The dynamics of free fermions in disguise

Balázs Pozsgay

Mathematical Physics for Quantum Science

Hangzhou, November 2024

Short message…

Simplest systems in many body dynamics:

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain $H = \sum_j X_j X_{j+1} + h_z Z_j$

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain $$H = \sum_j X_j X_{j+1} + h_z Z_j$$

- XYh model

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain $$H = \sum_j X_j X_{j+1} + h_z Z_j$$

- XYh model $$H = \sum_j J_x X_j X_{j+1} + J_y Y_j Y_{j+1} + h_z Z_j$$

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain $$H = \sum_j X_j X_{j+1} + h_z Z_j$$
- XYh model $$H = \sum_j J_x X_j X_{j+1} + J_y Y_j Y_{j+1} + h_z Z_j$$
- Selected (quantum) cellular automata

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain $H = \sum_j X_j X_{j+1} + h_z Z_j$

- XYh model $H = \sum_j J_x X_j X_{j+1} + J_y Y_j Y_{j+1} + h_z Z_j$

- Selected (quantum) cellular automata

- Kitaev chain

Simplest systems in many body dynamics:

FREE bosons and FREE fermions

Examples:

- Quantum Ising chain $$H = \sum_j X_j X_{j+1} + h_z Z_j$$
- XYh model $$H = \sum_j J_x X_j X_{j+1} + J_y Y_j Y_{j+1} + h_z Z_j$$
- Selected (quantum) cellular automata
- Kitaev chain
- Kitaev's honeycomb model (2D)

Questions:

Questions:

- Where do the fermions come from?

Questions:

- Where do the fermions come from?
- What is the most general class of free fermionic models?

The Jordan-Wigner transformation (1928)

The Jordan-Wigner transformation (1928)

$$[\sigma_j^+, \sigma_k^+] = [\sigma_j^+, \sigma_k^-] = [\sigma_j^-, \sigma_k^-] = 0$$

$$\{\sigma_j^+, \sigma_j^-\} = 1$$

The Jordan-Wigner transformation (1928)

$$[\sigma_j^+, \sigma_k^+] = [\sigma_j^+, \sigma_k^-] = [\sigma_j^-, \sigma_k^-] = 0$$

$$\{\sigma_j^+, \sigma_j^-\} = 1$$

$$c_j = \left(\prod_{k=1}^{j-1} Z_j \right) \sigma_j^-, \qquad c_j^\dagger = \left(\prod_{k=1}^{j-1} Z_j \right) \sigma_j^+$$

The Jordan-Wigner transformation (1928)

$$[\sigma_j^+, \sigma_k^+] = [\sigma_j^+, \sigma_k^-] = [\sigma_j^-, \sigma_k^-] = 0$$

$$\{\sigma_j^+, \sigma_j^-\} = 1$$

$$c_j = \left(\prod_{k=1}^{j-1} Z_j\right)\sigma_j^-, \qquad c_j^\dagger = \left(\prod_{k=1}^{j-1} Z_j\right)\sigma_j^+$$

$$\{c_j, c_k\} = 0, \qquad \{c_j^\dagger, c_k^\dagger\} = 0, \qquad \{c_j, c_k^\dagger\} = \delta_{jk}$$

The Jordan-Wigner transformation (1928)

$$[\sigma_j^+, \sigma_k^+] = [\sigma_j^+, \sigma_k^-] = [\sigma_j^-, \sigma_k^-] = 0$$

$$\{\sigma_j^+, \sigma_j^-\} = 1$$

$$c_j = \left(\prod_{k=1}^{j-1} Z_j \right) \sigma_j^-, \qquad c_j^\dagger = \left(\prod_{k=1}^{j-1} Z_j \right) \sigma_j^+$$

$$\{c_j, c_k\} = 0, \qquad \{c_j^\dagger, c_k^\dagger\} = 0, \qquad \{c_j, c_k^\dagger\} = \delta_{jk}$$

$$H = \left[\sum_{j,k} A_{jk} c_j c_k + B_{jk} c_j^\dagger c_k \right] + c.c.$$

The Jordan-Wigner transformation (1928)

$$[\sigma_j^+, \sigma_k^+] = [\sigma_j^+, \sigma_k^-] = [\sigma_j^-, \sigma_k^-] = 0$$

$$\{\sigma_j^+, \sigma_j^-\} = 1$$

$$c_j = \left(\prod_{k=1}^{j-1} Z_j\right)\sigma_j^-, \qquad c_j^\dagger = \left(\prod_{k=1}^{j-1} Z_j\right)\sigma_j^+$$

$$\{c_j, c_k\} = 0, \qquad \{c_j^\dagger, c_k^\dagger\} = 0, \qquad \{c_j, c_k^\dagger\} = \delta_{jk}$$

$$H = \left[\sum_{j,k} A_{jk} c_j c_k + B_{jk} c_j^\dagger c_k\right] + c.c.$$

Fermionic eigenmodes:

$$H = \sum_k \varepsilon_k \Psi_k^\dagger \Psi_k$$

$$\{\Psi_j, \Psi_k\} = 0, \qquad \{\Psi_j^\dagger, \Psi_k^\dagger\} = 0, \qquad \{\Psi_j, \Psi_k^\dagger\} = \delta_{jk}$$

- (generalized) Jordan-Wigner solvability:

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

$$H = \sum_j \alpha_j h_j, \qquad h_j \text{ are a product of Paulis}$$

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

$$H = \sum_j \alpha_j h_j, \qquad h_j \text{ are a product of Paulis}$$

$$h_j = i\chi_a \chi_b$$

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

When is a collection of h_j expressible as bilinears in Majorana fermions?

$$H = \sum_j \alpha_j h_j,$$ h_j are a product of Paulis

$$h_j = i\chi_a \chi_b$$

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

$$H = \sum_j \alpha_j h_j,$$ h_j are a product of Paulis

$$h_j = i\chi_a \chi_b$$

When is a collection of h_j expressible as bilinears in Majorana fermions?

A. Chapman and S. Flammia,
arXiv:2003.05465v2, Quantum 4, 278 (2020).
M. Ogura, Y. Imamura, N. Kameyama, K. Minami, M. Sato
arXiv:2003.13264, Phys. Rev. B 102, 245118 (2020)

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

When is a collection of h_j expressible as bilinears in Majorana fermions?

$$H = \sum_j \alpha_j h_j, \qquad h_j \text{ are a product of Paulis}$$

A. Chapman and S. Flammia,
arXiv:2003.05465v2, Quantum 4, 278 (2020).
M. Ogura, Y. Imamura, N. Kameyama, K. Minami, M. Sato
arXiv:2003.13264, Phys. Rev. B 102, 245118 (2020)

$$h_j = i\chi_a \chi_b$$

- Free fermions in disguise, Paul Fendley,

arXiv:1901.08078, J. Phys. A: Math. Theor. 52 (2019) 335002

$$H = \sum_j \alpha_j h_j, \qquad h_j = X_j X_{j+1} Z_{j+2}$$

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

When is a collection of h_j expressible as bilinears in Majorana fermions?

$$H = \sum_j \alpha_j h_j, \qquad h_j \text{ are a product of Paulis}$$

A. Chapman and S. Flammia,
arXiv:2003.05465v2, Quantum 4, 278 (2020).

$$h_j = i\chi_a \chi_b$$

M. Ogura, Y. Imamura, N. Kameyama, K. Minami, M. Sato
arXiv:2003.13264, Phys. Rev. B 102, 245118 (2020)

- Free fermions in disguise, Paul Fendley,

arXiv:1901.08078, J. Phys. A: Math. Theor. 52 (2019) 335002

$$H = \sum_j \alpha_j h_j, \qquad h_j = X_j X_{j+1} Z_{j+2}$$

$$H = \sum_k \varepsilon_k [\Psi_k^\dagger, \Psi_k] \qquad \{\Psi_j, \Psi_k\} = 0, \qquad \{\Psi_j^\dagger, \Psi_k^\dagger\} = 0, \qquad \{\Psi_j, \Psi_k^\dagger\} = \delta_{jk}$$

- (generalized) Jordan-Wigner solvability:

$$\chi_j = \chi_j^\dagger, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}$$

When is a collection of h_j expressible as bilinears in Majorana fermions?

$$H = \sum_j \alpha_j h_j,$$ h_j are a product of Paulis

A. Chapman and S. Flammia,
arXiv:2003.05465v2, Quantum 4, 278 (2020).
M. Ogura, Y. Imamura, N. Kameyama, K. Minami, M. Sato
arXiv:2003.13264, Phys. Rev. B 102, 245118 (2020)

$$h_j = i\chi_a\chi_b$$

- Free fermions in disguise, Paul Fendley,

arXiv:1901.08078, J. Phys. A: Math. Theor. 52 (2019) 335002

$$H = \sum_j \alpha_j h_j, \qquad h_j = X_j X_{j+1} Z_{j+2}$$

$$H = \sum_k \varepsilon_k [\Psi_k^\dagger, \Psi_k] \qquad \{\Psi_j, \Psi_k\} = 0, \qquad \{\Psi_j^\dagger, \Psi_k^\dagger\} = 0, \qquad \{\Psi_j, \Psi_k^\dagger\} = \delta_{jk}$$

h_j not bilinear in fermionic operators!

History:

- Fendley, Schoutens (2007)
- Fendley (2019): $$H = \sum_j X_j X_{j+1} Z_j$$
- Elman, Chapman, Flammia, Mann (2021+2023)
- Fendley, BP (2023): $$H = \sum_j X_j X_{j+1} Z_j + b Z_j Z_{j+2} + b^2 Z_j Y_{j+1} Y_{j+2}$$
- Chapman, Flammia, Kollár (2022)
- Alcaraz, Pimenta (6 papers, 2020-2023)
- BP (2024)
- Vona, Mestyán, BP (2024)

History:

- Fendley, Schoutens (2007)
- Fendley (2019): $$H = \sum_j X_j X_{j+1} Z_j$$
- Elman, Chapman, Flammia, Mann (2021+2023)
- Fendley, BP (2023): $$H = \sum_j X_j X_{j+1} Z_j + b Z_j Z_{j+2} + b^2 Z_j Y_{j+1} Y_{j+2}$$
- Chapman, Flammia, Kollár (2022)
- Alcaraz, Pimenta (6 papers, 2020-2023)
- BP (2024)
- Vona, Mestyán, BP (2024)

14 papers so far!

$$H = \sum_j \alpha_j h_j, \qquad h_j^2 = 1, \quad \{h_j, h_{j+1}\} = \{h_j, h_{j+2}\} = 0$$

$$[h_j, h_k] = 0 \text{ for } |j - k| > 2$$

$$H = \sum_j \alpha_j h_j, \qquad h_j^2 = 1, \quad \{h_j, h_{j+1}\} = \{h_j, h_{j+2}\} = 0$$

$$[h_j, h_k] = 0 \text{ for } |j - k| > 2$$

Transfer matrix: $T_M(u), \qquad [T_M(u), T_M(v)] = 0$

$$H = \sum_j \alpha_j h_j, \qquad h_j^2 = 1, \quad \{h_j, h_{j+1}\} = \{h_j, h_{j+2}\} = 0$$

$$[h_j, h_k] = 0 \text{ for } |j - k| > 2$$

Transfer matrix: $T_M(u), \qquad [T_M(u), T_M(v)] = 0$

Operator valued weighted independence polynomial of the frustration graph!

$$H = \sum_j \alpha_j h_j, \qquad h_j^2 = 1, \quad \{h_j, h_{j+1}\} = \{h_j, h_{j+2}\} = 0$$

$$[h_j, h_k] = 0 \text{ for } |j-k| > 2$$

Transfer matrix: $T_M(u), \qquad [T_M(u), T_M(v)] = 0$

Operator valued weighted independence polynomial of the frustration graph!

$$T_M(u) = \sum_n (-u)^n \sum H_{a_1} H_{a_2} \ldots H_{a_n}, \qquad H_j = \alpha_j h_j, \qquad [H_{a_j}, H_{a_k}] = 0$$

$$H = \sum_j \alpha_j h_j, \qquad h_j^2 = 1, \quad \{h_j, h_{j+1}\} = \{h_j, h_{j+2}\} = 0$$

$$[h_j, h_k] = 0 \text{ for } |j-k| > 2$$

Transfer matrix: $T_M(u), \qquad [T_M(u), T_M(v)] = 0$

Operator valued weighted independence polynomial of the frustration graph!

$$T_M(u) = \sum_n (-u)^n \sum H_{a_1} H_{a_2} \ldots H_{a_n}, \qquad H_j = \alpha_j h_j, \qquad [H_{a_j}, H_{a_k}] = 0$$

$$T_M(u) = T_{M-1}(u) - u b_M h_M T_{M-3}(u)$$

$$H = \sum_j \alpha_j h_j, \qquad h_j^2 = 1, \quad \{h_j, h_{j+1}\} = \{h_j, h_{j+2}\} = 0$$

$$[h_j, h_k] = 0 \text{ for } |j-k| > 2$$

Transfer matrix: $T_M(u), \qquad [T_M(u), T_M(v)] = 0$

Operator valued weighted independence polynomial of the frustration graph!

$$T_M(u) = \sum_n (-u)^n \sum H_{a_1} H_{a_2} \ldots H_{a_n}, \qquad H_j = \alpha_j h_j, \qquad [H_{a_j}, H_{a_k}] = 0$$

$$T_M(u) = T_{M-1}(u) - u b_M h_M T_{M-3}(u)$$

Inversion: $T_M(u) T_M(-u) = P_M(u^2)$

$$P_M(x) = P_{M-1}(x) - x b_M^2 P_{M-3}(x)$$

Edge operator: $\{\chi_0, h_1\} = 0,$ $[\chi_0, h_j] = 0, \quad j > 1$

Edge operator: $\{\chi_0, h_1\} = 0, \qquad [\chi_0, h_j] = 0, \quad j > 1$

Fermions: $\Psi_k \sim T(-u_k)\chi_0 T(u_k), \qquad P_M(u_k^2) = 0$

$$H = \sum_{k=1}^{S} \varepsilon_k [\Psi_k, \Psi_{-k}], \qquad \varepsilon_k = \frac{1}{u_k}$$

Edge operator: $\{\chi_0, h_1\} = 0,$ $[\chi_0, h_j] = 0, \quad j > 1$

Fermions: $\Psi_k \sim T(-u_k)\chi_0 T(u_k),$ $P_M(u_k^2) = 0$

$$H = \sum_{k=1}^{S} \varepsilon_k [\Psi_k, \Psi_{-k}], \qquad \varepsilon_k = \frac{1}{u_k}$$

Specializes to known results from Jordan-Wigner!

Common properties:

Common properties:

- Fermions NOT of Jordan-Wigner form

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings
- Large non-abelian symmetry algebra (SUSY+…)

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings
- Large non-abelian symmetry algebra (SUSY+…)
- Large homogeneous degeneracies

  number of fermionic modes $<$ volume

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings
- Large non-abelian symmetry algebra (SUSY+…)
- Large homogeneous degeneracies

  number of fermionic modes $<$ volume
- Good for quantum simulation: classically simulable!

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings
- Large non-abelian symmetry algebra (SUSY+…)
- Large homogeneous degeneracies

  number of fermionic modes $<$ volume
- Good for quantum simulation: classically simulable!
- Unitary quantum circuits possible

  ‘Free fermionic Trotterization’

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings
- Large non-abelian symmetry algebra (SUSY+…)
- Large homogeneous degeneracies

  number of fermionic modes $<$ volume
- Good for quantum simulation: classically simulable!
- Unitary quantum circuits possible

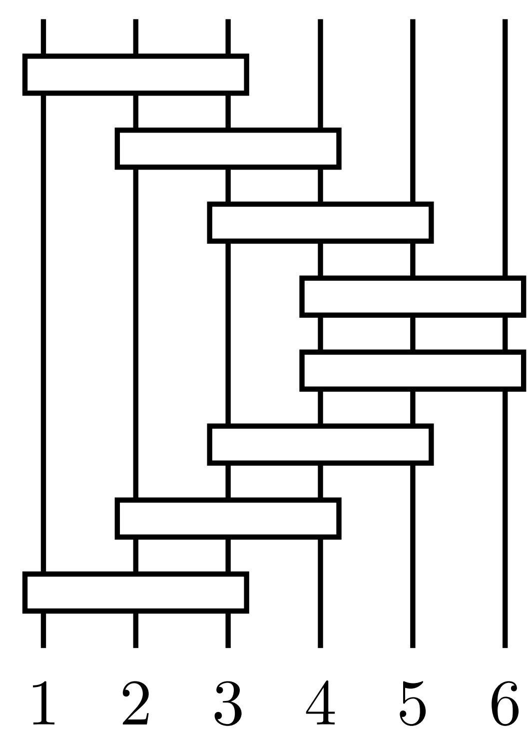
  'Free fermionic Trotterization'

Common properties:

- Fermions NOT of Jordan-Wigner form
- Open boundary conditions
- Integrable also with periodic b.c., but free fermions seem to be broken
- Large freedom in the couplings
- Large non-abelian symmetry algebra (SUSY+…)
- Large homogeneous degeneracies

  number of fermionic modes $<$ volume
- Good for quantum simulation: classically simulable!
- Unitary quantum circuits possible

  ‘Free fermionic Trotterization’



Computation of correlation functions

Computation of correlation functions

Correlations like $< h_1(t) h_1(0) >_{T=\infty}$

Computation of correlation functions

Correlations like $< h_1(t) h_1(0) >_{T=\infty}$

Needed: Solution of the inverse problem

Computation of correlation functions

Correlations like $< h_1(t) h_1(0) >_{T=\infty}$

Needed: Solution of the inverse problem

$$\chi_0 = \sum_{j=-S}^{S} C_j \Psi_j \qquad C_j = C_{-j} = \sqrt{\frac{P_{M\backslash 1}(u_j^2)}{-u_j^2 P'_M(u_j^2)}}$$

Computation of correlation functions

Correlations like $< h_1(t)h_1(0) >_{T=\infty}$

Needed: Solution of the inverse problem

$$\chi_0 = \sum_{j=-S}^{S} C_j \Psi_j \qquad C_j = C_{-j} = \sqrt{\frac{P_{M\setminus 1}(u_j^2)}{-u_j^2 P'_M(u_j^2)}}$$

$$h_1 = \alpha_1^{-1} \sum_{j,k=-S}^{S} \varepsilon_j C_j C_k \Psi_j \Psi_k$$

Computation of correlation functions

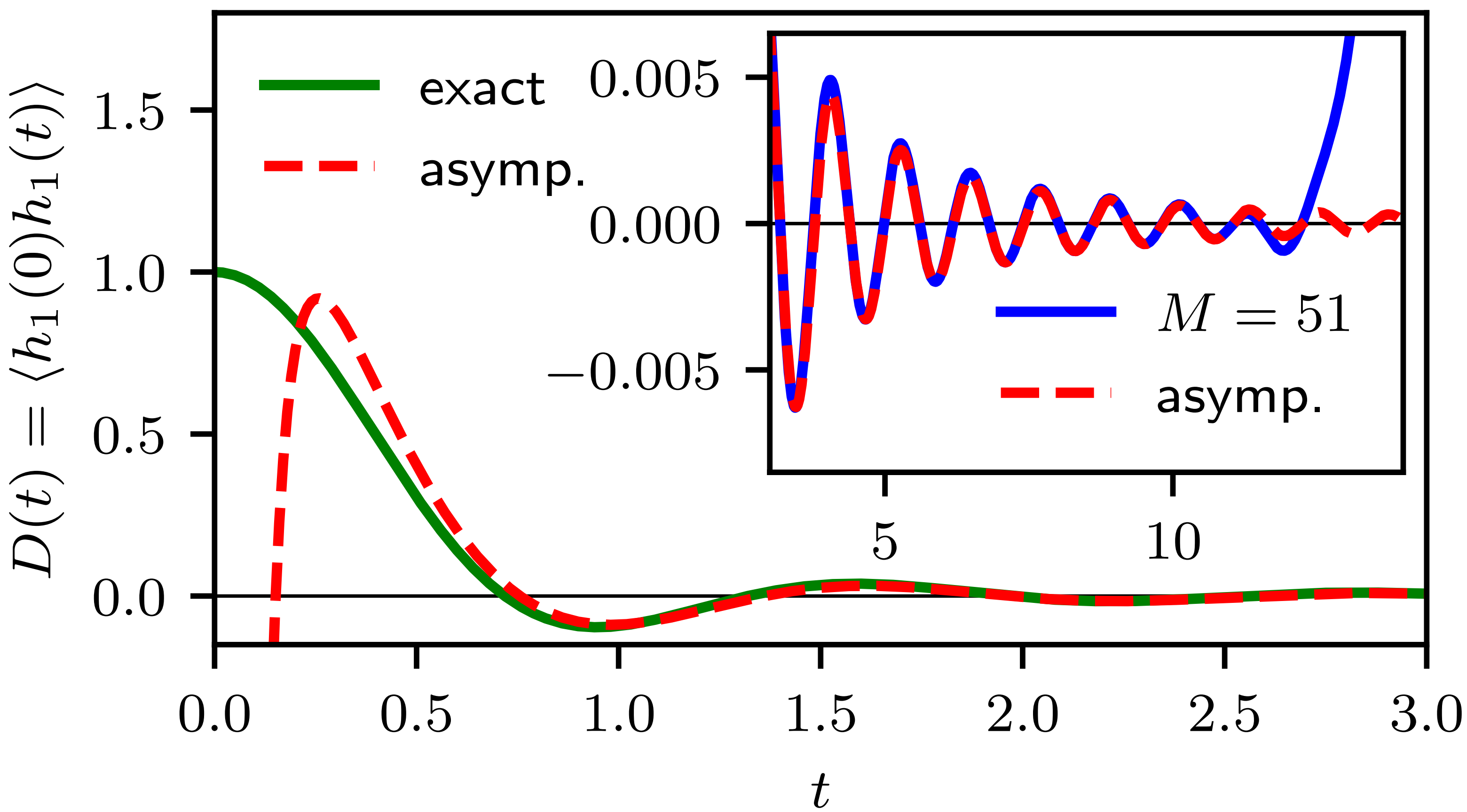
Correlations like $< h_1(t)h_1(0) >_{T=\infty}$

Needed: Solution of the inverse problem

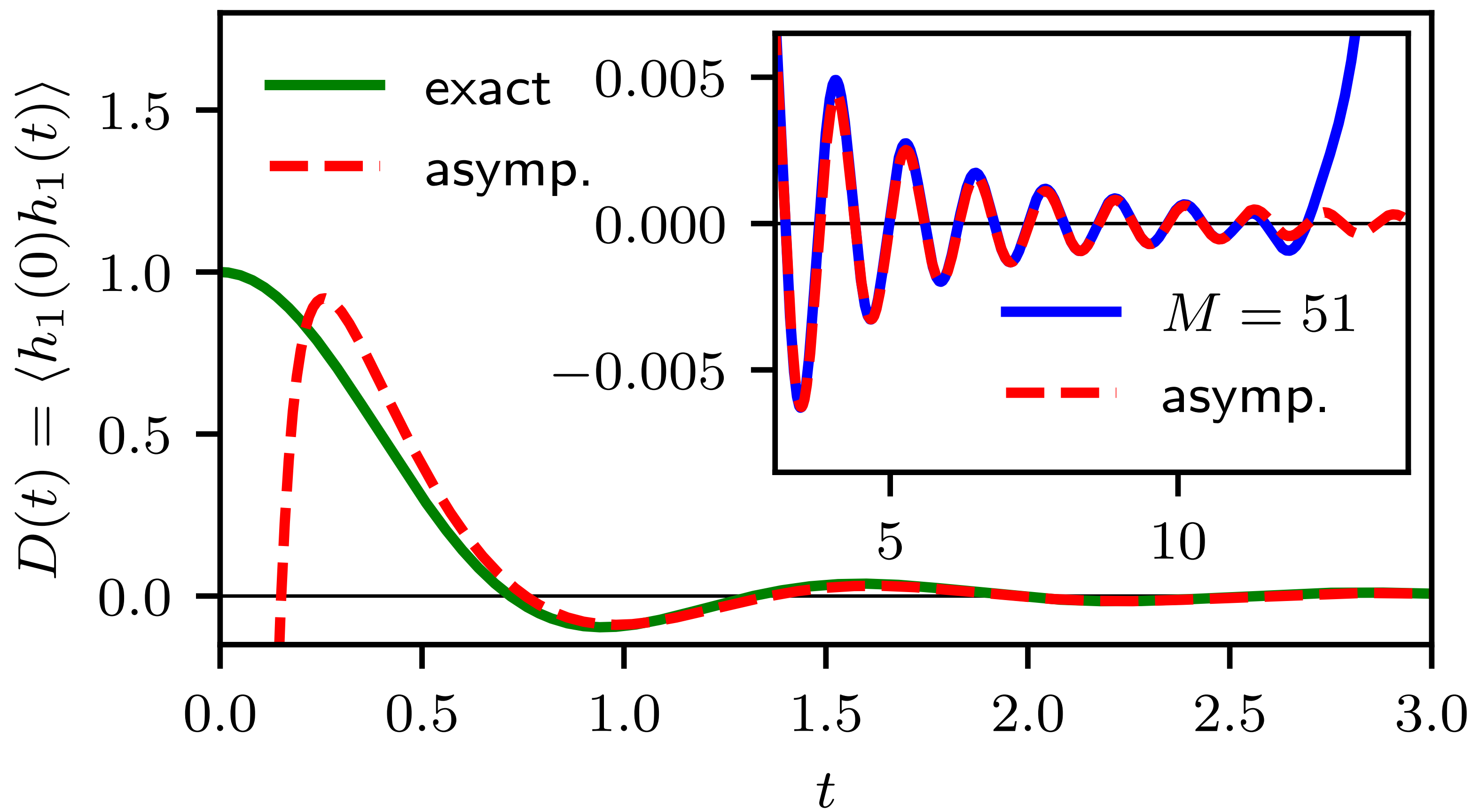
$$\chi_0 = \sum_{j=-S}^{S} C_j \Psi_j \qquad C_j = C_{-j} = \sqrt{\frac{P_{M\setminus 1}(u_j^2)}{-u_j^2 P'_M(u_j^2)}}$$

$$h_1 = \alpha_1^{-1} \sum_{j,k=-S}^{S} \varepsilon_j C_j C_k \Psi_j \Psi_k$$

$$< h_1(t)h_1(0) >_{T=\infty} = \frac{1}{4b_1^2} \sum_{k,\ell=0}^{S} C_k^2 C_\ell^2 \sum_{\sigma=\pm} (\varepsilon_k - \sigma\varepsilon_\ell)^2 \cos(\theta_k + \sigma\theta_\ell),$$

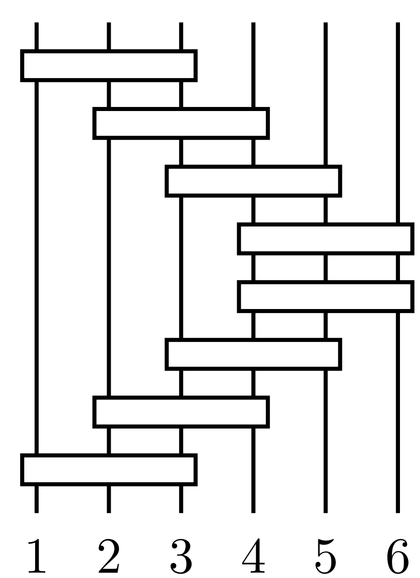


$$D(t) \approx c \frac{\sin\left(3\sqrt{3}t + 3\pi/4\right)}{t^{13/6}}$$



$$D(t) \approx c\frac{\sin\left(3\sqrt{3}t + 3\pi/4\right)}{t^{13/6}}$$

Extension to discrete time evolution!



Classical simulability of quantum processes

Classical simulability of quantum processes

$$< h_1(t)h_1(0) >_{T=\infty} \quad \to \quad < \Psi_0 | U^N | \Psi_0 >$$

Classical simulability of quantum processes

$$< h_1(t) h_1(0) >_{T=\infty} \quad \rightarrow \quad < \Psi_0 | U^N | \Psi_0 >$$

Idea:

$$h_2 h_3 h_5 \quad \rightarrow \quad | 01101 >$$

Classical simulability of quantum processes

$$< h_1(t)h_1(0) >_{T=\infty} \quad \rightarrow \quad < \Psi_0 | U^N | \Psi_0 >$$

Idea:

$$h_2h_3h_5 \quad \rightarrow \quad |01101>$$

$$|\Psi_0> = |10000\ldots>$$

Open questions:

Open questions:

- Simpler proofs…

Open questions:

- Simpler proofs…
- Finite depth circuits

Open questions:

- Simpler proofs…
- Finite depth circuits
- Full characterization of classically simulable processes

Thank you for the attention!