erivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# A Calogero model for the non-Abelian

## quantum Hall effect

Jean-Emile Bourgine

SIMIS & Fudan University

Workshop "Mathematical Physics for Quantum Science"

07-11-2024

[Based on [JEB-Matsuo 2401.03087]]

Introduction	
0000000	

erivation of the model 00000 Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### Hall effect

• The classical Hall effect has been discovered by Edwin Hall in 1879. It is the observation of a transverse voltage for a conductor in a magnetic field. It is due to the Lorentz force acting on the electrons.





Perivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Hall effect

• The classical Hall effect has been discovered by Edwin Hall in 1879. It is the observation of a transverse voltage for a conductor in a magnetic field. It is due to the Lorentz force acting on the electrons.



• As a result, the resistivity is expected to behave as follows



Derivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Quantum Hall effect

• The quantum Hall effect (QHE) refers to the observation of plateaux in the resistivity at low temperatures and strong magnetic field. The measured resistance is given by the formula

$$R_{xy} = rac{V_{\mathsf{Hall}}}{I} = rac{h}{e^2 
u}.$$

 $\rightarrow$  Plateaux are labeled by the filling factor  $\nu$ , it takes integer or fractional values.



→ Integer QHE observed by Klaus von Klitzing in 1980 in MOSFET.

---- Fractional QHE discovered by Daniel Tsui and Horst Störmer in 1982 in GaAs samples.

Perivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Integer quantum Hall effect

 The integer QHE has a simple quantum mechanical explanation. It is due to the impurities in the material that lift the macroscopic degeneracy of the Landau levels. Impurities also turn extended states into localized states. Since these states cannot carry a current, they do not contribute to the conductivity.

 $\Rightarrow$  Observe plateaux of length proportional to the amount of impurities!

Perivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Fractional quantum Hall effect

• The explanation of the **fractional** QHE is of a different nature and involve **interactions between electrons** which was previously neglected.

verivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Fractional quantum Hall effect

• The explanation of the **fractional** QHE is of a different nature and involve **interactions between electrons** which was previously neglected.

• The filling factor  $\nu$  takes a specific set of rational values. The first explanation of the FQHE was proposed by Laughlin in 1983. He wrote the expression of an effective ground state wave function for the values filling factors  $\nu = 1/k$  with k odd,

$$\psi(\mathbf{x}) = \prod_{a < b} (x_a - x_b)^k e^{-\frac{e\beta}{4\hbar} \sum_a |x_a|^2}.$$

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Fractional quantum Hall effect

• The explanation of the **fractional** QHE is of a different nature and involve **interactions between electrons** which was previously neglected.

• The filling factor  $\nu$  takes a specific set of rational values. The first explanation of the FQHE was proposed by Laughlin in 1983. He wrote the expression of an effective ground state wave function for the values filling factors  $\nu = 1/k$  with k odd,

$$\psi(\mathbf{x}) = \prod_{a < b} (x_a - x_b)^k e^{-\frac{eB}{4\hbar} \sum_a |x_a|^2}.$$

• The model exhibits excitations called quasi-holes with the wave function

$$\psi(\zeta_i, \mathbf{x}) = \prod_{i < j} (\zeta_i - \zeta_j)^{1/k} \times \prod_i \prod_a (\zeta_i - x_a) \times \prod_{a < b} (x_a - x_b)^k e^{-\frac{eB}{4\hbar} \sum_a |x_a|^2},$$

They carry a fraction of the charge of the electrons. They also obeys *a fractional statistics*, namely they are (Abelian) anyons!

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• Model of the fractional QHE with more general filling factors  $\nu = p/(k + pn)$  involve

**non-Abelian anyons**. The statistics of these anyons (braiding) is given in terms of a non-trivial matrix transformation.

---- This property is particularly interesting to implement quantum computations!

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

ermionic formalism

Discussion 00000

• Model of the fractional QHE with more general filling factors  $\nu = p/(k + pn)$  involve

**non-Abelian anyons**. The statistics of these anyons (braiding) is given in terms of a non-trivial matrix transformation.

---- This property is particularly interesting to implement quantum computations!

• In fact, anyons are **topologically protected quantum states**. They are very good candidates to realize fault-tolerant quantum processors!

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

ermionic formalism

Discussion 00000

• Model of the fractional QHE with more general filling factors  $\nu = p/(k + pn)$  involve non-Abelian anyons. The statistics of these anyons (braiding) is given in terms of a non-trivial

matrix transformation.

~> This property is particularly interesting to implement quantum computations!

• In fact, anyons are **topologically protected quantum states**. They are very good candidates to realize fault-tolerant quantum processors!

• So far, there have been no observation of non-Abelian anyons in condensed matter systems. But they have been realized recently on quantum processors.

[G. Q. AI and Collaborators, Nature 618 (2023) 264-269]

Derivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Motivations

- They are three main approaches to the fractional QHE:
  - Numerical calculations using realistic condensed matter models

Perivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Motivations

- They are three main approaches to the fractional QHE:
  - Numerical calculations using realistic condensed matter models
  - Direct proposal of wave functions with good physical properties.
    - $\rightsquigarrow$  Usually coming from a CFT correlator, e.g. WZW models with Kac-Moody symmetry.

Perivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Motivations

- They are three main approaches to the fractional QHE:
  - Numerical calculations using realistic condensed matter models
  - Direct proposal of wave functions with good physical properties.
    - $\rightsquigarrow$  Usually coming from a CFT correlator, e.g. WZW models with Kac-Moody symmetry.
  - Field theory (IR) description using 3d Chern-Simons theory
    - $\rightsquigarrow$  Abelian FQHE: U(1) Chern-Simons level k ( $\nu = 1/k$ )
    - → Non-Abelian FQHE: U(p) Chern-Simons level k ( $\nu = p/(k + p)$ ).

Perivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Motivations

- They are three main approaches to the fractional QHE:
  - Numerical calculations using realistic condensed matter models
  - Direct proposal of wave functions with good physical properties.
    - $\rightsquigarrow$  Usually coming from a CFT correlator, e.g. WZW models with Kac-Moody symmetry.
  - Field theory (IR) description using 3d Chern-Simons theory
    - $\rightsquigarrow$  Abelian FQHE: U(1) Chern-Simons level k ( $\nu = 1/k$ )
    - → Non-Abelian FQHE: U(p) Chern-Simons level k ( $\nu = p/(k + p)$ ).

 $\Rightarrow$  So, what's new?

rivation of the model

Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### • In [2401.03087], we introduce a new quantum model for the non-Abelian FQHE. It is

obtained as the diagonalization of a matrix model proposed by [Dorey, Tong, Turner 2016]. This model describes the vortices of 3d U(p) Chern-Simons theory.

rivation of the model

Spin Calogero model 00000 pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• In [2401.03087], we introduce a new quantum model for the non-Abelian FQHE. It is obtained as the diagonalization of a matrix model proposed by [Dorey, Tong, Turner 2016]. This model describes the vortices of 3d U(p) Chern-Simons theory.

• We recover some of the known wave functions obtained as CFT correlators. It is also possible to show the emergence of a Kac-Moody symmetry in the large N limit.

→ It relates the Chern-Simons and CFT approaches!

rivation of the model

Spin Calogero model 00000 pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• In [2401.03087], we introduce a new quantum model for the non-Abelian FQHE. It is obtained as the diagonalization of a matrix model proposed by [Dorey, Tong, Turner 2016]. This model describes the vortices of 3d U(p) Chern-Simons theory.

• We recover some of the known wave functions obtained as CFT correlators. It is also possible to show the emergence of a Kac-Moody symmetry in the large N limit.

→ It relates the Chern-Simons and CFT approaches!

• The Hamiltonian of our quantum system is a spin version of the Calogero Hamiltonian, a well known integrable system. It is expected to be also integrable.

→ Use the algebraic techniques of integrable systems to do exact calculations!

Derivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Outline

- 1. Introduction
- 2. Derivation of the model
- 3. Spin Calogero model
- 4. Spectrum and eigenfunctions
- 5. Fermionic formalism
- 6. Discussion

Derivation of the model ••••••

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

## 2. Derivation of the model

Derivation of the model 00000 Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### A bit of history...

• In 2001, Susskind proposed a description of the abelian FQHE using a non-commutative U(1) Chern-Simons theory at level k.

Derivation of the model 000000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### A bit of history...

• In 2001, Susskind proposed a description of the abelian FQHE using a non-commutative U(1) Chern-Simons theory at level k.

• Polychronakos introduced a U(N) matrix model as a regularization of Susskind's model to describe the microscopic dynamics of a droplet of N electrons. Diagonalizing the model, he found that the dynamics of eigenvalues is governed by the Calogero Hamiltonian.

Derivation of the model 000000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### A bit of history...

• In 2001, Susskind proposed a description of the abelian FQHE using a non-commutative U(1) Chern-Simons theory at level k.

• Polychronakos introduced a U(N) matrix model as a regularization of Susskind's model to describe the microscopic dynamics of a droplet of N electrons. Diagonalizing the model, he found that the dynamics of eigenvalues is governed by the Calogero Hamiltonian.

• In 2004, Tong re-interpreted Polychronakos's matrix model as a description of vortices in an Abelian commutative Chern-Simons theory.

Derivation of the model 000000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### A bit of history...

• In 2001, Susskind proposed a description of the abelian FQHE using a non-commutative U(1) Chern-Simons theory at level k.

• Polychronakos introduced a U(N) matrix model as a regularization of Susskind's model to describe the microscopic dynamics of a droplet of N electrons. Diagonalizing the model, he found that the dynamics of eigenvalues is governed by the Calogero Hamiltonian.

• In 2004, Tong re-interpreted Polychronakos's matrix model as a description of vortices in an Abelian commutative Chern-Simons theory.

• Following this interpretation, Dorey Tong and Turner introduced in 2016 an extension of Polychronakos's matrix model with an additional U(p) symmetry to render the dynamics of vortices in the non-Abelian Chern-Simons theory.

Derivation of the model 000000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### A bit of history...

• In 2001, Susskind proposed a description of the abelian FQHE using a non-commutative U(1) Chern-Simons theory at level k.

• Polychronakos introduced a U(N) matrix model as a regularization of Susskind's model to describe the microscopic dynamics of a droplet of N electrons. Diagonalizing the model, he found that the dynamics of eigenvalues is governed by the Calogero Hamiltonian.

• In 2004, Tong re-interpreted Polychronakos's matrix model as a description of vortices in an Abelian commutative Chern-Simons theory.

• Following this interpretation, Dorey Tong and Turner introduced in 2016 an extension of Polychronakos's matrix model with an additional U(p) symmetry to render the dynamics of vortices in the non-Abelian Chern-Simons theory.

 $\Rightarrow$  In [JEB-Matsuo 2401.03087], we diagonalized the matrix model and obtained a spin version of the Calogero Hamiltonian involving order k symmetric representations of U(p).

Derivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### DTT matrix model

• The matrix model introduced by [Dorey, Tong, Turner 2016] describes the moduli space of vortices. It involves a  $N \times N$  complex matrix Z(t) and p N-dimensional vectors  $\varphi_i(t)$  [Hanany-Tong 1996]

 $S = \int dt \left[ \frac{1}{2} iB \operatorname{tr}(Z^{\dagger} \mathcal{D}_{t} Z) + i \sum_{i=1}^{p} \varphi_{i}^{\dagger} \mathcal{D}_{t} \varphi_{i} - (k+p) \operatorname{tr} \alpha - \omega \operatorname{tr}(Z^{\dagger} Z) \right]$ 



 $\rightarrow$  It depends on the parameters *B* (magnetic field),  $\omega$  (strength confining potential) and *k* (Chern-Simons level)

Derivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### DTT matrix model

• The matrix model introduced by [Dorey, Tong, Turner 2016] describes the moduli space of vortices. It involves a  $N \times N$ complex matrix Z(t) and p N-dimensional vectors  $\varphi_i(t)$ [Hanany-Tong 1996]



$$S = \int dt \left[ \frac{1}{2} iB \operatorname{tr}(Z^{\dagger} \mathcal{D}_{t} Z) + i \sum_{i=1}^{p} \varphi_{i}^{\dagger} \mathcal{D}_{t} \varphi_{i} - (k+p) \operatorname{tr} \alpha - \omega \operatorname{tr}(Z^{\dagger} Z) \right]$$

 $\rightarrow$  It depends on the parameters *B* (magnetic field),  $\omega$  (strength confining potential) and *k* (Chern-Simons level)

• The model also contains a non-dynamical gauge field  $\alpha$  entering in the covariant derivatives  $D_t Z = \partial_t Z - i[\alpha, Z], D_t \varphi_i = \partial_t \varphi_i - i\alpha \varphi_i$ . This gauge field imposes the (classical) constraint

$$\frac{B}{2}[Z,Z^{\dagger}] + \sum_{i=1}^{p} \varphi_i \varphi_i^{\dagger} = (k+p) \mathbb{1}_N.$$

Derivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• In [Dorey, Tong, Turner 2016], the complex matrix Z is diagonalized. Instead, we followed Polychronakos's original approach and decomposed the matrix Z in terms of Hermitian matrices,

$$Z = X_1 + iX_2, \quad Z^{\dagger} = X_1 - iX_2$$

As a result, the action takes the form

$$S = \int dt \left( -\frac{B}{2} \operatorname{tr} \left( X_1 \dot{X}_2 - X_2 \dot{X}_1 \right) + i \sum_{i=1}^p \varphi_i^{\dagger} \dot{\varphi}_i - \omega \operatorname{tr} (X_1^2 + X_2^2) + \operatorname{tr} \left[ \alpha G(X_1, X_2, \varphi_i, \varphi_i^{\dagger}) \right] \right),$$
  
with  $G(X_1, X_2, \varphi_i, \varphi_i^{\dagger}) = -iB[X_1, X_2] + \sum_{i=1}^p \varphi_i \varphi_i^{\dagger} - (k+p) \mathbb{1}_N,$ 

Derivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Diagonalization

• The diagonalization is done following the usual method. Exploiting the U(N) invariance, we decompose  $X_1(t) = \Omega(t)^{\dagger} x(t) \Omega(t)$  with  $(x)_{a,b} = x_a \delta_{a,b}$ ,  $\Omega \in SU(N)$ . We derive the momenta and move to the Hamiltonian frame,

$$\mathcal{H} = \omega \operatorname{tr}(x^2 + (X_2)^2) = \omega \sum_{a=1}^{N} \left( x_a^2 + \frac{p_a^2}{B^2} \right) + 2 \frac{\omega}{B^2} \sum_{a < b} \frac{(J_{a,b} - \Pi_{a,b}^+)(J_{b,a} - \Pi_{a,b}^-)}{(x_a - x_b)^2},$$

where  $\prod_{a,b}^{\pm}$  are momenta associated to angular degrees of freedom, and

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad p_{a} = \frac{\delta \mathcal{L}}{\delta \dot{x}_{a}} = B X_{2\ a,a}$$

Derivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Diagonalization

• The diagonalization is done following the usual method. Exploiting the U(N) invariance, we decompose  $X_1(t) = \Omega(t)^{\dagger} x(t) \Omega(t)$  with  $(x)_{a,b} = x_a \delta_{a,b}$ ,  $\Omega \in SU(N)$ . We derive the momenta and move to the Hamiltonian frame,

$$\mathcal{H} = \omega \operatorname{tr}(x^2 + (X_2)^2) = \omega \sum_{a=1}^{N} \left( x_a^2 + \frac{p_a^2}{B^2} \right) + 2 \frac{\omega}{B^2} \sum_{a < b} \frac{(J_{a,b} - \Pi_{a,b}^+)(J_{b,a} - \Pi_{a,b}^-)}{(x_a - x_b)^2},$$

where  $\Pi_{a,b}^{\pm}$  are momenta associated to angular degrees of freedom, and

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad p_{a} = rac{\delta \mathcal{L}}{\delta \dot{x}_{a}} = B X_{2\ a,a}$$

• Then, we impose the canonical quantization conditions. The gauge constraint imposed on physical states implies

$$\Pi_{a,b}^{\pm} \ket{\text{phys}} = 0, \quad J_{a,a} \ket{\text{phys}} = k \ket{\text{phys}}.$$

#### $\rightsquigarrow$ The parameter k identified to a mode number is quantized!

Derivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• Due to the Vandermonde determinant in the measure coming from the diagonalization

 $dX_1 = d\Omega d \times \Delta(x)$ , the momentum  $p_a$  acts on wave functions as

$$p_a = -i\Delta(\mathbf{x})^{-1}\partial_a\Delta(\mathbf{x}), \quad \Delta(\mathbf{x}) = \prod_{\substack{a,b=1\\a < b}}^N (x_a - x_b).$$

Derivation of the model

Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• Due to the Vandermonde determinant in the measure coming from the diagonalization

 $dX_1 = d\Omega d \times \Delta(x)$ , the momentum  $p_a$  acts on wave functions as

$$p_a = -i\Delta(\mathbf{x})^{-1}\partial_a\Delta(\mathbf{x}), \quad \Delta(\mathbf{x}) = \prod_{\substack{a,b=1\\a < b}}^N (x_a - x_b).$$

• Taking these facts into account, we find the Hamiltonian acting on physical states,

$$\mathcal{H} = \frac{\omega}{B^2} \left( \sum_{a=1}^N \left( -\Delta(\mathbf{x})^{-1} \partial_a^2 \Delta(\mathbf{x}) + B^2 x_a^2 \right) + 2 \sum_{a < b} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2} \right),$$

 $\rightsquigarrow$  The Hamiltonian  $\mathcal{H}$  of our model is obtained upon rescaling by a factor  $\omega^{-1}B^2$ , and a conjugation with the Vandermonde  $\mathcal{H} \to \Delta(x)\mathcal{H}\Delta(x)^{-1}$ .

Derivation of the model

Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• Due to the Vandermonde determinant in the measure coming from the diagonalization

 $dX_1 = d\Omega d \times \Delta(x)$ , the momentum  $p_a$  acts on wave functions as

$$p_a = -i\Delta(\mathbf{x})^{-1}\partial_a\Delta(\mathbf{x}), \quad \Delta(\mathbf{x}) = \prod_{\substack{a,b=1\\a < b}}^N (x_a - x_b).$$

• Taking these facts into account, we find the Hamiltonian acting on physical states,

$$\mathcal{H} = \frac{\omega}{B^2} \left( \sum_{a=1}^N \left( -\Delta(\mathbf{x})^{-1} \partial_a^2 \Delta(\mathbf{x}) + B^2 x_a^2 \right) + 2 \sum_{a < b} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2} \right),$$

 $\rightsquigarrow$  The Hamiltonian  $\mathcal{H}$  of our model is obtained upon rescaling by a factor  $\omega^{-1}B^2$ , and a conjugation with the Vandermonde  $\mathcal{H} \to \Delta(x)\mathcal{H}\Delta(x)^{-1}$ .

#### $\Rightarrow$ Let's examine this quantum system more closely!

Derivation of the model

Spin Calogero model •0000

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

## 3. Spin Calogero model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Definition of the model

- The model depends of the following parameters:
  - $N \in \mathbb{Z}^{\geq 0}$  the number of vortices in Chern-Simons theory
  - $p \in \mathbb{Z}^{>0}$  the rank of the non-Abelian U(p) symmetry
  - $k \in \mathbb{Z}^{\geq 0}$  the level of the Chern-Simons theory (filling factor  $\nu = p/(k+p)$ ).
  - $B \in \mathbb{R}$  the magnetic field (strength of the confining potential)

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Definition of the model

- The model depends of the following parameters:
  - $N \in \mathbb{Z}^{\geq 0}$  the number of vortices in Chern-Simons theory
  - $p \in \mathbb{Z}^{>0}$  the rank of the non-Abelian U(p) symmetry
  - $k \in \mathbb{Z}^{\geq 0}$  the level of the Chern-Simons theory (filling factor  $\nu = p/(k+p)$ ).
  - $B \in \mathbb{R}$  the magnetic field (strength of the confining potential)

• The model describes N particles of coordinates  $x_a \in \mathbb{R}$  carrying a representation of the U(p) symmetry. The spin structure is introduced using the bosonic oscillators  $\varphi_{i,a}^{\dagger}$ ,  $\varphi_{i,a}$ ,

 $[\varphi_{i,a}, \varphi_{i,b}^{\dagger}] = \delta_{i,j} \delta_{a,b}, \quad i, j = 1 \cdots p, \quad a, b = 1 \cdots N.$ 

 $\rightsquigarrow$  At level k, each particle carries a state built from k oscillators  $\varphi_{i_1,a}^{\dagger} \cdots \varphi_{i_k,a}^{\dagger} |\emptyset\rangle$ .

 $\Rightarrow$  The model involves higher order symmetric representations of the U(p) symmetry!
In	tr	od	u	ct	io	n	
0	0	0	0	0	0	0	0

ivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### • The Hamiltonian of the model reads

$$\mathcal{H} = \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2},$$

where  $J_{a,b}$  are the generators of  $\mathfrak{gl}(N)$  acting on the spin components,

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad [J_{a,b}, J_{c,d}] = \delta_{a,d} J_{c,b} - \delta_{b,c} J_{a,d}.$$

In	tr	od	u	ct	io	n	
0	0	0	0	0	0	0	0

ivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### • The Hamiltonian of the model reads

$$\mathcal{H} = \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2},$$

where  $J_{a,b}$  are the generators of  $\mathfrak{gl}(N)$  acting on the spin components,

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad [J_{a,b}, J_{c,d}] = \delta_{a,d} J_{c,b} - \delta_{b,c} J_{a,d}.$$

• The model exhibits a global U(p) invariance generated by

$$\mathcal{K}_{i,j} = \sum_{a=1}^{N} \varphi_{i,a}^{\dagger} \varphi_{j,a}, \quad [\mathcal{K}_{i,j}, \mathcal{K}_{k,l}] = \delta_{j,k} \mathcal{K}_{i,l} - \delta_{i,l} \mathcal{K}_{k,j}$$

In	tr	od	u	ct	io	n	
0	0	0	0	0	0	0	0

ivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### • The Hamiltonian of the model reads

$$\mathcal{H} = \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2},$$

where  $J_{a,b}$  are the generators of  $\mathfrak{gl}(N)$  acting on the spin components,

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad [J_{a,b}, J_{c,d}] = \delta_{a,d} J_{c,b} - \delta_{b,c} J_{a,d}.$$

• The model exhibits a global U(p) invariance generated by

$$\mathcal{K}_{i,j} = \sum_{a=1}^{N} \varphi_{i,a}^{\dagger} \varphi_{j,a}, \quad [\mathcal{K}_{i,j}, \mathcal{K}_{k,l}] = \delta_{j,k} \mathcal{K}_{i,l} - \delta_{i,l} \mathcal{K}_{k,j}.$$

 $\Rightarrow$  Let's examine some specific values of p and k!

Perivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# **Specializations**

• For p = 1, we recover the Calogero Hamiltonian describing the Abelian model

$$\mathcal{H}^{(p=1)} = \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{k(k+1)}{(x_a - x_b)^2}.$$

→ This is a well-known integrable system!

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Specializations

• For p = 1, we recover the Calogero Hamiltonian describing the Abelian model

$$\mathcal{H}^{(p=1)} = \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a$$

→ This is a well-known integrable system!

• When k = 1, particles carry a fundamental representation and the interaction term simplifies. We recover the spin-Calogero model involving the permutation of spins  $P_{a,b,r}$ 

$$\mathcal{H}^{(k=1)} = \sum_{a} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{1 + P_{a,b}}{(x_a - x_b)^2}$$

 $\rightarrow$  This model is known to be integrable and has Yangian symmetry  $Y(sl_p)!$ 

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Specializations

• For p = 1, we recover the Calogero Hamiltonian describing the Abelian model

$$\mathcal{H}^{(p=1)} = \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a$$

→ This is a well-known integrable system!

• When k = 1, particles carry a fundamental representation and the interaction term simplifies. We recover the spin-Calogero model involving the permutation of spins  $P_{a,b,r}$ 

$$\mathcal{H}^{(k=1)} = \sum_{a} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{1 + P_{a,b}}{(x_a - x_b)^2}$$

 $\rightarrow$  This model is known to be integrable and has Yangian symmetry  $Y(sl_p)!$ 

• When k = 0, particles carry no spin degree of freedom and the interaction term vanishes.

 $\rightsquigarrow$  We are left with N decoupled Harmonic oscillators!

In	tro	odu	ctic	n
0	0	oc	00	00

Perivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Simplification

• To simplify our analysis, we also perform the following technical manipulations:

• Conjugation of the Hamiltonian,

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_{a} x_{a}^{2}} \mathcal{H} e^{-\frac{B}{2}\sum_{a} x_{a}^{2}} = \sum_{a} \left( -\frac{\partial^{2}}{\partial x_{a}^{2}} + 2Bx_{a}\frac{\partial}{\partial x_{a}} \right) + NB + \sum_{a \neq b} \frac{J_{a,b}J_{b,a}}{(x_{a} - x_{b})^{2}}.$$

 $\rightsquigarrow$  Wave functions have a common Gaussian factor  $e^{-\frac{B}{2}\sum_{a}x_{a}^{2}}$  omitted here.

Derivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Simplification

- To simplify our analysis, we also perform the following technical manipulations:
  - Conjugation of the Hamiltonian,

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_{a} x_{a}^{2}} \mathcal{H} e^{-\frac{B}{2}\sum_{a} x_{a}^{2}} = \sum_{a} \left( -\frac{\partial^{2}}{\partial x_{a}^{2}} + 2Bx_{a}\frac{\partial}{\partial x_{a}} \right) + NB + \sum_{a \neq b} \frac{J_{a,b}J_{b,a}}{(x_{a} - x_{b})^{2}}.$$

- $\rightsquigarrow$  Wave functions have a common Gaussian factor  $e^{-\frac{B}{2}\sum_{a} x_{a}^{2}}$  omitted here.
- Rescaling of the coordinates  $x_a \to B^{-1/2} x_a$  and energies  $\tilde{\mathcal{H}} \to B^{-1} \tilde{\mathcal{H}}$  to set B = 1.

Derivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Simplification

- To simplify our analysis, we also perform the following technical manipulations:
  - Conjugation of the Hamiltonian,

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_{a} x_{a}^{2}} \mathcal{H} e^{-\frac{B}{2}\sum_{a} x_{a}^{2}} = \sum_{a} \left( -\frac{\partial^{2}}{\partial x_{a}^{2}} + 2Bx_{a}\frac{\partial}{\partial x_{a}} \right) + NB + \sum_{a \neq b} \frac{J_{a,b}J_{b,a}}{(x_{a} - x_{b})^{2}}.$$

- $\rightsquigarrow$  Wave functions have a common Gaussian factor  $e^{-\frac{B}{2}\sum_{a} x_{a}^{2}}$  omitted here.
- Rescaling of the coordinates  $x_a \to B^{-1/2} x_a$  and energies  $\tilde{\mathcal{H}} \to B^{-1} \tilde{\mathcal{H}}$  to set B = 1.

 $\Rightarrow$  Next step: diagonalization of the model!

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# 4. Spectrum and eigenfunctions

erivation of the model 00000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Wedge states

• We use a polynomial representation for the spin states,

$$\varphi_{i_1,a}^{\dagger}\cdots\varphi_{i_k,a}^{\dagger}|\emptyset\rangle \rightarrow y_{i_1,a}\cdots y_{i_k,a}, \quad \varphi_{i,a} \rightarrow \partial_{i,a} = \frac{\partial}{\partial y_{i,a}}.$$

 $\rightsquigarrow$  States are expressed as polynomial wave functions  $\Psi(x, y)$ .

They are homogeneous of total degree k in  $y_{i,a}$  for each particle a.

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### Wedge states

• We use a polynomial representation for the spin states,

$$\varphi_{i_1,a}^{\dagger}\cdots\varphi_{i_k,a}^{\dagger}|\emptyset\rangle\rightarrow y_{i_1,a}\cdots y_{i_k,a},\quad \varphi_{i,a}\rightarrow\partial_{i,a}=\frac{\partial}{\partial y_{i,a}}.$$

 $\rightsquigarrow$  States are expressed as polynomial wave functions  $\Psi(x, y)$ .

They are homogeneous of total degree k in  $y_{i,a}$  for each particle a.

• To express the wave functions, we first define the wedge states as N imes N determinants,

$$\mathcal{Y}_{r}(\mathbf{x}, \mathbf{y}) = [y_{i_{1}} \times^{n_{1}} \wedge \dots \wedge y_{i_{N}} \times^{n_{N}}] = \begin{vmatrix} y_{i_{1},1}(x_{1})^{n_{1}} & \cdots & y_{i_{N},1}(x_{1})^{n_{N}} \\ \vdots & \vdots \\ y_{i_{1},N}(x_{N})^{n_{1}} & \cdots & y_{i_{N},N}(x_{N})^{n_{N}} \end{vmatrix}$$

They are labeled by *N*-tuple integers  $r = (r_1, \dots, r_N)$ , ordered as  $0 \le r_1 < r_2 < \dots < r_N$ , which decompose as  $r_a = n_a p + i_a - 1$  with  $i_a \in [\![1, p]\!]$  under Euclidean division.

→ It vanishes trivially when two columns coincide (i.e.  $r_a = r_b$  for  $a \neq b$ ).

ivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• For  $k \in \mathbb{Z}^{>0}$  fixed, we introduce the following set of wave functions

$$\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\mathbf{r}^{(\alpha)}}(\mathbf{x},\mathbf{y}), \quad \Delta(\mathbf{x}) = \prod_{a < b} (x_a - x_b).$$

They are labeled by k tuples of p integers (ordered as before),  $t = \{r^{(1)}, \dots, r^{(k)}\}$ . We also assume that re-ordering of  $r^{(\alpha)}$  in t does not give a new element.

ivation of the model 0000 Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• For  $k \in \mathbb{Z}^{>0}$  fixed, we introduce the following set of wave functions

$$\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\mathbf{r}^{(\alpha)}}(\mathbf{x},\mathbf{y}), \quad \Delta(\mathbf{x}) = \prod_{a < b} (x_a - x_b)$$

They are labeled by k tuples of p integers (ordered as before),  $t = \{r^{(1)}, \dots, r^{(k)}\}$ . We also assume that re-ordering of  $r^{(\alpha)}$  in t does not give a new element.

• There is a natural grading obtained by acting with the operator  $\mathcal{D} = \sum_{a=1}^{N} x_a \partial_a$ ,

$$\mathcal{D}\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = |\mathfrak{r}|\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}), \quad |\mathfrak{r}| = \sum_{lpha=1}^{k} \sum_{a=1}^{N} \lfloor r_{a}^{(lpha)}/
ho 
floor = \sum_{lpha=1}^{k} \sum_{a=1}^{N} n_{a}^{(lpha)}.$$

ivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• For  $k \in \mathbb{Z}^{>0}$  fixed, we introduce the following set of wave functions

$$\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\mathbf{r}^{(\alpha)}}(\mathbf{x},\mathbf{y}), \quad \Delta(\mathbf{x}) = \prod_{a < b} (x_a - x_b)$$

They are labeled by k tuples of p integers (ordered as before),  $v = \{r^{(1)}, \dots, r^{(k)}\}$ . We also assume that re-ordering of  $r^{(\alpha)}$  in v does not give a new element.

• There is a natural grading obtained by acting with the operator  $\mathcal{D} = \sum_{a=1}^{N} x_a \partial_a$ ,

$$\mathcal{D}\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = |\mathfrak{r}|\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}), \quad |\mathfrak{r}| = \sum_{\alpha=1}^{k} \sum_{a=1}^{N} \lfloor r_{a}^{(\alpha)}/\rho \rfloor = \sum_{\alpha=1}^{k} \sum_{a=1}^{N} n_{a}^{(\alpha)}.$$

#### ▲ These wave functions are not linearly independent!

This can be seen e.g. from the existence of Plücker relations between determinants

$$\sum_{a=0}^{N} (-1)^{a} [z^{r_{1}} \wedge \cdots \wedge z^{r_{N-1}} \wedge z^{s_{a}}] [z^{s_{0}} \wedge \cdots \wedge z^{s_{n'}} \wedge \cdots \wedge z^{s_{N}}] = 0,$$

for arbitrary pair  $r=(r_1,\cdots,r_{N-1})$  and  $s=(s_0,\cdots,s_N)$ , and Uglov's notation  $z_a^{r_b}=x_a^{n_b}y_{i_b,a}$ .

 $\Rightarrow$  We will come back to this problem in the next section.

erivation of the model 00000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Main result

• We proved that the Hamiltonian  $\mathcal H$  has a triangular action on functions  $\psi_{\mathfrak r}(x,y)$ ,

$$ilde{\mathcal{H}}\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = E(\mathfrak{r})\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) + \sum_{|\mathfrak{s}|=|\mathfrak{r}|-2} C(\mathfrak{r},\mathfrak{s})\psi_{\mathfrak{s}}(\mathbf{x},\mathbf{y}),$$

where  $C(\mathfrak{r},\mathfrak{s})$  is a coefficient. This can be inverted to write down the eigenfunctions,

$$\Psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = \psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) + \sum_{|\mathfrak{s}| \leq |\mathfrak{r}|-2} D(\mathfrak{r},\mathfrak{s})\psi_{\mathfrak{s}}(\mathbf{x},\mathbf{y}).$$

 $\rightsquigarrow$  We deduce the energy spectrum  $E(\mathfrak{r}) = 2|\mathfrak{r}| + N$ .

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Remark: $\mathfrak{sl}_2$ -triplet

• [This remark is due to Yehao Zhou.] Recall the expression of our Hamiltonian,

$$\begin{split} \mathcal{H} &= \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + \sum_{\substack{a,b=1\\a\neq b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2}, \quad \mathcal{D} = \sum_a x_a \partial_a \\ \tilde{\mathcal{H}} &= e^{\frac{B}{2} \sum_a x_a^2} \mathcal{H} e^{-\frac{B}{2} \sum_a x_a^2} = \sum_a -\frac{\partial^2}{\partial x_a^2} + \sum_{a\neq b} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2} + NB + 2B\mathcal{D}. \end{split}$$

rivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### Remark: $\mathfrak{sl}_2$ -triplet

• [This remark is due to Yehao Zhou.] Recall the expression of our Hamiltonian,

$$\begin{split} \mathcal{H} &= \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + \sum_{\substack{a,b=1\\a\neq b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2}, \quad \mathcal{D} = \sum_a x_a \partial_a \\ \tilde{\mathcal{H}} &= e^{\frac{B}{2} \sum_a x_a^2} \mathcal{H} e^{-\frac{B}{2} \sum_a x_a^2} = \sum_a -\frac{\partial^2}{\partial x_a^2} + \sum_{a\neq b} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2} + NB + 2B\mathcal{D}. \end{split}$$

• If we define

$$e = \frac{1}{2} \sum_{a} x_a^2, \quad f = -\frac{1}{2} \sum_{a} \frac{\partial^2}{\partial x_a^2} + \frac{1}{2} \sum_{\substack{a,b=1\\a\neq b}}^{N} \frac{J_{a,b}J_{b,a}}{(x_a - x_b)^2}, \quad h = \mathcal{D} + \frac{N}{2}$$

our main result implies that they form an sl2-triplet!

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

rivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### Remark: *sl*<sub>2</sub>-triplet

• [This remark is due to Yehao Zhou.] Recall the expression of our Hamiltonian,

$$\begin{aligned} \mathcal{H} &= \sum_{a=1}^{N} \left( -\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + \sum_{\substack{a,b=1\\a\neq b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2}, \quad \mathcal{D} &= \sum_a x_a \partial_a \\ \tilde{\mathcal{H}} &= e^{\frac{B}{2} \sum_a x_a^2} \mathcal{H} e^{-\frac{B}{2} \sum_a x_a^2} = \sum_a -\frac{\partial^2}{\partial x_a^2} + \sum_{a\neq b} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2} + NB + 2B\mathcal{D}. \end{aligned}$$

• If we define

$$e = \frac{1}{2} \sum_{a} x_a^2, \quad f = -\frac{1}{2} \sum_{a} \frac{\partial^2}{\partial x_a^2} + \frac{1}{2} \sum_{\substack{a,b=1\\a\neq b}}^{N} \frac{J_{a,b}J_{b,a}}{(x_a - x_b)^2}, \quad h = \mathcal{D} + \frac{N}{2}$$

our main result implies that they form an sl2-triplet!

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

• In particular, we have

 $\mathcal{H}=2f+2B^2e,\quad \tilde{\mathcal{H}}=e^{Be}\mathcal{H}e^{-Be}=2f+2Bh.$ 

vation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• This remark has interesting implications for the wave functions. Recall that the product of wedge states diagonalize the grading operator

$$\mathcal{D}\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = |\mathfrak{r}|\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}), \quad \text{i.e.} \quad h\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y}) = \left(\frac{N}{2} + |\mathfrak{r}|\right)\psi_{\mathfrak{r}}(\mathbf{x},\mathbf{y})$$

→ Our Hamiltonian can be written as

$$\tilde{\mathcal{H}} = 2Be^{rac{f}{2B}}he^{-rac{f}{2B}}$$

and so its eigenfunctions have the form

$$\Psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = e^{\frac{f}{2B}}\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y})$$

where f is a nilpotent operator. This is indeed what we observed...

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Ground states

• We deduce the ground states wave function from the triangular action. When k = 1, the ground state is unique if p divides N and corresponds to  $r_0 = (0, 1, 2, \dots, N-1)$ ,

 $\mathcal{Y}_{r_0}(\mathbf{x},\mathbf{y}) = [y_1 \wedge \cdots \wedge y_p \wedge y_1 \times \wedge \cdots \wedge y_p \times \wedge \cdots \wedge y_1 \times^{m-1} \wedge \cdots \wedge y_p \times^{m-1}].$ 

 $\rightarrow$  The ground state is also a singlet for k > 1 if p|N, with the wave function  $\psi_{\tau_0} = \Delta(\mathcal{Y}_{r_0})^k$ .

rivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### Ground states

• We deduce the ground states wave function from the triangular action. When k = 1, the ground state is unique if p divides N and corresponds to  $r_0 = (0, 1, 2, \dots, N-1)$ ,

 $\mathcal{Y}_{r_0}(\mathbf{x},\mathbf{y}) = [y_1 \wedge \cdots \wedge y_p \wedge y_1 \times \wedge \cdots \wedge y_p \times \wedge \cdots \wedge y_1 \times^{m-1} \wedge \cdots \wedge y_p \times^{m-1}].$ 

 $\rightsquigarrow$  The ground state is also a singlet for k > 1 if p|N, with the wave function  $\psi_{r_0} = \Delta(\mathcal{Y}_{r_0})^k$ .

Example: for N = 4, p = 2 and k = 1 we have

$$\mathcal{Y}_{r}(\mathbf{x},\mathbf{y}) = [y_{1} \wedge y_{2} \wedge y_{1} \times \wedge y_{2} \times] = \begin{vmatrix} y_{1,1} & y_{2,1} & y_{1,1} \times 1 & y_{2,1} \times 1 \\ y_{1,2} & y_{2,2} & y_{1,2} \times 2 & y_{2,2} \times 2 \\ y_{1,3} & y_{2,3} & y_{1,3} \times 3 & y_{2,3} \times 3 \\ y_{1,4} & y_{2,4} & y_{1,4} \times 4 & y_{2,4} \times 4 \end{vmatrix}.$$

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• When N = mp + q with  $0 < q \le p - 1$ , the ground state for k = 1 is  $\binom{p}{q}$ -fold degenerate. It corresponds to a choice of spin for q extra particles,

 $\mathcal{Y}_{r_0}(\mathbf{x}, \mathbf{y}) = [y_1 \wedge \dots \wedge y_p \wedge y_1 \times \wedge \dots \wedge y_p \times \wedge \dots \wedge y_1 \times^{m-1} \wedge \dots \wedge y_p \times^{m-1} \wedge y_{i_1} \times^m \wedge \dots \wedge y_{i_q} \times^m].$ 

→ The corresponding ground state energy is  $E(r_0) = pm(m-1) + 2qm$ .

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

When N = mp + q with 0 < q ≤ p − 1, the ground state for k = 1 is (<sup>p</sup><sub>q</sub>)-fold degenerate.
 It corresponds to a choice of spin for q extra particles,

 $\mathcal{Y}_{r_0}(\mathbf{x}, \mathbf{y}) = [y_1 \wedge \dots \wedge y_p \wedge y_1 \times \wedge \dots \wedge y_p \times \wedge \dots \wedge y_1 \times^{m-1} \wedge \dots \wedge y_p \times^{m-1} \wedge y_{i_1} \times^m \wedge \dots \wedge y_{i_q} \times^m].$ 

→ The corresponding ground state energy is  $E(r_0) = pm(m-1) + 2qm$ .

• When k > 1, ground states wave functions are products of k determinants,

$$\psi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{x}) \prod_{\alpha=1}^k \mathcal{Y}_{\mathbf{r}_0^{(\alpha)}}(\mathbf{x},\mathbf{y}), \quad \mathfrak{r}_0 = (\mathbf{r}_0^{(1)}, \cdots, \mathbf{r}_0^{(k)}).$$

→ The corresponding energy is simply  $E(\mathfrak{r}_0) = kE(\mathbf{r}_0)$ .

erivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Knizhnik-Zamolodchikov equation

• We show that the ground state wave functions obey the KZ equation

$$(p+k)\partial_a\Phi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y})=\sum_{b\neq a}\frac{J_{a,b}J_{b,a}}{x_a-x_b}\Phi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y}).$$

with  $\Phi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{x})^{-1}\psi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y}).$ 

 $\rightarrow$  Indicates a connection with  $\mathfrak{sl}(p)$  Kac-Moody symmetry!

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

## Generalized statistics

• In general, the wedge states  $\mathcal{Y}_{r_0}(x, y)$  do not vanish when two coordinates coincide  $x_a \to x_b$ due to the presence of the spin variables  $y_{i,a}$ . But they do vanish when p + 1 particles approach each other since at least two particles will have the same spin.

→ Wedge states describe particles obeying a generalized exclusion principle!

erivation of the model 00000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

### Generalized statistics

• In general, the wedge states  $\mathcal{Y}_{r_0}(x, y)$  do not vanish when two coordinates coincide  $x_a \to x_b$  due to the presence of the spin variables  $y_{i,a}$ . But they do vanish when p+1 particles approach each other since at least two particles will have the same spin.

→ Wedge states describe particles obeying a generalized exclusion principle!

• Using a similar argument, we can introduce wave functions for holes of coordinate  $\zeta$  by combining p particles of different spins. We find the  $(N + p) \times (N + p)$  determinant



erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Example

• Many known FQHE wave functions can be recovered from  $\psi_{\mathfrak{r}}(\mathbf{x}, \mathbf{y})$ .

For instance, taking k = 2, p = 2, N = 2m, we find a singlet ground state  $r_0 = (r_0, r_0)$  with  $r_0 = \{0, 1, \dots, 2m - 1\}.$ 

 $\rightsquigarrow$  The corresponding wave function reproduces the Moore-Read wave function,

$$\Phi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y}) = (\mathcal{Y}_{r_0}(\mathbf{x},\mathbf{y}))^2 \propto \mathsf{Pf}\left(\frac{(y_{1a}y_{2b} - y_{2a}y_{1b})^2}{x_a - x_b}\right) \prod_{a < b} (x_a - x_b).$$



erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Example

• Many known FQHE wave functions can be recovered from  $\psi_{\mathfrak{r}}(\mathbf{x}, \mathbf{y})$ .

For instance, taking k = 2, p = 2, N = 2m, we find a singlet ground state  $r_0 = (r_0, r_0)$  with  $r_0 = \{0, 1, \dots, 2m - 1\}.$ 

 $\rightsquigarrow$  The corresponding wave function reproduces the Moore-Read wave function,

$$\Phi_{\mathfrak{r}_0}(\mathbf{x},\mathbf{y}) = (\mathcal{Y}_{r_0}(\mathbf{x},\mathbf{y}))^2 \propto \mathsf{Pf}\left(\frac{(y_{1a}y_{2b} - y_{2a}y_{1b})^2}{x_a - x_b}\right) \prod_{a < b} (x_a - x_b).$$

- Two main remaining questions at this stage:
  - How to get rid of extra states due to Plücker-like relations between determinants?
  - Can we observe the emergence of Kac-Moody symmetry at large N?

 $\Rightarrow$  We will use a free fermion formalism to address these questions.

erivation of the model

Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# 5. Free fermion formalism

erivation of the model 00000 Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Fermions at finite N

• The symmetries of the wedge states for k > 1 can be understood by introducing a set of (non-relativistic) free fermion oscillators,

$$\left\{\psi_n^{i,\alpha},\bar{\psi}_m^{j,\beta}\right\} = \delta_{i,j}\delta_{\alpha,\beta}\delta_{n,m}, \quad \left\{\psi_n^{i,\alpha},\psi_m^{j,\beta}\right\} = \left\{\bar{\psi}_n^{i,\alpha},\bar{\psi}_m^{j,\beta}\right\} = 0\,.$$

The fermionic modes carry three types of indices,

- a spin label  $i, j \in \llbracket 1, p \rrbracket$ ,
- a Chern-Simons level  $\alpha, \beta \in \llbracket 1, k \rrbracket$ ,
- a mode number  $n \in \mathbb{Z}^{\geq 0}$ .

erivation of the model 00000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# Fermions at finite N

• The symmetries of the wedge states for k > 1 can be understood by introducing a set of (non-relativistic) free fermion oscillators,

$$\left\{\psi_n^{i,\alpha},\bar{\psi}_m^{j,\beta}\right\} = \delta_{i,j}\delta_{\alpha,\beta}\delta_{n,m}, \quad \left\{\psi_n^{i,\alpha},\psi_m^{j,\beta}\right\} = \left\{\bar{\psi}_n^{i,\alpha},\bar{\psi}_m^{j,\beta}\right\} = 0.$$

The fermionic modes carry three types of indices,

- a spin label  $i, j \in \llbracket 1, p \rrbracket$ ,
- a Chern-Simons level  $\alpha, \beta \in \llbracket 1, k \rrbracket$ ,
- a mode number  $n \in \mathbb{Z}^{\geq 0}$ .

• The Fock space is built from the action of  $\bar{\psi}_n^{i,\alpha}$  on vacuum  $|0\rangle$  annihilated by modes  $\psi_n^{i,\alpha}$ . It is graded by  $\hat{N}_{\alpha} = \sum_{n=0}^{\infty} \sum_{i=1}^{p} \bar{\psi}_n^{i,\alpha} \psi_n^{i,\alpha}$  which counts the modes  $\bar{\psi}_n^{i,\alpha}$  for each  $\alpha$ . It decomposes accordingly into

$$\mathcal{F}_{(N_1,\cdots,N_k)}^{(k)} = \left\{ \prod_{\alpha=1}^k \left( \bar{\psi}_{n_1}^{i_1^{(\alpha)},\alpha} \cdots \bar{\psi}_{n_{N_\alpha}^{(\alpha)}}^{i_{N_\alpha}^{(\alpha)},\alpha} \right) |0\rangle, \quad n_{a_\alpha}^{(\alpha)} \in \mathbb{Z}^{\geq 0}, \ i_{a_\alpha}^{(\alpha)} \in \llbracket 1, p \rrbracket, \quad a_\alpha = 1 \cdots N_\alpha \right\}$$

 $\rightsquigarrow$  We also introduce the dual state such that  $\langle 0|ar{\psi}_n^{i,lpha}=0.$ 

ivation of the model 0000 Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

- These fermionic modes can be used to represent loop algebras with positive modes:
  - $\widehat{\mathfrak{u}}(k)_+$  (or  $\mathfrak{gl}(k)[z]$ ) generated by  $L_n^{\alpha\beta}$ ,  $\alpha, \beta \in \llbracket 1, k \rrbracket$ ,  $n \ge 0$ ,

$$L_n^{\alpha\beta} = \sum_{\ell=0}^{\infty} \sum_{i=1}^{p} \bar{\psi}_{n+\ell}^{i,\alpha} \psi_{\ell}^{i,\beta}, \quad \left[ L_n^{\alpha\beta}, L_m^{\gamma\delta} \right] = \delta_{\beta,\gamma} L_{n+m}^{\alpha\delta} - \delta_{\alpha,\delta} L_{n+m}^{\gamma\beta}.$$

- $\widehat{\mathfrak{u}}(p)_+$  generated by  $K_n^{ij} = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^k \overline{\psi}_{n+\ell}^{i,\alpha} \psi_n^{j,\alpha}$ ,  $i, j \in \llbracket 1, p \rrbracket$ ,  $n \ge 0$ ,
- $\widehat{\mathfrak{u}}(pk)_+$  generated by  $M_n^{i+p(\alpha-1),j+p(\beta-1)} = \sum_{\ell=0}^{\infty} \overline{\psi}_{n+\ell}^{i,\alpha} \psi_{\ell}^{j,\beta}$ ,
- $\widehat{\mathfrak{u}}(1)_+$  generated by  $J_n = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^k \sum_{i=1}^p \overline{\psi}_{n+\ell}^{i,\alpha} \psi_n^{i,\alpha} = \sum_{\alpha=1}^k L_n^{\alpha\alpha} = \sum_{i=1}^p K_n^{ii}$  $\rightsquigarrow$  We note that  $\left[L_n^{\alpha\beta}, K_m^{ij}\right] = 0.$

ivation of the model 0000 Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

- These fermionic modes can be used to represent loop algebras with positive modes:
  - $\widehat{\mathfrak{u}}(k)_+$  (or  $\mathfrak{gl}(k)[z]$ ) generated by  $L_n^{\alpha\beta}$ ,  $\alpha, \beta \in \llbracket 1, k \rrbracket$ ,  $n \ge 0$ ,

$$L_n^{\alpha\beta} = \sum_{\ell=0}^{\infty} \sum_{i=1}^{p} \bar{\psi}_{n+\ell}^{i,\alpha} \psi_{\ell}^{i,\beta}, \quad \left[ L_n^{\alpha\beta}, L_m^{\gamma\delta} \right] = \delta_{\beta,\gamma} L_{n+m}^{\alpha\delta} - \delta_{\alpha,\delta} L_{n+m}^{\gamma\beta}.$$

- $\widehat{\mathfrak{u}}(p)_+$  generated by  $K_n^{ij} = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^k \overline{\psi}_{n+\ell}^{i,\alpha} \psi_n^{j,\alpha}$ ,  $i, j \in \llbracket 1, p \rrbracket$ ,  $n \ge 0$ ,
- $\widehat{\mathfrak{u}}(pk)_+$  generated by  $M_n^{i+p(\alpha-1),j+p(\beta-1)} = \sum_{\ell=0}^{\infty} \overline{\psi}_{n+\ell}^{i,\alpha} \psi_{\ell}^{j,\beta}$ ,
- $\widehat{\mathfrak{u}}(1)_+$  generated by  $J_n = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^k \sum_{i=1}^p \overline{\psi}_{n+\ell}^{i,\alpha} \psi_n^{i,\alpha} = \sum_{\alpha=1}^k L_n^{\alpha\alpha} = \sum_{i=1}^p K_n^{ii}$
- $\rightsquigarrow$  We note that  $\left[L_n^{\alpha\beta}, K_m^{ij}\right] = 0.$
- We have the decomposition [T. Nakanishi, A. Tsuchiya 1992]

$$\widehat{\mathfrak{u}}(pk)_+ \supset \widehat{\mathfrak{su}}(p)_+ \oplus \widehat{\mathfrak{su}}(k)_+ \oplus \widehat{\mathfrak{u}}(1)_+$$

where  $\widehat{\mathfrak{su}}(k)_+$  is defined using traceless generators  $\overline{L}_n^{\alpha\beta} = L_n^{\alpha\beta} - \frac{\delta_{\alpha,\beta}}{k} \sum_{\gamma} L_n^{\gamma\gamma}$ .

erivation of the model 00000 Spin Calogero mode 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# $\widehat{\mathfrak{su}}(k)_+$ -invariance

• Wedge states can be realized as projection of states in the fermionic Fock space  $\mathcal{F}_{(N,\cdots,N)}^{(k)}$ ,

$$\prod_{\alpha=1}^{k} [x_{1}^{n_{1}^{(\alpha)}} y_{i_{1}^{(\alpha)}} \wedge \dots \wedge x_{N}^{n_{N}^{(\alpha)}} y_{i_{N}^{(\alpha)}}] = \langle \mathbf{x}, \mathbf{y} | \prod_{\alpha=1}^{k} \left( \bar{\psi}_{1}^{i_{1}^{(\alpha)}, \alpha} \cdots \bar{\psi}_{n_{N}^{(\alpha)}}^{i_{N}^{(\alpha)}, \alpha} \right) | \mathbf{0} \rangle$$

with the projector  $\langle x, y |$  constructed as follows,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} | &= \langle 0 | \Psi(x_N, y_N) \cdots \Psi(x_1, y_1) \\ \Psi(x_a, y_a) &= \prod_{\alpha=1}^k \psi^{\alpha}(x_a, y_a), \quad \psi^{\alpha}(x_a, y_a) = \sum_{n=0}^\infty \sum_{i=1}^p \psi_n^{i, \alpha} x_a^n y_{i, a}. \end{aligned}$$

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

# $\widehat{\mathfrak{su}}(k)_+$ -invariance

• Wedge states can be realized as projection of states in the fermionic Fock space  $\mathcal{F}_{(N,\cdots,N)}^{(k)}$ ,

$$\prod_{\alpha=1}^{k} [x_{1}^{n_{1}^{(\alpha)}} y_{i_{1}^{(\alpha)}} \wedge \dots \wedge x_{N}^{n_{N}^{(\alpha)}} y_{i_{N}^{(\alpha)}}] = \langle \mathbf{x}, \mathbf{y} | \prod_{\alpha=1}^{k} \left( \bar{\psi}_{1}^{i_{1}^{(\alpha)}, \alpha} \cdots \bar{\psi}_{n_{N}^{(\alpha)}}^{i_{N}^{(\alpha)}, \alpha} \right) | \mathbf{0} \rangle$$

with the projector  $\langle x, y |$  constructed as follows,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} | &= \langle 0 | \Psi(x_N, y_N) \cdots \Psi(x_1, y_1) \\ \Psi(x_a, y_a) &= \prod_{\alpha=1}^k \psi^{\alpha}(x_a, y_a), \quad \psi^{\alpha}(x_a, y_a) = \sum_{n=0}^\infty \sum_{i=1}^p \psi^{i, \alpha}_n x^n_a y_{i, a}. \end{aligned}$$

• Linear relations between product of determinants follow from the  $\hat{\mathfrak{su}}(k)_+$ -invariance

$$\langle \boldsymbol{x}, \boldsymbol{y} | \bar{L}_n^{\alpha\beta} = 0$$
erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### $\widehat{\mathfrak{su}}(k)_+$ -invariance

• Wedge states can be realized as projection of states in the fermionic Fock space  $\mathcal{F}_{(N,\dots,N)}^{(k)}$ ,

$$\prod_{\alpha=1}^{k} [x_{1}^{n_{1}^{(\alpha)}} y_{i_{1}^{(\alpha)}} \wedge \dots \wedge x_{N}^{n_{N}^{(\alpha)}} y_{i_{N}^{(\alpha)}}] = \langle \mathbf{x}, \mathbf{y} | \prod_{\alpha=1}^{k} \left( \bar{\psi}_{1}^{i_{1}^{(\alpha)}, \alpha} \cdots \bar{\psi}_{n_{N}^{(\alpha)}}^{i_{N}^{(\alpha)}, \alpha} \right) | \mathbf{0} \rangle$$

with the projector  $\langle x, y |$  constructed as follows,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} | &= \langle 0 | \Psi(x_N, y_N) \cdots \Psi(x_1, y_1) \\ \Psi(x_a, y_a) &= \prod_{\alpha=1}^k \psi^{\alpha}(x_a, y_a), \quad \psi^{\alpha}(x_a, y_a) = \sum_{n=0}^\infty \sum_{i=1}^p \psi^{i, \alpha}_n x^n_a y_{i, a}. \end{aligned}$$

• Linear relations between product of determinants follow from the  $\hat{\mathfrak{su}}(k)_+$ -invariance

$$\langle \boldsymbol{x}, \boldsymbol{y} | \bar{L}_n^{lpha eta} = 0$$

⇒ Resolve problem of state overcounting by taking the quotient of  $\mathcal{F}_{N}^{(k)}$  under  $\widehat{\mathfrak{su}}(k)_{+}$ -symmetry!

L

ivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

Example: For k = 2, we have three series of generators  $L_n^{12}$ ,  $L_n^{21}$  and  $L_n^{11} - L_n^{22}$  for  $\widehat{\mathfrak{su}}(2)_+$ . Fermionic states have the form  $\overline{\psi}_{n_1}^{i_1,1} \cdots \overline{\psi}_{n_{N_1}}^{i_{N_1},1} \overline{\psi}_{m_1}^{j_1,2} \cdots \overline{\psi}_{m_{N_2}}^{i_{N_2},2} |0\rangle$ .

 $\sim$  Consider the action of  $L_n^{21}$  on the following states

$$\begin{split} & \sum_{n}^{21} \bar{\psi}_{n_{1}}^{i_{1},1} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{m_{1}}^{j_{1},2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \\ &= (-1)^{N} \sum_{a=1}^{N+1} (-1)^{a-1} \bar{\psi}_{n_{1}}^{i_{1},1} \cdots \bar{\psi}_{n_{a}}^{j_{a},2} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{n_{a}+n}^{i_{a},2} \bar{\psi}_{m_{1}}^{j_{1},2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \end{split}$$

L

ivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

Example: For k = 2, we have three series of generators  $L_n^{12}$ ,  $L_n^{21}$  and  $L_n^{11} - L_n^{22}$  for  $\widehat{\mathfrak{su}}(2)_+$ . Fermionic states have the form  $\overline{\psi}_{n_1}^{i_1,1} \cdots \overline{\psi}_{n_{N_1}}^{i_{N_1},1} \overline{\psi}_{m_1}^{j_1,2} \cdots \overline{\psi}_{m_{N_2}}^{i_{N_2},2} |0\rangle$ .

 $\sim$  Consider the action of  $L_n^{21}$  on the following states

$$\begin{split} & \stackrel{21}{n} \bar{\psi}_{n_{1}}^{i_{1},1} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{m_{1}}^{i_{1},2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \\ &= (-1)^{N} \sum_{a=1}^{N+1} (-1)^{a-1} \bar{\psi}_{n_{1}}^{i_{1},1} \cdots \bar{\psi}_{n_{a}}^{j_{a},2} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{n_{a}+n}^{i_{a},2} \bar{\psi}_{m_{1}}^{j_{1},2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \end{split}$$

 $\rightarrow$  After projection on  $\langle x, y |$ , we find a generalization of Plücker relations,

 $\sum_{a=1}^{N+1} (-1)^{a} [y_{i_{1}} x^{n_{1}} \wedge \dots \wedge y_{i_{d}} x^{n_{d}} \wedge \dots \wedge y_{i_{N+1}} x^{n_{N+1}}] \cdot [y_{i_{a}} x^{n_{a}+n} \wedge y_{j_{1}} x^{m_{1}} \wedge \dots \wedge y_{j_{N-1}} x^{m_{N-1}}] = 0.$ 

L

ivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

Example: For k = 2, we have three series of generators  $L_n^{12}$ ,  $L_n^{21}$  and  $L_n^{11} - L_n^{22}$  for  $\widehat{\mathfrak{su}}(2)_+$ . Fermionic states have the form  $\overline{\psi}_{n_1}^{i_1,1} \cdots \overline{\psi}_{n_{N_1}}^{i_{N_1},1} \overline{\psi}_{m_1}^{j_1,2} \cdots \overline{\psi}_{m_{N_2}}^{i_{N_2},2} |0\rangle$ .

 $\sim$  Consider the action of  $L_n^{21}$  on the following states

$$\begin{split} & \stackrel{21}{n} \bar{\psi}_{n_{1}}^{i_{1},1} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{m_{1}}^{i_{1},2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \\ &= (-1)^{N} \sum_{a=1}^{N+1} (-1)^{a-1} \bar{\psi}_{n_{1}}^{i_{1},1} \cdots \bar{\psi}_{n_{a}}^{j_{a},2} \cdots \bar{\psi}_{n_{N+1}}^{i_{N+1},1} \bar{\psi}_{n_{a}+n}^{i_{a},2} \bar{\psi}_{m_{1}}^{j_{1},2} \cdots \bar{\psi}_{m_{N-1}}^{j_{N-1},2} |0\rangle \end{split}$$

 $\rightarrow$  After projection on  $\langle x, y |$ , we find a generalization of Plücker relations,

$$\sum_{a=1}^{N+1} (-1)^a [y_{i_1} x^{n_1} \wedge \cdots \wedge y_{j_a} x^{n_a} \wedge \cdots \wedge y_{i_{N+1}} x^{n_{N+1}}] \cdot [y_{i_a} x^{n_a+n} \wedge y_{j_1} x^{m_1} \wedge \cdots \wedge y_{j_{N-1}} x^{m_{N-1}}] = 0.$$

 $\rightsquigarrow$  We find the same relation for  $L_n^{12}$  while  $L_n^{11} - L_n^{22}$  produces a trivial identity.

erivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Fermions in large N

• We can use this fermionic description to see the emergence of the Kac-Moody symmetry in the large N limit (recall N = # number of vortices).

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Fermions in large N

• We can use this fermionic description to see the emergence of the Kac-Moody symmetry in the large N limit (recall N = # number of vortices).

• The symmetries of the model in the large N limit are described using the fermionic modes

$$\left\{\psi_r^{i,\alpha},\bar{\psi}_s^{j,\beta}\right\} = \delta_{i,j}\delta_{\alpha,\beta}\delta_{r+s,0}, \quad \left\{\psi_r^{i,\alpha},\psi_s^{j,\beta}\right\} = \left\{\bar{\psi}_r^{i,\alpha},\bar{\psi}_s^{j,\beta}\right\} = 0.$$

 $\rightsquigarrow \quad \text{Still carry spin } i,j \text{ and CS } \alpha,\beta \text{ indices, but now have half-integer mode indices} \\ r,s \in \mathbb{Z}+\frac{1}{2}.$ 

 $\stackrel{\rightsquigarrow}{\longrightarrow} \ \ \text{The Fock space is obtained by the action of negative modes } \psi_{-r}^{i,\alpha}, \ \bar{\psi}_{-r}^{i,\alpha} \ \text{on the vacuum} \\ |\emptyset\rangle, \ \text{with } \psi_{r}^{i,\alpha}|\emptyset\rangle = \bar{\psi}_{r}^{i,\alpha}|\emptyset\rangle = 0, \ (r > 0).$ 

rivation of the model

Spin Calogero mode 00000 pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

- The fermionic Fock space  $\mathcal{F}_{\infty}^{(p,k)}$  admits the action of the following Kac-Moody algebra,
  - $\widehat{\mathfrak{u}}(pk)_1$  generated by  $M_n^{i+(\alpha-1)p;j+(\beta-1)p} = \sum_{s \in \mathbb{Z}+1/2} : \overline{\psi}_s^{i\alpha} \psi_{n-s}^{j,\beta} :$ ,
  - $\hat{\mathfrak{u}}(p)_k$  generated by  $K_n^{ij} = \sum_{s \in \mathbb{Z}+1/2} \sum_{\alpha=1}^k : \bar{\psi}_s^{i\alpha} \psi_{n-s}^{j,\alpha} :$
  - $\widehat{\mathfrak{u}}(k)_{\rho}$  generated by  $L_{n}^{\alpha\beta} = \sum_{s \in \mathbb{Z}+1/2} \sum_{i=1}^{\rho} : \overline{\psi}_{s}^{i\alpha} \psi_{n-s}^{i,\beta} :.$
  - $\widehat{\mathfrak{u}}(1)_{pk}$  generated by  $J_n = \sum_i K_n^{ii} = \sum_{\alpha} L_n^{\alpha\alpha}$ .
- $\Rightarrow$  We have the following decomposition as a conformal embedding,

 $\widehat{\mathfrak{su}}(pk)_1 \supset \widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{su}}(k)_p.$ 

where  $\widehat{\mathfrak{su}}(p)_k$  and  $\widehat{\mathfrak{su}}(k)_p$  are again defined using traceless generators.

rivation of the model

Spin Calogero mode 00000 pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

- The fermionic Fock space  $\mathcal{F}_{\infty}^{(p,k)}$  admits the action of the following Kac-Moody algebra,
  - $\widehat{\mathfrak{u}}(pk)_1$  generated by  $M_n^{i+(\alpha-1)p;j+(\beta-1)p} = \sum_{s \in \mathbb{Z}+1/2} : \overline{\psi}_s^{i\alpha} \psi_{n-s}^{j,\beta} :$ ,
  - $\hat{u}(p)_k$  generated by  $K_n^{ij} = \sum_{s \in \mathbb{Z}+1/2} \sum_{\alpha=1}^k : \bar{\psi}_s^{i\alpha} \psi_{n-s}^{j,\alpha} :$
  - $\widehat{\mathfrak{u}}(k)_{\rho}$  generated by  $L_{n}^{\alpha\beta} = \sum_{s \in \mathbb{Z}+1/2} \sum_{i=1}^{\rho} : \overline{\psi}_{s}^{i\alpha} \psi_{n-s}^{i,\beta} :.$
  - $\widehat{\mathfrak{u}}(1)_{pk}$  generated by  $J_n = \sum_i K_n^{ii} = \sum_{\alpha} L_n^{\alpha\alpha}$ .
- $\Rightarrow$  We have the following decomposition as a conformal embedding,

 $\widehat{\mathfrak{su}}(pk)_1 \supset \widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{su}}(k)_p.$ 

where  $\widehat{\mathfrak{su}}(p)_k$  and  $\widehat{\mathfrak{su}}(k)_p$  are again defined using traceless generators.

• Accordingly, the fermionic Fock space  $\mathcal{F}^{(p,k)}_\infty$  can be decomposed as

$$\mathcal{F}_{\infty}^{(p,k)} = \bigoplus_{\sigma \in \mathbb{Z}} \mathcal{F}_{\infty,\sigma}^{(p,k)}, \quad \mathcal{F}_{\sigma}^{p,k} \cong \bigoplus_{\lambda} \mathcal{W}_{\lambda}^{\widehat{\mathfrak{su}}(p)_k} \otimes \mathcal{W}_{\lambda'}^{\widehat{\mathfrak{su}}(k)_p} \otimes \mathcal{W}_{\sigma}^{\widehat{\mathfrak{u}}(1)}.$$

where  $\sigma \in \mathbb{Z}$  is the  $\hat{\mathfrak{u}}(1)_{pk}$  charge,  $\mathcal{W}^{\hat{\mathfrak{g}}_k}_{\boldsymbol{\lambda}}$  is an irreducible representation of  $\hat{\mathfrak{g}}_k$  labeled by certain partitions  $\lambda$ , and  $\lambda'$  is the transposed of  $\lambda$ .

vation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

- The large N limit of wedge states can be obtained by
  - **()** Factoring out a monomial in  $x_a$  (and then discarding it), eg for the vacua at k = 1,

$$\mathcal{Y}_{r_0}(\mathbf{x},\mathbf{y}) = \left(\prod_{a=1}^N x_a^{m-1}\right) \left[y_1 x^{-m+1} \wedge \cdots \wedge y_p x^{-m+1} \wedge \cdots \wedge y_1 \wedge \cdots \wedge y_p \wedge xy_{i_1} \wedge \cdots \wedge xy_{i_q}\right]$$

Q Re-ordering the wedges starting from the higher exponent for x,

 $\mathcal{Y}_{r_0}(\mathbf{x}, \mathbf{y}) \propto [(y_{i_1} \mathbf{x}) \wedge \cdots \wedge (y_{i_a} \mathbf{x}) \wedge y_1 \wedge \cdots \wedge y_p \wedge y_1 \mathbf{x}^{-1} \wedge \cdots \wedge y_p \mathbf{x}^{-1} \cdots \wedge y_1 \mathbf{x}^{-m+1} \wedge \cdots \wedge y_p \mathbf{x}^{-m+1}].$ 

 $\rightsquigarrow$  The large N limit, obtained as  $m \to \infty$  for N = mp + q, produces an infinite wedge product,

$$\mathcal{Y}_{\mathbf{r}_0}^{\infty}(\mathbf{x},\mathbf{y}) = \left[ (y_{i_1}x) \wedge \cdots \wedge (y_{i_q}x) \wedge y_1 \wedge \cdots \wedge y_p \wedge y_1x^{-1} \wedge \cdots \wedge y_px^{-1} \wedge \cdots \right].$$

vation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

- The large N limit of wedge states can be obtained by
  - **()** Factoring out a monomial in  $x_a$  (and then discarding it), eg for the vacua at k = 1,

$$\mathcal{Y}_{r_0}(\mathbf{x},\mathbf{y}) = \left(\prod_{a=1}^N x_a^{m-1}\right) \left[y_1 x^{-m+1} \wedge \cdots \wedge y_p x^{-m+1} \wedge \cdots \wedge y_1 \wedge \cdots \wedge y_p \wedge xy_{i_1} \wedge \cdots \wedge xy_{i_q}\right]$$

Q Re-ordering the wedges starting from the higher exponent for x,

 $\mathcal{Y}_{r_0}(\mathbf{x}, \mathbf{y}) \propto [(y_{i_1} x) \wedge \cdots \wedge (y_{i_n} x) \wedge y_1 \wedge \cdots \wedge y_p \wedge y_1 x^{-1} \wedge \cdots \wedge y_p x^{-1} \cdots \wedge y_1 x^{-m+1} \wedge \cdots \wedge y_p x^{-m+1}].$ 

 $\rightsquigarrow$  The large N limit, obtained as  $m \to \infty$  for N = mp + q, produces an infinite wedge product,

$$\mathcal{Y}_{\mathbf{r}_0}^{\infty}(\mathbf{x},\mathbf{y}) = \left[ (y_{i_1} \times) \wedge \cdots \wedge (y_{i_q} \times) \wedge y_1 \wedge \cdots \wedge y_p \wedge y_1 \times^{-1} \wedge \cdots \wedge y_p \times^{-1} \wedge \cdots \right].$$

• Formally, these wedge products can be obtained as

$$\prod_{\alpha=1}^{k} \mathcal{Y}^{\infty}_{\boldsymbol{r}^{(\alpha)}_{0}}(\boldsymbol{x},\boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{y} | \prod_{\alpha=1}^{k} \prod_{r=1}^{q} \bar{\psi}^{i^{(\alpha)},\alpha}_{-1/2} | \emptyset \rangle.$$

with the projector  $\langle x, y |$  defined as a certain limit of the previous one.

Spin Calogero model 00000 Spectrum and eigenfunctions

Discussion 00000

• In large N limit, the  $\hat{\mathfrak{su}}(k)_+$ -invariance of the projector  $\langle x, y \rangle$  is expected to be replaced by

 $\langle \mathbf{x}, \mathbf{y} | \overline{L}_n^{\alpha\beta} = 0, \quad n \leq 0,$ 

If so, the decomposition of  $\widehat{\mathfrak{su}}(pk)_1$  implies that linearly-independent eigenfunctions are spanned by  $\widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{u}}(1)_{pk}$ . The extra  $\widehat{\mathfrak{u}}(1)_{pk}$  factor can be introduced by an extra free boson.

rivation of the model

Spin Calogero model

Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

• In large N limit, the  $\hat{\mathfrak{su}}(k)_+$ -invariance of the projector  $\langle x, y \rangle$  is expected to be replaced by

 $\langle \mathbf{x}, \mathbf{y} | \overline{L}_n^{\alpha\beta} = 0, \quad n \leq 0,$ 

If so, the decomposition of  $\widehat{\mathfrak{su}}(pk)_1$  implies that linearly-independent eigenfunctions are spanned by  $\widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{u}}(1)_{pk}$ . The extra  $\widehat{\mathfrak{u}}(1)_{pk}$  factor can be introduced by an extra free boson.

 $\Rightarrow$  In this way, we recover the Kac-Moody symmetry of the model!

Spin Calogero model 00000 Discussion 00000

• In large N limit, the  $\widehat{\mathfrak{su}}(k)_+$ -invariance of the projector  $\langle x, y |$  is expected to be replaced by

 $\langle \mathbf{x}, \mathbf{y} | \overline{L}_n^{\alpha\beta} = 0, \quad n \leq 0,$ 

If so, the decomposition of  $\widehat{\mathfrak{su}}(pk)_1$  implies that linearly-independent eigenfunctions are spanned by  $\widehat{\mathfrak{su}}(p)_k \oplus \widehat{\mathfrak{u}}(1)_{pk}$ . The extra  $\widehat{\mathfrak{u}}(1)_{pk}$  factor can be introduced by an extra free boson.

 $\Rightarrow$  In this way, we recover the Kac-Moody symmetry of the model!

Remark: The fields  $\Psi(x_a, y_a)$  satisfy the  $\widehat{u}(p)_k$  primary field condition,

 $[K_n^{ij}, \Psi(x_a, y_a)] = -x_a^{-n} y_{i,a} \partial_{j,a} \Psi(x_a, y_a).$ 

→ It would be nice to relate it to the KZ equation!

erivation of the model

Spin Calogero mode

Spectrum and eigenfunctions

Fermionic formalism

Discussion •0000

### 6. Discussion

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Summary

• Starting from DTT's matrix model, we derived the Hamiltonian describing the dynamics of vortices in 3d U(p) Chern-Simons theory at level k.

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Summary

• Starting from DTT's matrix model, we derived the Hamiltonian describing the dynamics of vortices in 3d U(p) Chern-Simons theory at level k.

• We introduced a class of wave functions constructed as determinants involving both coordinates and spin dependence called wedge states.

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Summary

• Starting from DTT's matrix model, we derived the Hamiltonian describing the dynamics of vortices in 3d U(p) Chern-Simons theory at level k.

• We introduced a class of wave functions constructed as determinants involving both coordinates and spin dependence called wedge states.

• We have shown that the action of the Hamiltonian on wedge states is triangular. We deduced the energy spectrum and ground state wave functions.

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion

#### Summary

• Starting from DTT's matrix model, we derived the Hamiltonian describing the dynamics of vortices in 3d U(p) Chern-Simons theory at level k.

• We introduced a class of wave functions constructed as determinants involving both coordinates and spin dependence called wedge states.

• We have shown that the action of the Hamiltonian on wedge states is triangular. We deduced the energy spectrum and ground state wave functions.

• We proved that ground state wave functions obey a Knizhnik-Zamolodchikov equation.

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Summary

• Starting from DTT's matrix model, we derived the Hamiltonian describing the dynamics of vortices in 3d U(p) Chern-Simons theory at level k.

• We introduced a class of wave functions constructed as determinants involving both coordinates and spin dependence called wedge states.

• We have shown that the action of the Hamiltonian on wedge states is triangular. We deduced the energy spectrum and ground state wave functions.

• We proved that ground state wave functions obey a Knizhnik-Zamolodchikov equation.

• We introduced a fermionic construction for wedge states at finite N and explained the overcounting of states for k > 1 by the presence of a  $\widehat{\mathfrak{su}(k)}_+$  loop algebra symmetry.

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Summary

• Starting from DTT's matrix model, we derived the Hamiltonian describing the dynamics of vortices in 3d U(p) Chern-Simons theory at level k.

• We introduced a class of wave functions constructed as determinants involving both coordinates and spin dependence called wedge states.

• We have shown that the action of the Hamiltonian on wedge states is triangular. We deduced the energy spectrum and ground state wave functions.

• We proved that ground state wave functions obey a Knizhnik-Zamolodchikov equation.

• We introduced a fermionic construction for wedge states at finite N and explained the overcounting of states for k > 1 by the presence of a  $\widehat{\mathfrak{su}(k)}_+$  loop algebra symmetry.

• We discussed the large N limit of the fermionic description and observed the emergence of the  $\widehat{\mathfrak{su}(p)}_k$  Kac-Moody symmetry.



### Epilogue

Discussion

• Until recently, the main open question was the integrability of the model. But new results have been obtained by [Hu, Li, Ye, Zhou 2409.12486] using a geometric construction of the Hilbert space, and the action of the Deformed Double Current Algebra.

# Introduction Derivation of the model Spin Calogero model Spectrum and eigenfunctions Coordination October Coordina

• Until recently, the main open question was the integrability of the model. But new results have been obtained by [Hu, Li, Ye, Zhou 2409.12486] using a geometric construction of the Hilbert space, and the action of the Deformed Double Current Algebra.

- Proof of integrability and Yangian invariance  $(\mathcal{H} = 2e^{\frac{1}{2}ad_f}\mathcal{D} \text{ so } [\mathcal{H}, e^{\frac{1}{2}ad_f}Y(\mathfrak{gl}_p)] = 0).$ 

# Introduction Derivation of the model Spin Calogero model Spectrum and eigenfunctions Coordination Coordinatio Coordination Coordination

• Until recently, the main open question was the integrability of the model. But new results have been obtained by [Hu, Li, Ye, Zhou 2409.12486] using a geometric construction of the Hilbert space, and the action of the Deformed Double Current Algebra.

- Proof of integrability and Yangian invariance  $(\mathcal{H} = 2e^{\frac{1}{2}ad_f}\mathcal{D} \text{ so } [\mathcal{H}, e^{\frac{1}{2}ad_f}Y(\mathfrak{gl}_p)] = 0).$
- Proof of emergence of Kac-Moody symmetry from DDCA generators as  $N 
  ightarrow \infty.$

# Introduction Derivation of the model Spin Calogero model Spectrum and eigenfunctions Fermionic formalism Discussion

• Until recently, the main open question was the integrability of the model. But new results have been obtained by [Hu, Li, Ye, Zhou 2409.12486] using a geometric construction of the Hilbert space, and the action of the Deformed Double Current Algebra.

- Proof of integrability and Yangian invariance  $(\mathcal{H} = 2e^{\frac{1}{2}ad_f}\mathcal{D} \text{ so } [\mathcal{H}, e^{\frac{1}{2}ad_f}Y(\mathfrak{gl}_p)] = 0).$
- Proof of emergence of Kac-Moody symmetry from DDCA generators as  $N 
  ightarrow \infty.$
- Study of the Calogero-Sutherland version of the model,

$$\mathcal{H}_{CS} = -\sum_{a=1}^{N} \left( x_a \frac{\partial}{\partial x_a} \right)^2 + 2\sum_{\substack{a,b=1\\a < b}}^{N} \frac{x_a x_b}{(x_a - x_b)^2} J_{a,b} J_{b,a}.$$

→ Spectrum of the Hamiltonian (labeled by Gelfand-Tsetlin patterns).

### 

• Until recently, the main open question was the integrability of the model. But new results have been obtained by [Hu, Li, Ye, Zhou 2409.12486] using a geometric construction of the Hilbert space, and the action of the Deformed Double Current Algebra.

- Proof of integrability and Yangian invariance  $(\mathcal{H} = 2e^{\frac{1}{2}ad_f}\mathcal{D} \text{ so } [\mathcal{H}, e^{\frac{1}{2}ad_f}Y(\mathfrak{gl}_p)] = 0).$
- Proof of emergence of Kac-Moody symmetry from DDCA generators as  $N 
  ightarrow \infty.$
- Study of the Calogero-Sutherland version of the model,

-

$$\mathcal{H}_{CS} = -\sum_{a=1}^{N} \left( x_a \frac{\partial}{\partial x_a} \right)^2 + 2\sum_{\substack{a,b=1\\a < b}}^{N} \frac{x_a x_b}{(x_a - x_b)^2} J_{a,b} J_{b,a}.$$

→ Spectrum of the Hamiltonian (labeled by Gelfand-Tsetlin patterns).

rivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Open questions

Holes: How do we include the holes in this description? In the abelian case,

$$\mathcal{H}_{\text{total}} = \mathcal{H}_g(\mathbf{x}) - g\mathcal{H}_{\frac{1}{g}}(\mathbf{y}), \quad \mathcal{H}_g(\mathbf{x}) = \sum_{a} \left( -\frac{\partial^2}{\partial x_a^2} + x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{g(g-1)}{(x_a - x_b)^2}.$$

How is the Hamiltonian modified in the non-Abelian case? Corresponding matrix model?

erivation of the model

Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Open questions

Holes: How do we include the holes in this description? In the abelian case,

$$\mathcal{H}_{\text{total}} = \mathcal{H}_g(\mathbf{x}) - g\mathcal{H}_{\frac{1}{g}}(\mathbf{y}), \quad \mathcal{H}_g(\mathbf{x}) = \sum_{a} \left( -\frac{\partial^2}{\partial x_a^2} + x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{g(g-1)}{(x_a - x_b)^2}.$$

How is the Hamiltonian modified in the non-Abelian case? Corresponding matrix model?

Observables: How can we exploit integrability to compute physically relevant quantities? E.g. correlation functions, entanglement entropy,...

erivation of the model 00000 Spin Calogero model 00000 Spectrum and eigenfunctions

Fermionic formalism

Discussion 00000

#### Open questions

Holes: How do we include the holes in this description? In the abelian case,

$$\mathcal{H}_{\text{total}} = \mathcal{H}_g(\mathbf{x}) - g\mathcal{H}_{\frac{1}{g}}(\mathbf{y}), \quad \mathcal{H}_g(\mathbf{x}) = \sum_{a} \left( -\frac{\partial^2}{\partial x_a^2} + x_a^2 \right) + 2 \sum_{\substack{a,b=1\\a < b}}^{N} \frac{g(g-1)}{(x_a - x_b)^2}.$$

How is the Hamiltonian modified in the non-Abelian case? Corresponding matrix model?

Observables: How can we exploit integrability to compute physically relevant quantities? E.g. correlation functions, entanglement entropy,...

Deformations: It is possible to introduce trigonometric/elliptic deformations, relativistic deformations, and  $\beta$ -deformations of the coupling. Generalization of Uglov's construction of Calogero-Sutherland wave functions?

rivation of the model

Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

Algebras: This type of quantum models have also connections with the AGT correspondence (e.g. [Estienne, Pasquier, Santachiara, Serban]). Toroidal quantum groups are known to play an interesting role in this context!

 $\Rightarrow$  It would be interesting to investigate the interplay between these algebraic structures and the braiding of non-Abelian anyons!

rivation of the model

Spin Calogero model

pectrum and eigenfunctions

Fermionic formalism

Discussion 00000

Algebras: This type of quantum models have also connections with the AGT correspondence (e.g. [Estienne, Pasquier, Santachiara, Serban]). Toroidal quantum groups are known to play an interesting role in this context!

 $\Rightarrow$  It would be interesting to investigate the interplay between these algebraic structures and the braiding of non-Abelian anyons!



Thank you !!!