k_{\mathrm{P}(j)})\right]}. \tag{6}$$

To see this, just calculate $a(\mathrm{P})/a(\mathrm{P}^*)$ and use $\Theta(p,q) = -\Theta(q,p)$.

It remains to solve the periodic boundary conditions (3), in order to determine the wave vectors k_j. We only need to solve

$$f(0, x_2, x_3, \cdots, x_n) = f(x_2, x_3, \cdots, x_n, N),$$

as the other equation is equivalent to

$$f(1, x_2+1, x_3+1, \cdots, x_n+1) = f(x_2+1, x_3+1, \cdots, x_n+1, N+1),$$

which after substituting Bethe Ansatz (4) reduces to the first one after taking out a common factor $\exp(\mathrm{i}\sum_j k_j)$ from every term. Therefore, both give

$$\begin{aligned}\sum_{\mathrm{P}} a(\mathrm{P}) \exp\left(\mathrm{i}\sum_{j=2}^{n} k_{\mathrm{P}(j)} x_j\right) &= \sum_{\mathrm{P}'} a(\mathrm{P}') \exp\left(\mathrm{i}\sum_{j=1}^{n-1} k_{\mathrm{P}'(j)} x_{j+1} + \mathrm{i}k_{\mathrm{P}'(n)} N\right) \\ &= \sum_{\mathrm{P}} a(\mathrm{P}') \exp\left(\mathrm{i}\sum_{j=2}^{n} k_{\mathrm{P}(j)} x_j + \mathrm{i}k_{\mathrm{P}(1)} N\right),\end{aligned}$$

identifying $\mathrm{P}'(n) = \mathrm{P}(1)$ and $\mathrm{P}'(j) = \mathrm{P}(j+1)$ for $j = 1, \cdots, n-1$.

Equating coefficients of similar exponentials,

$$a(\mathrm{P}) = a(\mathrm{P}') \exp\left(\mathrm{i}k_{\mathrm{P}(1)} N\right) = a(\mathrm{PP}_n) \exp\left(\mathrm{i}k_{\mathrm{P}(1)} N\right),$$

introducing the cyclic permutations P_ℓ satisfying

$$\mathrm{P}_\ell(j) = j+1, \quad \text{for } j < \ell, \qquad \mathrm{P}_\ell(\ell) = 1, \qquad \mathrm{P}_\ell(j) = j, \quad \text{for } j > \ell,$$

so that $\mathrm{P}' = \mathrm{PP}_n$.

In cycle notation, indicating how objects in an ordered set move their positions, $\mathrm{P}_\ell = (1\,2\,3 \cdots \ell)^{-1}$, i.e., $1 \leftarrow 2 \leftarrow \cdots \leftarrow \ell \leftarrow 1$, and P_1 is the identity $\mathrm{P}_1(j) \equiv j$. Let also $\mathrm{P}_{i,j} = (ij)$ be the interchange $i \leftrightarrow j$. Then one can easily check

$$\mathrm{P}_\ell = \mathrm{P}_{\ell-1}\mathrm{P}_{1,\ell} \qquad \text{or} \qquad (123\cdots\underline{\ell-1})^{-1}\,(1\ell) = (123\cdots\ell)^{-1}.$$

Therefore,

$$\mathrm{P}' = \mathrm{PP}_n = \mathrm{PP}_{1,n}\mathrm{P}_{1,n-1}\cdots\mathrm{P}_{1,3}\mathrm{P}_{1,2},$$

and

$$\frac{a(\mathrm{P}')}{a(\mathrm{P})} = \frac{a(\mathrm{PP}_n)}{a(\mathrm{P})} = \frac{a(\mathrm{PP}_n)}{a(\mathrm{PP}_{n-1})}\,\frac{a(\mathrm{PP}_{n-1})}{a(\mathrm{PP}_{n-2})}\cdots\frac{a(\mathrm{PP}_2)}{a(\mathrm{P})} = \prod_{\ell=2}^{n}\frac{a(\mathrm{PP}_\ell)}{a(\mathrm{PP}_{\ell-1})}.$$

Action of the various permutations PP_ℓ, ($PP_1 \equiv P$):

	1	2	3	4	$\cdots$	$\ell-2$	$\ell-1$	ℓ	$\cdots$	$n-1$	n
P	P(1)	P(2)	P(3)	P(4)	$\cdots$	$P(\ell-2)$	$P(\ell-1)$	$P(\ell)$	$\cdots$	$P(n-1)$	$P(n)$
PP_2	P(2)	P(1)	P(3)	P(4)	$\cdots$	$P(\ell-2)$	$P(\ell-1)$	$P(\ell)$	$\cdots$	$P(n-1)$	$P(n)$
PP_3	P(2)	P(3)	P(1)	P(4)	$\cdots$	$P(\ell-2)$	$P(\ell-1)$	$P(\ell)$	$\cdots$	$P(n-1)$	$P(n)$
PP_4	P(2)	P(3)	P(4)	P(1)	$\cdots$	$P(\ell-2)$	$P(\ell-1)$	$P(\ell)$	$\cdots$	$P(n-1)$	$P(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$PP_{\ell-1}$	P(2)	P(3)	P(4)	P(5)	$\cdots$	$P(\ell-1)$	$P(1)$	$P(\ell)$	$\cdots$	$P(n-1)$	$P(n)$
PP_ℓ	P(2)	P(3)	P(4)	P(5)	$\cdots$	$P(\ell-1)$	$P(\ell)$	$P(1)$	$\cdots$	$P(n-1)$	$P(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
PP_n	P(2)	P(3)	P(4)	P(5)	$\cdots$	$P(\ell-1)$	$P(\ell)$	$P(\ell+1)$	$\cdots$	$P(n)$	$P(1)$

The actions of $PP_{\ell-1}$ and PP_ℓ differ only by the interchange of P(1) and $P(\ell)$.

We had for P and P^* only differing by their actions on j and $j+1$,

$$\frac{a(P)}{a(P^*)} = -e^{-i\Theta(p,q)}, \quad p = k_{P(j)} = k_{P^*(j+1)}, \quad q = k_{P(j+1)} = k_{P^*(j)}.$$

Therefore,

$$\frac{a(PP_{\ell-1})}{a(PP_\ell)} = -e^{-i\Theta(p,q)} = -e^{-i\Theta(k_{P(1)},k_{P(\ell)})} = -e^{i\Theta(k_{P(\ell)},k_{P(1)})},$$

after use of

$$p = k_{PP_{\ell-1}(\ell-1)} = k_{PP_\ell(\ell)} = k_{P(1)}, \quad q = k_{PP_{\ell-1}(\ell)} = k_{PP_\ell(\ell-1)} = k_{P(\ell)},$$

so that

$$e^{-ik_{P(1)}N} = \frac{a(P')}{a(P)} = \frac{a(PP_n)}{a(P)} = \prod_{\ell=2}^{n} \frac{a(PP_\ell)}{a(PP_{\ell-1})} = \prod_{\ell=2}^{n} \big[-e^{-i\Theta(k_{P(\ell)},k_{P(1)})}\big].$$

Replacing $\mathrm{P}(1) \Longrightarrow j$ and $\mathrm{P}(\ell) \Longrightarrow \ell$ and using $\Theta(p,p)=0$, we have

$$\boxed{\mathrm{e}^{\mathrm{i}k_j N} = (-1)^{n-1}\prod_{\ell=1}^{n}\mathrm{e}^{\mathrm{i}\Theta(k_\ell,k_j)}, \quad \text{for } j=1,2,\cdots,n}. \tag{7}$$

Taking the logarithm we find, following Yang and Yang [Phys. Rev. **150**, 321–327 (1966)],

$$\boxed{\begin{aligned} Nk_j = 2\pi I_j + \sum_{\ell=1}^{n}\Theta(k_\ell,k_j), \quad j=1,\cdots,n,\\ \text{with } I_j \text{ integer for } n \text{ odd},\\ \text{half-integer for } n \text{ even.}\end{aligned}} \tag{8}$$

Just like in Kaufman's 1949 solution of the two-dimensional Ising model, we have two sectors, (n even or odd), nowadays called the Ramond and Neveu-Schwarz sectors.

Continuum Limit of Quantum Lattice Gas

Remember we started with

$$\mathcal{H}_{\mathrm{QLG}} = -\frac{\hbar^2}{2ma^2}\sum_{i=1}^{N}(\sigma_{i+1}^{-}+\sigma_{i-1}^{-}-2\sigma_i^{-})\sigma_i^{+} - 2\Delta\sum_{i=1}^{N}\sigma_i^{-}\sigma_i^{+}\sigma_{i+1}^{-}\sigma_{i+1}^{+},$$

from which we got

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2ma^2}\sum_{j=1}^{n}\frac{\Delta^2}{\Delta x_j^2} - 2\Delta\sum_{j=1}^{n}\delta_{x_j,x_{j+1}}, \quad \hat{\mathcal{H}}f = E_{\mathrm{QLG}}f.$$

Before we chose dimensionless variables setting

$$\frac{\hbar^2}{2ma^2} = 1.$$

but now we have reintroduced the lattice constant a, so that we can take the continuum limit letting $a \downarrow 0$.

Let us replace the coordinates and other variables as follows:

position :	x_j	$\longrightarrow$	$x_j^{(a)} = a x_j$
lattice :	$1, 2, \cdots, N$	$\longrightarrow$	$a, 2a, \cdots, Na = L$
wave function :	$f(x_1, x_2, \cdots, x_n)$	$\longrightarrow$	$f^{(a)}(x_1^{(a)}, x_2^{(a)}, \cdots, x_n^{(a)})$
domain :	$1 \leqslant x_1 < x_2 < \cdots < x_n$	$\longrightarrow$	$0 < a \leqslant x_1^{(a)} < x_2^{(a)} < \cdots < x_n^{(a)} \leqslant Na = L$
energy :	$E_{\mathrm{QLG}}(N, n, \Delta)$	$\longrightarrow$	$\frac{\hbar^2}{2ma^2} E_{\mathrm{QLG}}(N, n, \Delta) \equiv E_{\mathrm{QLG}}^{(a)}(N, n, \Delta)$
wave vector :	k_j	$\longrightarrow$	$k_j^{(a)} = \frac{k_j}{a}$
interaction :	Δ	$\longrightarrow$	$\Delta^{(a)} = 1 - \frac{1}{2} ac$
units :	$\frac{\hbar^2}{2ma^2} = 1$	$\longrightarrow$	$\frac{\hbar^2}{2m} = 1$

Then in the limit $a \downarrow 0$, $N \to \infty$, such that $Na = L$,

$$\frac{1}{a^2} \frac{\Delta^2 f}{\Delta x_j^2} = \frac{\Delta^2 f^{(a)}}{\left(\Delta x_j^{(a)}\right)^2}$$

$$= \frac{f^{(a)}(\cdots, x_j^{(a)} + a, \cdots) + f^{(a)}(\cdots, x_j^{(a)} - a, \cdots) - 2 f^{(a)}(\cdots, x_j^{(a)}, \cdots)}{a^2}$$

$$\xrightarrow{a \downarrow 0} \quad \frac{\partial^2 f^{(0)}}{\partial x_j^{(0)2}},$$

so that equation (1) transforms as

$$-\sum_{j=1}^{n} \frac{\Delta^2 f}{\Delta x_j^2} = E_{\mathrm{QLG}} f \quad \longrightarrow \quad -\sum_{j=1}^{n} \frac{\partial^2 f^{(0)}}{\partial x_j^{(0)2}} = E_L^{(0)}(n, c) f^{(0)},$$

or, dropping the superscripts (0),

$$\boxed{-\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = E_L(n, c) f}. \tag{1'}$$

We can also rewrite

$$f(x_1,\cdots,x_j+1,x_{j+1},\cdots,x_n)+f(x_1,\cdots,x_j,x_{j+1}-1,\cdots,x_n) \\ =2\Delta\, f(x_1,\cdots,x_j,x_{j+1},\cdots,x_n), \quad \text{whenever } x_{j+1}=x_j, \tag{2}$$

as

$$\begin{aligned} \frac{\Delta f^{(a)}}{\Delta x_j^{(a)}}+\frac{\Delta f^{(a)}}{-\Delta x_{j+1}^{(a)}} &= \frac{f^{(a)}(\cdots,x_j^{(a)}+a,x_{j+1}^{(a)},\cdots)-f^{(a)}(\cdots,x_j^{(a)},x_{j+1}^{(a)},\cdots)}{a} \\ &\quad + \frac{f^{(a)}(\cdots,x_j^{(a)},x_{j+1}^{(a)}-a,\cdots)-f^{(a)}(\cdots,x_j^{(a)},x_{j+1}^{(a)},\cdots)}{a} \\ &= \frac{2(\Delta-1)}{a} f^{(a)}(\cdots,x_j^{(a)},x_{j+1}^{(a)},\cdots), \quad \text{for } x_{j+1}^{(a)}=x_j^{(a)}+a\,. \end{aligned}$$

In the limit $a\downarrow 0$ this becomes

$$\left(\frac{\partial f^{(0)}}{\partial x_j^{(0)}}-\frac{\partial f^{(0)}}{\partial x_{j+1}^{(0)}}\right)\Bigg|_{x_{j+1}^{(0)}\downarrow x_j^{(0)}} = -c\, f^{(0)}(x_1^{(0)},\cdots,x_n^{(0)})\Bigg|_{x_{j+1}^{(0)}\downarrow x_j^{(0)}},$$

or, again dropping the superscripts (0),

$$\boxed{\frac{\partial f}{\partial x_{j+1}}-\frac{\partial f}{\partial x_j}\bigg|_{x_{j+1}=x_j} = c\, f\Big|_{x_{j+1}=x_j}}\,. \tag{2$'$}$$

Even more directly, in the limit $a\downarrow 0$,

$$\begin{aligned} &f(0,x_2,x_3,\cdots,x_n)\equiv f(x_2,x_3,\cdots,x_n,N), \\ &f(x_1,x_2,\cdots,x_{n-1},N+1)\equiv f(1,x_1,x_2,\cdots,x_{n-1}). \end{aligned} \tag{3}$$

becomes

$$\boxed{f(0,x_2,x_3,\cdots,x_n)\equiv f(x_2,x_3,\cdots,x_n,L)}\,. \tag{3$'$}$$

Equations (1′), (2′) and (3′) are equations (2.1a), (2.4a) and (2.8a) in the paper of E.H. Lieb and W. Liniger [Phys. Rev. **130**, 1605–1616 (1963)].

Therefore, in the continuum limit described the quantum lattice gas model and the XXZ model reduce to the Bose gas with delta interaction.

The Bethe Ansatz for the wave function

$$f(x_1,\cdots,x_n) = \frac{1}{\mathcal{N}} \sum_{\mathrm{P}} a(\mathrm{P}) \exp\left(\mathrm{i}\sum_{j=1}^{n} k_{\mathrm{P}(j)} x_j\right), \tag{4}$$

agrees with (2.9) of Lieb and Liniger.

Using $k_j^{(a)} = k_j/a$ or $k_j = a k_j^{(a)}$, we rewrite

$$\tan\frac{\Theta(p,q)}{2} = \frac{\tan\frac{q}{2} - \tan\frac{p}{2}}{(1-\frac{1}{\Delta}) + (1+\frac{1}{\Delta})\tan\frac{q}{2}\tan\frac{p}{2}}, \tag{5}$$

as

$$\tan\frac{\Theta^{(a)}(p^{(a)},q^{(a)})}{2} = \frac{\tan(a\frac{q^{(a)}}{2}) - \tan(a\frac{p^{(a)}}{2})}{(1-\frac{1}{\Delta}) + (1+\frac{1}{\Delta})\tan(a\frac{q^{(a)}}{2})\tan(a\frac{p^{(a)}}{2})}$$
$$\longrightarrow \quad \frac{1}{-\frac{1}{2}ac}\,\frac{a}{2}\,(q^{(0)} - p^{(0)}) = -\frac{q^{(0)} - p^{(0)}}{c},$$

and

$$E_{\mathrm{QLG}} = \sum_{j=1}^{n} 4\sin^2\frac{k_j}{2} \qquad \text{as} \qquad E_{\mathrm{QLG}}^{(a)}(N,n,\Delta) = \frac{\hbar^2}{2ma^2}\sum_{j=1}^{n} 4\sin^2\frac{a k_j^{(a)}}{2},$$

so that

$$\Theta^{(0)}(p,q) = -2\arctan\left(\frac{q-p}{c}\right) = \theta(q-p), \tag{5'}$$

$$E_{\mathrm{QLG}}^{(0)} = \frac{\hbar^2}{2m}\sum_{j=1}^{n} k_j^2 = E_{\mathrm{LL}},$$

agreeing with (2.13b) and (2.10) of Lieb and Liniger. Finally, we similarly get

$$a(\mathrm{P}) = (-1)^{\mathrm{P}} \prod_{\ell<j} \exp\left[-\frac{\mathrm{i}}{2}\theta(k_{\mathrm{P}(j)} - k_{\mathrm{P}(\ell)})\right]. \tag{6'}$$

$$\mathrm{e}^{\mathrm{i}k_j L} = (-1)^{n-1}\prod_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}\theta(k_j - k_\ell)}, \quad \text{for } j = 1,2,\cdots,n. \tag{7'}$$

$$\begin{aligned} L k_j =& 2\pi I_j + \sum_{\ell=1}^{n}\theta(k_j - k_\ell), \quad j = 1,\cdots,n, \\ & \text{with } I_j \text{ integer for } n \text{ odd}, \quad \text{half-integer for } n \text{ even}. \end{aligned} \tag{8'}$$