Quantum integrable system associated with the G_2 exceptional Lie algebra

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Contents

• 1920, Wilhelm Lenz, to explain the physical phenomenon that the magnetism of magnet disappears when the temperature is larger than a critical value

$$H = J \sum_{j=1}^{N} \sigma_j \sigma_{j+1}$$

1. two kinds of states in nature: ordered and disordered.

paramagnetic and ferromagnetic states

normal and superfluid states

normal and superconductive states

2. the transition between two states.

1920, Ising, 1D

1944, Onsager, 2D

Open a new field: Ising model has many applications in many fields such as the folding of DNA in organisms, the spread of viruses, artificial intelligence, activation and deactivation of brain nerve cells, weather forecast, forest fire, social sciences, finance.

- 1928, Heisenberg model

$$H = J \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$$

spin exchanging interaction

quantum magnetism: AFM, FM

coordinate Bethe ansatz

quantum inverse scattering method

spinon, fractional excitations

anisotropic couplings, magnetic ordered states, quantum phase transitions

$$H = -\sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z)$$

$|\Delta| < 1$, gapless phase; $|\Delta| > 1$, gapped phase; $|\Delta| = 1$, critical points.

$\Delta = -1$, first order phase transition; $\Delta = 1$, KT phase transition

- 1931, Bethe ansatz

$$H = \frac{1}{2} J \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} - \frac{1}{2} h \sum_{j=1}^{N} \sigma_j^z,$$

Eigenstates

$$\begin{aligned}
|0\rangle &= |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_N, \\
|k\rangle &= \sum_{x=1}^{N} \psi(x) S_x^- |0\rangle, \\
|k_1, k_2\rangle &= \sum_{x_1, x_2 = 1}^{N} \psi(x_1, x_2) S_{x_1}^- S_{x_2}^- |0\rangle, \\
|k_1, \cdots, k_M\rangle &= \sum_{\{x_1, \cdots, x_M\}=1}^{N} \psi(x_1, \cdots, x_M) S_{x_1}^- \cdots S_{x_M}^- |0\rangle.
\end{aligned}$$

Eigen-equation

$$\begin{aligned}
J \sum_{j=1}^{M} \sum_{\delta=\pm 1} &\psi(x_1, \cdots, x_j + \delta, \cdots, x_M) + (h - 2J) M \psi(x_1, \cdots, x_M) \\
&+ 2J \sum_{i<j} \psi(x_1, \cdots, x_M)[\delta_{x_i, x_j+1} + \delta_{x_i, x_j-1}] = (E - E_0) \psi(x_1, \cdots, x_M).
\end{aligned}$$

Bethe ansatz

$$\psi(x_1, \cdots, x_M) = \sum_{P,Q} A_P e^{i\sum_{j=1}^{M} k_{P_j} x_{Q_j}} \theta(x_{Q_1} < \cdots < x_{Q_M}),$$

where P and Q are the permutations.

When only two filliped spins are neighbor, substituting the Bethe ansatz into eigen-equation, we obtain the scattering matrix

$$S_{P_i P_{i+1}} = \frac{A_{P_1 \cdots P_{i+1} P_i \cdots P_M}}{A_{P_1 \cdots P_i P_{i+1} \cdots P_M}} = -\frac{e^{i(k_{P_i} + k_{P_{i+1}})} + 1 - 2e^{ik_{P_{i+1}}}}{e^{i(k_{P_i} + k_{P_{i+1}})} + 1 - 2e^{ik_{P_i}}}.$$

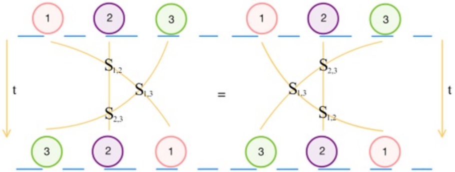
When more then two filliped spins are neighbor, the Bethe wave functions are also the eigen-wave functions of the Hamiltonian.

Ideas:

1. The multi-particle scattering process can be decomposed into the product of two-particle scatterings.

$$S_{12\cdots M}(k_1, k_2, \cdots, k_M) = S_{12}(K_1, k_2)S_{1,3}(k_1, k_3)\cdots S_{1,M}(k_1, k_M).$$

2. The two-particle scattering should satisfy the Yang-Baxter equation



$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}.$$

From the eigen-equation, the eigenvalues of the Hamiltonian are

$$E - E_0 = 2J \sum_{j=1}^{M} (\cos k_j - 1) + Mh.$$

From $\psi(\cdots, x_j + N, \cdots) = \psi(\cdots, x_j, \cdots)$, the k_j should satisfy the Bethe ansatz equations

$$e^{ik_j N} = \prod_{l \neq j}^{M} S_{jl}^{-1}, \quad j = 1, \cdots, M.$$

Explanations:

◇ Diagonalization a 2^N matrix $\rightarrow$ Solving N algebraic equation, which can be done.

◇ Thermodynamic limit, density of state, physical quantities, finite temperature

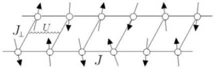
Exactly solvable models:

1. interacting particles with δ -function

$$H=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+2c\sum_{i<j}\delta(x_i-x_j)$$

2. spin chain and spin ladder

$$H=\frac{1}{2}J\sum_{j=1}^{N}\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}-\frac{1}{2}h\sum_{j=1}^{N}\sigma_j^z$$



3. Hubbard, supersymmetry t-J, Kondo

$$H=\sum_{i,j=1}^{N}\sum_{\sigma=\uparrow,\downarrow}t_{i,j}(C_{i,\sigma}^\dagger C_{j,\sigma}+h.c.)+U\sum_{i=1}^{N}n_{i\uparrow}n_{i\downarrow}$$

$$H=\sum_{j=1}^{N}\left\{-t\mathcal{P}\sum_{\sigma=\pm1}(c_{j,\sigma}^\dagger c_{j+1,\sigma}+H.c.)\mathcal{P}+J(\mathbf{S}_j\mathbf{S}_{j+1}-\frac{1}{4}n_n n_{j+1})\right\}$$

4. τ_2, Chiral Potts, vertex model

$$H_{P_9}=-\sum_{l=1}^{N/2}\sum_{k=1}^{8}\Omega_l^k-\sum_{l=1}^{N/2-1}\sum_{k=1}^{8}R_l^k R_{l+1}^{9-k}.$$

5. long range interaction

Gaudin model (1/r)

$$\hat{H}_n=2gS_n^z+\sum_{m\neq n}^{N}\sum_{\alpha}\frac{2S_n^\alpha S_n^\alpha}{\delta_n-\delta_m}$$

Calogero-Sutherland model ($1/r^2$,continue case)

$$H_{CM}=-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+\sum_{j,k=1,j\neq k}^{N}\frac{\lambda^2-\lambda P_{jk}}{(x_i-x_j)^2}$$

Haldane-Shastry model ($1/r^2$, lattice case)

$$H_{HS}=\sum_{j<l}^{N}\frac{1}{\sin^2\frac{\pi}{N}(j-l)}\mathbf{S}_j\cdot\mathbf{S}_l$$

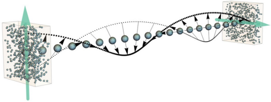
• Now, quantum integrable system (number of degrees of freedom = number of conserved quantities) is an important branch of condensed matter physics, statistical physics, theoretical and mathematical physics.

- ◇ New physics by the well-known integrable models
- ◇ New integrable models
- ◇ New method to obtain the exact solutions

• The quantum integrable systems with $U(1)$ symmetry have been studied extensively.

• There exists another kind of quantum integrable systems which donot possess the $U(1)$ symmetry.

Examples: the anisotropic XYZ Heisenberg model with periodic boundary condition, and some models with anti-periodic boundary condition or the unparallel boundary fields.



In these cases, the $U(1)$ symmetry is broken.

$\diamond$ Lacking the reference state

Due to the $U(1)$ symmetry-broken, there is no obvious reference state. Traditional Bethe ansatz does not work. Although the model has been proved to be integrable, the exact solutions are difficult to be obtained.

Methods: off-diagonal Bethe ansatz, Separation of variables, Onsager algebra, $\cdots$

- Basic idea of ODBA is the polynomial analysis

The Hamiltonian is generated by

$$H = c_2^{-1} \frac{\partial \ \ln t(u)}{\partial u}\Big|_{u=0,\{\theta_j\}=0} - c_0,$$

where $t(u)$ is transfer matrix, the generating functional of all the conserved quantities

$$t(u) = tr_0 T_0(u), \quad periodic$$

$$t(u) = tr_0\{M T_0(u)\}, \quad anti-periodic, \ M : twisted \ matrix$$

$$t(u) = tr_0\{K_0^+(u) T_0(u) K_0^-(u) \hat{T}_0(u)\}, \quad open$$

$$T_0(u) = R_{0,N}(u-\theta_N) R_{0,N-1}(u-\theta_{N-1}) \cdots R_{0,1}(u-\theta_1),$$

$$\hat{T}_0(u) = R_{0,1}(u+\theta_1) R_{0,2}(u+\theta_2) \cdots R_{0,N}(u+\theta_N),$$

$$K_0^{\pm}(u) : boundary \ reflection \ matrix$$

$$[H, t(u)] = 0.$$

From the definition, we know that $t(u)$ is a operator polynomial of u. If the degree of $t(u)$ is M, the value of $t(u)$ can be determined by the $M+1$ constraints of $t(u)$.

In order to obtain these constraints, we use the method of fusion.

The main idea of fusion is as follows. If the R-matrix of the system has a degenerate point $-\alpha$, at which the R-matrix can be written as $R_{12}(-\alpha) = P_{12}^{(d)} S_{12}$, where $P_{12}^{(d)}$ is a d-dimensional projector and S_{12} is a constant matrix. The Yang-Baxter equation at the degenerate point gives

$$R_{12}(-\alpha)R_{13}(u-\alpha)R_{23}(u) = R_{23}(u)R_{13}(u-\alpha)R_{12}(-\alpha). \tag{1}$$

Multiplying Eq.(1) with the projector $P_{12}^{(d)}$ from left and using the property $P_{12}^{(d)} R_{12}(-\alpha) = R_{12}(-\alpha)$, we have

$$R_{12}(-\alpha)R_{13}(u-\alpha)R_{23}(u) = P_{12}^{(d)} R_{23}(u)R_{13}(u-\alpha)R_{12}(-\alpha). \tag{2}$$

Comparing the right hand sides of Eqs.(1) and (2), we obtain

$$P_{12}^{(d)} R_{23}(u) R_{13}(u-\alpha) P_{12}^{(d)} = R_{23}(u) R_{13}(u-\alpha) P_{12}^{(d)} \equiv R_{\langle 12\rangle 3}(u) \equiv R_{\bar{1}3}(u). \tag{3}$$

Explanations:

1) The operator $R_{23}(u)R_{13}(u-\alpha)$ defined in the tensor space $V_1 \otimes V_2 \otimes V_3$ can be projected into the subspace $V_{\langle 12\rangle} \otimes V_3$, where $V_{\langle 12\rangle}$ is the projected d-dimensional subspace of $V_1 \otimes V_2$.

2) The fused R-matrix $R_{\bar{1}2}(u)$ also satisfies the Yang-Baxter equation, which means that the fusion does not broke the integrability.

3) Fusion is used to obtain the high-dimensional representation of certain algebras.

4) If we take the fusion both in auxiliary and in quantum (physical) spaces, from the resulted R-matrix $R_{\langle 12\rangle\langle 34\rangle}(u)$ we can construct some new interesting integrable models.

The $D_2^{(2)}$ spin chain model is one of the most representative integrable system associated with quantum algebra beyond A-series.

It has many applications in the string theory and black hole.

The exact solution of the $D_2^{(2)}$ spin chain is also the foundation to solve the high rank $D_n^{(2)}$ models with nested analytical methods.

The exact solution of the system with periodic boundary condition has been obtained by Reshetikhin by using the analytical Bethe ansatz [Lett. Math. Phys. 14 (1987) 235].

Later, Martins solves the model by using the algebraic Bethe ansatz [Phys. Rev. E 59 (1999) 7220].

The $D_2^{(2)}$ model with diagonal boundary fields was exactly solved via both the coordinate Bethe ansatz [Nucl. Phys. B 583 (2000) 721] and the analytical Bethe ansatz [Nucl. Phys. B 924 (2017) 86; Nucl. Phys. B 938 (2019) 266].

Recently, Robertson, Pawelkiewicz, Jacobsen and Saleur reported that the R-matrix of $D_2^{(2)}$ model is related to the antiferromagnetic Potts model and the staggered XXZ spin chain [JHEP 05 (2020) 144].

Following this idea, Nepomechie and Retore obtained the exact solutions of the $D_2^{(2)}$ model by using the factorization identities and algebraic Bethe ansatz [JHEP 03 (2021) 089].

The R-matrix of $D_2^{(2)}$ model is the 16×16 one

$$
\begin{aligned}
R_{12}(u) = e^{-2(u+2\eta)}\Big\{&(e^{2u}-e^{4\eta})(e^{2u}-e^{4\eta})\sum_{\alpha\neq 2,3}[e_1]^\alpha_\alpha\otimes[e_2]^\alpha_\alpha + e^{2\eta}(e^{2u}-1)(e^{2u}-e^{4\eta}) \\
&\times\sum_{\substack{\alpha\neq\beta,\beta' \\ \alpha \text{ or } \beta\neq 2,3}}[e_1]^\alpha_\alpha\otimes[e_2]^\beta_\beta - \frac{1}{2}(e^{4\eta}-1)(e^{2u}-e^{4\eta})\Big[(e^u+1)\Big(\sum_{\alpha=1,\beta=2,3} + e^u\sum_{\alpha=4,\beta=2,3}\Big) \\
&\times\Big([e_1]^\alpha_\beta\otimes[e_2]^\beta_\alpha + [e_1]^{\beta'}_{\alpha'}\otimes[e_2]^{\alpha'}_{\beta'}\Big) + (e^u-1)\Big(-\sum_{\alpha=1,\beta=2,3} + e^u\sum_{\alpha=4,\beta=2,3}\Big) \\
&\times\Big([e_1]^\alpha_\beta\otimes[e_2]^{\beta'}_\alpha + [e_1]^{\beta'}_{\alpha'}\otimes[e_2]^{\alpha'}_\beta\Big)\Big] + \sum_{\alpha,\beta\neq 2,3}a_{\alpha\beta}(u)[e_1]^\alpha_\beta\otimes[e_2]^{\alpha'}_{\beta'} + \frac{1}{2}\sum_{\alpha\neq 2,3,\beta=2,3} \\
&\times\Big[b^+_\alpha(u)\Big([e_1]^\alpha_\beta\otimes[e_2]^{\alpha'}_{\beta'} + [e_1]^{\beta'}_{\alpha'}\otimes[e_2]^\beta_\alpha\Big) + b^-_\alpha(u)\Big([e_1]^\alpha_\beta\otimes[e_2]^{\alpha'}_\beta + [e_1]^\beta_{\alpha'}\otimes[e_2]^\beta_\alpha\Big)\Big] \\
&+\sum_{\alpha=2,3}\Big(c^+(u)[e_1]^\alpha_\alpha\otimes[e_1]^{\alpha'}_{\alpha'} + c^-(u)[e_1]^\alpha_\alpha\otimes[e_2]^\alpha_\alpha + d^+(u)[e_1]^\alpha_{\alpha'}\otimes[e_2]^{\alpha'}_\alpha \\
&+d^-(u)[e_1]^\alpha_{\alpha'}\otimes[e_2]^\alpha_{\alpha'}\Big)\Big\},
\end{aligned}
$$

which satisfies the Yang-Baxter equation

$$
R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).
$$

The reflection matrix at one side of the $D_2^{(2)}$ chain is quantified as

$$K_k^-(u) = \begin{pmatrix} k_{11}(u) & k_{12}(u) & k_{13}(u) & k_{14}(u) \\ k_{21}(u) & k_{22}(u) & k_{23}(u) & k_{24}(u) \\ k_{31}(u) & k_{32}(u) & k_{33}(u) & k_{34}(u) \\ k_{41}(u) & k_{42}(u) & k_{43}(u) & k_{44}(u) \end{pmatrix},$$

which has the non-diagonal matrix elements. Thus the spin of a quasi-particle could be changed after the boundary reflection, and the $U(1)$ symmetry of the system is broken. The reflection matrix satisfies the reflection equation

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v).$$

The reflection matrix at the other side is

$$K_k^+(u) = M_k K_k^-(-u+2\eta)|_{\{s,s_1,s_2\}\to\{s',s_1',s_2'\}},$$

which satisfies the dual reflection equation

$$\begin{aligned} &R_{12}(-u+v)K_1^+(u)M_1^{-1}R_{21}(-u-v+4\eta)M_1K_2^+(v) \\ &\quad = K_2^+(v)M_1R_{12}(-u-v+4\eta)M_1^{-1}K_1^+(u)R_{21}(-u+v). \end{aligned} \tag{4}$$

We should note that the reflection matrix $K^-(u)$ and its dual one $K^+(u)$ can not be diagonalized simultaneously.

The transfer matrix of the $D_2^{(2)}$ model with open boundary condition is defined as

$$t(u) = tr_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\}.$$

From the Yang-Baxter equation, reflection equation and the dual one, one can prove that the transfer matrices with different spectral parameters commutate with each other

$$[t(u), t(v)] = 0.$$

Thus all the expansion coefficients of $t(u)$ with respect to u are commutative. All the coefficients and their combinations are the conserved quantities.

The Hamiltonian of the $D_2^{(2)}$ model is generated by the transfer matrix as

$$H = \frac{\partial \ln t(u)}{2\partial u}|_{u=0,\{\theta_j\}=0} - \frac{tr_0 K_0^+(0)'}{2tr_0 K_0^+(0)} = \sum_{k=1}^{N-1} H_{kk+1} + \frac{K_N^-(0)'}{2K_N^-(0)} + \frac{tr_0\{K_0^+(0)H_{10}\}}{tr_0 K_0^+(0)},$$

where $H_{k\,k+1}$ quantifies the nearest neighbor interaction in the bulk, and the rest terms characterize the boundary reflections.

Factorization

The spin of $D_2^{(2)}$ chain at j-th site has four components.

The four-dimensional space can be regarded as the tensor of two two-dimensional spaces. Then the R-matrix of $D_2^{(2)}$ model can be factorized as the product of R-matrices of the anisotropic XXZ spin chain with suitable global transformation

$$R_{12}(u) = 2^4[S \otimes S]\tilde{R}_{1'4'}(u+i\pi)\tilde{R}_{1'3'}(u)\tilde{R}_{2'4'}(u)\tilde{R}_{2'3'}(u-i\pi)[S \otimes S]^{-1},$$

$$S = \begin{pmatrix} 1 & & & \\ & \frac{\cosh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & -\frac{\sinh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & \\ & -\frac{\sinh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & -\frac{\cosh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & \\ & & & 1 \end{pmatrix},$$

$$\tilde{R}_{1'2'}(u) = \begin{pmatrix} \sinh(-\frac{u}{2}+\eta) & & & \\ & \sinh\frac{u}{2} & e^{-\frac{u}{2}}\sinh\eta & \\ & e^{\frac{u}{2}}\sinh\eta & \sinh\frac{u}{2} & \\ & & & \sinh(-\frac{u}{2}+\eta) \end{pmatrix}. \quad (5)$$

The reflection matrices can also be factorized as

$$K_1^+(u) = [\rho_s(i\pi)]^{-\frac{1}{2}} S \tilde{R}_{2'1'}(i\pi)\tilde{K}_{2'}^+(u)\tilde{M}_{2'}^{-1}\tilde{R}_{1'2'}(-2u+4\eta-i\pi)\tilde{M}_{2'}\tilde{K}_{1'}^+(u)S^{-1},$$

$$K_1^-(u) = [\rho_s(i\pi)]^{-\frac{1}{2}} S \tilde{K}_{1'}^-(u+i\pi)\tilde{R}_{2'1'}(2u+i\pi)\tilde{K}_{2'}^-(u)\tilde{R}_{1'2'}(-i\pi)S^{-1},$$

where $\tilde{K}_{k'}^{\pm}(u)$ are the 2×2 generic non-diagonal reflection matrices of the XXZ spin chain

$$\tilde{K}_{k'}^+(u) = \tilde{M}_{k'}\, \tilde{K}_{k'}^-(-u+2\eta)|_{\{s,s_1,s_2\}\to\{s',s_1',s_2'\}},$$

$$\tilde{K}_{k'}^-(u) = \begin{pmatrix} -e^{-\frac{u}{2}}\sinh(\frac{u}{2}-s) & s_1\sinh u \\ s_2\sinh u & e^{\frac{u}{2}}\sinh(\frac{u}{2}+s) \end{pmatrix}. \quad (6)$$

Based on the R-matrix (5) and reflection matrices (6), we construct the transfer matrix $\tilde{t}(u)$ of the inhomogeneous XXZ spin chain as

$$\tilde{t}(u) = tr_{0'}\{\tilde{K}^{+}_{0'}(u)\tilde{T}_{0'}(u)\tilde{K}^{-}_{0'}(u)\hat{\tilde{T}}_{0'}(u)\},$$

where $\tilde{T}_{0'}(u)$ and $\hat{\tilde{T}}_{0'}(u)$ are the monodromy matrices

$$\begin{aligned}\tilde{T}_{0'}(u) &= \tilde{R}_{0'1'}(u-\theta_1)\tilde{R}_{0'2'}(u-\theta_2)\cdots\tilde{R}_{0'(2N)'}(u-\theta_{2N}),\\ \hat{\tilde{T}}_{0'}(u) &= \tilde{R}_{0'(2N)'}(u+\theta_{2N})\tilde{R}_{0'(2N-1)'}(u+\theta_{2N-1})\cdots\tilde{R}_{0'1'}(u+\theta_1),\end{aligned} \tag{7}$$

and $\{\theta_j|j=1,\cdots,2N\}$ are the inhomogeneous parameters.

We should note that the quantum space of transfer matrix $\tilde{t}(u)$ for the XXZ spin chain and that of $t(u)$ for the $D_2^{(2)}$ model should be the same. Then the number of sites in Eq.(7) is extended to $2N$ to ensure the dimension of Hilbert space is 4^N.

It is easy to prove that the transfer matrices $\tilde{t}(u)$ with different spectral parameters commutate with each other

$$[\tilde{t}(u), \tilde{t}(v)] = 0.$$

From above factorization, we conclude that if the inhomogeneous parameters are staggered, i.e., $\theta_j = 0$ for the odd j and $\theta_j = i\pi$ for the even j, the transfer matrix of the $D_2^{(2)}$ spin chain can be factorized as the product of transfer matrices of two staggered XXZ spin chains with fixed spectral difference

$$t(u) = 2^{8N} \rho_s(2u + i\pi - 2\eta)\, \tilde{t}_s(u + i\pi)\, \tilde{t}_s(u), \tag{8}$$

where $\tilde{t}_s(u) = \tilde{t}(u)|_{\{\theta_j\}=\{0,\, i\pi\}}$.

Because $\tilde{t}_s(u + i\pi)$ and $\tilde{t}_s(u)$ have common eigenstates. Acting Eq.(8) on a common eigenstate, we obtain

$$\Lambda(u) = 2^{8N} \rho_s(2u + i\pi - 2\eta)\, \tilde{\Lambda}_s(u + i\pi)\, \tilde{\Lambda}_s(u), \tag{9}$$

where $\Lambda(u)$, $\tilde{\Lambda}_s(u + i\pi)$ and $\tilde{\Lambda}_s(u)$ are the eigenvalues of the transfer matrices $t(u)$, $\tilde{t}_s(u + i\pi)$ and $\tilde{t}_s(u)$, respectively.

Therefore, the eigenvalue of $D_2^{(2)}$ spin chain can be obtained by using the exact solution of the anisotropic XXZ spin chain.

Off-diagonal Bethe ansatz solution

Now, we derive the eigenvalues of transfer matrix $\tilde{t}_s(u)$ of the staggered XXZ spin chain, which can be achieved by diagonalizing the transfer matrix $\tilde{t}(u)$ of the inhomogeneous XXZ spin chain with non-diagonal boundary reflection.

The corresponding inhomogeneous XXZ spin chain still does not have the obvious reference state, thus we use the off-diagonal Bethe ansatz.

From the definition, we know that the transfer matrix $\tilde{t}(u)$ is a trigonometric operator polynomial of u with the degree $4N+4$,

$$\tilde{t}(u) = \hat{O}_{2N+2} \sinh^{4N+4} u + \hat{O}_{4N+3} \sinh^{4N+3} u + \cdots + \hat{O}_0$$

Acting it on an eigenstate, the corresponding eigenvalue is

$$\tilde{\Lambda}(u) = O_{2N+2} \sinh^{4N+4} u + O_{4N+3} \sinh^{4N+3} u + \cdots + O_0$$

Our task is to determine the coefficients, which can be determined by the values of $\tilde{\Lambda}(u)$ at $4N+5$ points.

Now, we seek these $4N+5$ constraints. At the points of $\pm\eta$, the R-matrix becomes

$$R_{12}(\pm\eta) \sim P_{12}^{(\pm)},$$

where $P_{1,2}^{(-)}$ is the one-dimensional antisymmetric projector and $P_{1,2}^{(+)}$ is the three-dimensional symmetric projector.

By using fusion technique, we obtain following operators product identities

$$\begin{aligned}
\tilde{t}(\pm\theta_j)\,\tilde{t}(\pm\theta_j+2\eta) &= \frac{4\sinh(\pm\theta_j-2\eta)\sinh(\pm\theta_j+2\eta)}{\alpha\alpha'\sinh(\pm\theta_j-\eta)\sinh(\pm\theta_j+\eta)}\cosh\frac{\pm\theta_j-\alpha_1}{2} \\
&\times\cosh\frac{\pm\theta_j+\alpha_1}{2}\cosh\frac{\pm\theta_j-\alpha_2}{2}\cosh\frac{\pm\theta_j+\alpha_2}{2}\cosh\frac{\pm\theta_j-\alpha_1'}{2}\cosh\frac{\pm\theta_j+\alpha_1'}{2} \\
&\times\cosh\frac{\pm\theta_j-\alpha_2'}{2}\cosh\frac{\pm\theta_j+\alpha_2'}{2}\prod_{i=1}^{M}\sinh\frac{\pm\theta_j-\theta_i-2\eta}{2} \\
&\times\sinh\frac{\pm\theta_j-\theta_i+2\eta}{2}\sinh\frac{\pm\theta_j+\theta_i-2\eta}{2}\sinh\frac{\pm\theta_j+\theta_i+2\eta}{2},\quad j=1,\cdots,2N. \qquad (10)
\end{aligned}$$

We see that the product of two transfer matrices with fixed spectral parameters is the quantum determinant at the point of $u=\theta_j$.

Please note that the fusion identities (10) hold only at the discrete inhomogeneous points.

Besides, from the direct calculation, we also obtain the values of $\tilde{t}(u)$ at the points of $u = 0, 2\eta, i\pi$ as

$$\tilde{t}(0) = \tilde{t}(2\eta) = 2\cosh\eta \sinh s \sinh s' \prod_{j=1}^{2N} \rho_s(\theta_j),$$

$$\tilde{t}(i\pi) = 2\cosh\eta \cosh s \cosh s' \prod_{j=1}^{2N} \rho_s(\theta_j + i\pi). \quad (11)$$

The asymptotic behavior of $\tilde{t}(u)$ when the spectral parameter tends to infinity reads

$$\tilde{t}(u)|_{u\to\pm\infty} = -2^{-4N-2} e^{\pm[(2N+2)(u-\eta)]}(e^{-\eta} s_1 s_2' + e^{\eta} s_2 s_1'). \quad (12)$$

Thus the above $4N$ fusion identities (10) and 5 additional conditions (11)-(12) give us sufficient information to determine the eigenvalue $\tilde{\Lambda}(u)$ of $\tilde{t}(u)$.

The eigenvalue $\tilde{\Lambda}(u)$ can be expressed as the inhomogeneous $T-Q$ relation

$$\begin{aligned}\tilde{\Lambda}(u) &= \frac{2\sinh(u-2\eta)}{\sinh(u-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{u+\alpha_1}{2}\cosh\frac{u+\alpha_2}{2}\cosh\frac{u+\alpha_1'}{2}\cosh\frac{u+\alpha_2'}{2}a(u)\frac{Q(u+2\eta)}{Q(u)} \\ &+ \frac{2\sinh u}{\sinh(u-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{u-2\eta-\alpha_1}{2}\cosh\frac{u-2\eta-\alpha_2}{2}\cosh\frac{u-2\eta-\alpha_1'}{2} \\ &\times\cosh\frac{u-2\eta-\alpha_2'}{2}d(u)\frac{Q(u-2\eta)}{Q(u)} + x\sinh u\sinh(u-2\eta)\frac{a(u)d(u)}{Q(u)},\end{aligned} \tag{13}$$

where the functions $Q(u)$, $a(u)$, $d(u)$ and parameter x are

$$Q(u) = \prod_{l=1}^{2N}\sinh\frac{1}{2}(u-\mu_l)\sinh\frac{1}{2}(u+\mu_l-2\eta),$$

$$a(u) = \prod_{j=1}^{2N}\sinh\frac{1}{2}(u-\theta_j-2\eta)\sinh\frac{1}{2}(u+\theta_j-2\eta) = d(u-2\eta),$$

$$x = -2\sqrt{s_1s_2s_1's_2'}\cosh[(2N+1)\eta+\frac{\alpha_1+\alpha_2+\alpha_1'+\alpha_2'}{2}] - (e^{-\eta}s_1s_2'+e^{\eta}s_2s_1').$$

Because $\tilde{\Lambda}(u)$ is a polynomial, the singularities of right hand side of Eq.(13) should be cancelled with each other, which gives that the Bethe roots $\{\mu_l\}$ should satisfy the Bethe ansatz equations

$$\frac{2\sinh(\mu_l-2\eta)}{\sinh(\mu_l-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{\mu_l+\alpha_1}{2}\cosh\frac{\mu_l+\alpha_2}{2}\cosh\frac{\mu_l+\alpha_1'}{2}\cosh\frac{\mu_l+\alpha_2'}{2}\frac{Q(\mu_l+2\eta)}{d(\mu_l)}$$
$$+\frac{2\sinh\mu_l}{\sinh(\mu_l-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{\mu_l-2\eta-\alpha_1}{2}\cosh\frac{\mu_l-2\eta-\alpha_2}{2}\cosh\frac{\mu_l-2\eta-\alpha_1'}{2}$$
$$\times\cosh\frac{\mu_l-2\eta-\alpha_2'}{2}\frac{Q(\mu_l-2\eta)}{a(\mu_l)}=-x\sinh\mu_l\sinh(\mu_l-2\eta),\qquad l=1,\cdots,2N. \tag{14}$$

Some remarks:

1. By solving the Bethe ansatz equations, we obtain the values of Bethe roots $\{\mu_l\}$. Substituting these values into the inhomogeneous $T-Q$ relation (13), we obtain the eigenvalue $\tilde{\Lambda}(u)$.

2. The different sets of solutions of Bethe roots would give different eigenvalues.

3. Based on the numerical calculation and analytical analysis with the help of Bézout theorem, the $T-Q$ relation can generate all the eigenvalues of $\tilde{t}(u)$.

4. The eigenvalue $\tilde{\Lambda}(u)$ has the well-defined quasi-inhomogeneous limit $\{\theta_j\}=\{0,\, i\pi\}$.

5. Conclusion: The eigenvalue of the transfer matrix $t(u)$ of the $D_2^{(2)}$ spin chain is

$$\Lambda(u)=2^{8N}\rho_s(2u+i\pi-2\eta)\,\tilde{\Lambda}_s(u+i\pi)\,\tilde{\Lambda}_s(u),$$

where

$$\tilde{\Lambda}_s(u)=\tilde{\Lambda}(u)|_{\{\theta_j\}=\{0,\,i\pi\}},\quad \tilde{\Lambda}_s(u+i\pi)=\tilde{\Lambda}(u+i\pi)|_{\{\theta_j\}=\{0,\,i\pi\}}.$$

The G_2 symmetry is the smallest possible exceptional Lie algebra besides the automorphism group of the algebra of octonions.

Its relation to Clifford algebras and spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups has been studied.

The holonomy group G_2 is also associated with the compact Riemannian manifolds with special geometric structure, such as Spin-7 manifolds or nearly Kähler manifolds.

These manifolds play important roles as ingredients for compactifications in string theory, topology and M-theory. In addition, the G_2 model has applications in quantum logic, special relativity and supersymmetry.

$7^2 \times 7^2$ R-matrix

$$
\begin{aligned}
R_{12}(u) = {} & a(u) \sum_{i=1, i\neq 4}^{7} (E_i^i \otimes E_i^i) + \bar{a}(u)(E_4^4 \otimes E_4^4) + c(u) \sum_{i=1, i\neq 4}^{7} (E_i^i \otimes E_4^4 + E_4^4 \otimes E_i^i) \\
& + e(u) \sum_{i=1}^{3} (E_i^i \otimes E_{\bar{i}}^{\bar{i}} + E_{\bar{i}}^{\bar{i}} \otimes E_i^i) + b(u) \sum_{i=2}^{3} (E_1^1 \otimes E_i^i + E_i^i \otimes E_1^1 + E_7^7 \otimes E_{\bar{i}}^{\bar{i}} + E_{\bar{i}}^{\bar{i}} \otimes E_7^7 \\
& + E_i^i \otimes E_{i+3}^{i+3} + E_{i+3}^{i+3} \otimes E_i^i) + d(u) \sum_{i=5}^{6} (E_1^1 \otimes E_i^i + E_i^i \otimes E_1^1 + E_7^7 \otimes E_{\bar{i}}^{\bar{i}} + E_{\bar{i}}^{\bar{i}} \otimes E_7^7 \\
& + E_i^i \otimes E_{\bar{i}+3}^{\bar{i}+3} + E_{\bar{i}}^{\bar{i}} \otimes E_{i-3}^{i-3}) + g_1(u) \sum_{i=2}^{3} (E_i^1 \otimes E_1^i + E_1^i \otimes E_i^1 + E_{\bar{i}}^7 \otimes E_7^{\bar{i}} + E_7^{\bar{i}} \otimes E_{\bar{i}}^7 \\
& + E_{i+3}^i \otimes E_i^{i+3} + E_i^{i+3} \otimes E_{i+3}^i) + g_6(u) \sum_{i=5}^{6} (E_i^1 \otimes E_1^i + E_1^i \otimes E_i^1 + E_{\bar{i}}^7 \otimes E_7^{\bar{i}} + E_7^{\bar{i}} \otimes E_{\bar{i}}^7 \\
& + E_{\bar{i}+3}^i \otimes E_i^{\bar{i}+3} + E_{i-3}^{\bar{i}} \otimes E_{\bar{i}}^{i-3}) + g_4(u) \sum_{i=1, i\neq 4}^{7} (E_i^4 \otimes E_4^i + E_4^i \otimes E_i^4) \\
& + g_8(u) \sum_{i=1}^{3} (E_{\bar{i}}^i \otimes E_i^{\bar{i}} + E_i^{\bar{i}} \otimes E_{\bar{i}}^i) + g_5(u) \sum_{i=1}^{3} \xi_i (E_4^i \otimes E_4^{\bar{i}} + E_4^{\bar{i}} \otimes E_4^i + E_i^4 \otimes E_{\bar{i}}^4 + E_{\bar{i}}^4 \otimes E_i^4) \\
& + g_3(u) \sum_{i=2}^{3} [E_{i+3}^i \otimes E_{\bar{i}-3}^{\bar{i}} + E_{\bar{i}-3}^{\bar{i}} \otimes E_{i+3}^i - \xi_i (E_i^1 \otimes E_{\bar{i}}^7 + E_{\bar{i}}^7 \otimes E_i^1 + E_1^i \otimes E_7^{\bar{i}} + E_7^{\bar{i}} \otimes E_1^i)] \\
& + g_7(u) \sum_{i=2}^{3} [E_{\bar{i}-3}^i \otimes E_{i+3}^{\bar{i}} + E_{i+3}^{\bar{i}} \otimes E_{\bar{i}-3}^i - \xi_i (E_{\bar{i}}^1 \otimes E_i^7 + E_i^7 \otimes E_{\bar{i}}^1 + E_1^{\bar{i}} \otimes E_7^i + E_7^i \otimes E_1^{\bar{i}})] \\
& + g_2(u) \sum_{i\langle j, l \langle k, i\neq \bar{l}, j\neq \bar{k}, i+k=j+l=5,6,7,9,10,11} (E_j^i \otimes E_l^k + E_i^j \otimes E_k^l - E_l^i \otimes E_j^k - E_k^j \otimes E_i^l \\
& + E_l^k \otimes E_j^i + E_k^l \otimes E_j^i - E_j^k \otimes E_l^i - E_i^l \otimes E_k^j).
\end{aligned}
$$

The R-matrix satisfies:

$$\begin{aligned}
\text{YBE}: &\qquad R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v),\\
\text{regularity}: &\qquad R_{12}(0) = \rho_{12}(0)^{\frac{1}{2}}\mathcal{P}_{12},\\
\text{unitary}: &\qquad R_{12}(u)R_{21}(-u) = a_1(u)a_1(-u) \equiv \rho_{12}(u),\\
\text{crossing symmetry}: &\qquad R_{12}(u) = -V_1 R_{21}^{t_1}(-u-6)V_1^{-1} = -V_2^{t_2} R_{21}^{t_2}(-u-6)[V_2^{t_2}]^{-1},\\
\text{crossing unitary}: &\qquad R_{12}^{t_1}(u)R_{21}^{t_1}(-u-12) = -\rho_{12}(u+6) \equiv \tilde{\rho}_{12}(u).
\end{aligned}$$

The R-matrix can also be written as

$$
\begin{aligned}
R_{12}(u) \quad = \quad & (u-1)(u+4)(u-6)P_{12}^{(1)} + (u+1)(u-4)(u+6)P_{12}^{(7)} \\
& +(u-1)(u+4)(u+6)P_{12}^{(14)} + (u+1)(u+4)(u+6)P_{12}^{(27)},
\end{aligned}
$$

where $P_{12}^{(d)}$ are d-dimensional projectors, where $d = 1, 7, 14, 27$. Thus the R-matrix can degenerate into the projectors at ceratin points of the spectral parameter.

For example, if $u = -6$, we have

$$
R_{12}(-6) = P_{12}^{(1)} \times S_1,
$$

where S_1 is an irrelevant constant matrix, $P_{12}^{(1)}$ is the one-dimensional projector

$$
P_{12}^{(1)} = |\psi_0\rangle\langle\psi_0|,
$$

the vector $|\psi_0\rangle$ is

$$
|\psi_0\rangle = \frac{1}{\sqrt{7}}(|17\rangle - |26\rangle + |35\rangle - |44\rangle + |53\rangle - |62\rangle + |71\rangle),
$$

and $\{|j\rangle, j = 1, \cdots, 7\}$ are the orthogonal bases of the spaces $\mathbf{V}_1$ or $\mathbf{V}_2$.

Reflection matrix

$$K^{-}(u) = 1 + Mu, \quad M = \begin{pmatrix} c_{11} & 0 & 0 & 0 & c_1 & c_2 & 0 \\ 0 & c_{22} & c_3 & 0 & 0 & 0 & -c_2 \\ 0 & c_3 & c_{33} & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & c_{33} & -c_3 & 0 \\ c_2 & 0 & 0 & 0 & -c_3 & c_{22} & 0 \\ 0 & -c_2 & c_1 & 0 & 0 & 0 & c_{11} \end{pmatrix},$$

where

$$c_{11} = \frac{c_1 c_3}{c_2} + \frac{c_2 c_3}{c_1} - 2, \quad c_{22} = 2 - \frac{c_2 c_3}{c_1}, \quad c_{33} = 2 - \frac{c_1 c_3}{c_2}, \quad \frac{c_1 c_3}{c_2} + \frac{c_2 c_3}{c_1} + \frac{c_1 c_2}{c_3} = 4.$$

Thus there are two free parameters.

The boundary reflections at the other end of the chain is characterized by the dual reflection matrix

$$K^{+}(u) = K^{-}(-u-6)|_{\{c_1,c_2,c_3\} \to \{\tilde{c}_1,\tilde{c}_2,\tilde{c}_3\}}.$$

The model Hamiltonian can be obtained by taking the derivative of the logarithm of the transfer matrix $t(u)$ as

$$
\begin{aligned}
H &= \frac{\partial \ln t(u)}{\partial u}\Big|_{u=0,\{\theta_j\}=0} \\
&= \sum_{k=1}^{N-1} H_{kk+1} + \frac{1}{2}K_N^{-\prime}(0) + \frac{tr_0\{K_0^+(0)H_{10}\}}{tr_0 K_0^+(0)} + \mathrm{constant}.
\end{aligned}
$$

Explanations:

1. The Hamiltonian can be expressed by the spin-3 operators.

2. The reflection matrices $K^{\pm}(u)$ have the non-diagonal elements, which means that the number of particles with fixed spin-component are not conserved, although the total number is conserved. The $K^{\pm}(u)$ can not be diagonalized simultaneously.

3. The non-diagonal boundary reflections can be achieved by applying the nonparallel boundary magnetic fields to the chain.

4. The twisted magnetic fields will induce the helix spin states. Then it is hard to construct the reference state.

Now, we solve the $t(u)$ and H by using the polynomial analysis and fusion.

From the definition of transfer matrix, we know that $t(u)$ is a operator polynomial of u with the degree $6N + 2$.

Besides, by using the crossing relation of the fundamental R-matrix, we can show that the transfer matrix $t(u)$ satisfies the crossing relation

$$t(u) = t(-u - 6).$$

Thus the number of unknowns in the polynomial is reduced to $3N + 2$, which means that the value of $t(u)$ can be determined by $3N + 2$ constraints.

We use the fusion method to obtain these constraints.

Step 1

When $u=-6$, we get an one-dimensional projector $P_{12}^{(1)}$. By using the fusion

$$
\begin{aligned}
&P_{21}^{(1)}R_{13}(u)R_{23}(u-6)P_{21}^{(1)} = a(u)e(u-6)\times \mathrm{id},\\
&P_{12}^{(1)}R_{31}(u)R_{32}(u-6)P_{12}^{(1)} = a(u)e(u-6)\times \mathrm{id},\\
&P_{21}^{(1)}K_1^-(u)R_{21}(2u-6)K_2^-(u-6)P_{12}^{(1)} =\\
&\qquad 4(u-1)(u-6)(2u-5)(2u-1)(2u+1)\times id,\\
&P_{12}^{(1)}K_2^+(u-6)R_{12}(-2u-2\kappa+6)K_1^+(u)P_{21}^{(1)} =\\
&\qquad -4(u+1)(u+6)(2u+5)(2u-1)(2u+1)\times id,
\end{aligned}
$$

We obtain that the product of the transfer matrices satisfies the relation

$$
\begin{aligned}
t(\theta_j)\,t(\theta_j-6) &= 4^2\frac{(\theta_j-1)(\theta_j-6)(\theta_j+1)(\theta_j+6)}{(\theta_j-2)(\theta_j-3)(\theta_j+2)(\theta_j+3)}\\
&\times(\theta_j-\frac{1}{2})(\theta_j-\frac{5}{2})(\theta_j+\frac{1}{2})(\theta_j+\frac{5}{2})\prod_{i=1}^{N}\rho_{12}(\theta_j-\theta_i)\rho_{12}(\theta_j+\theta_i)\times \mathrm{id},\ j=1,\cdots,N.
\end{aligned}
$$

Step 2

When $u=-1$, we get a 15-dimensional projector $P_{12}^{(15)}$. The fusion with 15-dimensional projector gives

$$P_{12}^{(15)}R_{23}(u)R_{13}(u-1)P_{12}^{(15)}=(u-1)(u+1)(u+4)(u+6)R_{\bar{1}3}(u-\frac{1}{2}),$$

$$P_{21}^{(15)}R_{32}(u)R_{31}(u-1)P_{21}^{(15)}=(u-1)(u+1)(u+4)(u+6)R_{3\bar{1}}(u-\frac{1}{2}),$$

where the subscript $\bar{1}$ means the 15-dimensional fused space and $R_{\bar{1}3}(u)$ is the $(15\times 7)\times(15\times 7)$-dimensional fused R-matrix. The matrix elements of $R_{\bar{1}3}(u)$ are the polynomials of u, and the maximum degree is 2.

The fused R-matrix has the properties

$$R_{\bar{1}2}(u)R_{2\bar{1}}(-u)=\rho_{\bar{1}2}(u)=(u+\frac{7}{2})(u+\frac{11}{2})(u-\frac{7}{2})(u-\frac{11}{2}),$$

$$R_{\bar{1}2}(u)^{t_{\bar{1}}}R_{2\bar{1}}(-u-12)^{t_{\bar{1}}}=\tilde{\rho}_{\bar{1}2}(u)=\rho_{\bar{1}2}(u+6),$$

$$R_{\bar{1}2}(u)=V_{\bar{1}}R_{2\bar{1}}^{t_{\bar{1}}}(-u-6)[V_{\bar{1}}]^{-1},\quad V_{\bar{1}}=P_{12}^{(15)}V_2V_1P_{12}^{(15)},$$

$$R_{2\bar{1}}(u)=V_{\bar{1}}^{t_{\bar{1}}}R_{\bar{1}2}^{t_{\bar{1}}}(-u-6)[V_{\bar{1}}^{t_{\bar{1}}}]^{-1},$$

and satisfies the Yang-Baxter equation

$$R_{\bar{1}2}(u-v)R_{\bar{1}3}(u)R_{23}(v)=R_{23}(v)R_{\bar{1}3}(u)R_{\bar{1}2}(u-v).$$

The 15-dimensional fusion of reflection matrices gives

$$
\begin{aligned}
&P_{12}^{(15)}K_2^-(u)R_{12}(2u-1)K_1^-(u-1)P_{21}^{(15)}\\
&\qquad=8(u-\frac{1}{2})(u+\frac{1}{2})(u+\frac{5}{2})(u-1)K_{\bar{1}}^-(u-\frac{1}{2}),\\
&P_{21}^{(15)}K_1^+(u-1)R_{21}(-2u-2\kappa+1)K_2^+(u)P_{12}^{(15)}\\
&\qquad=8(u+\frac{5}{2})(u+\frac{9}{2})(u+\frac{11}{2})(u+6)K_{\bar{1}}^+(u-\frac{1}{2}),
\end{aligned}
$$

where $K_{\bar{1}}^{\mp}(u)$ are the 15×15-dimensional fused reflection matrices. The matrix elements of $K_{\bar{1}}^{\mp}(u)$ are the polynomials of u and the maximum degree is 1. Besides, $K_{\bar{1}}^{\mp}(u)$ satisfy the reflection equations

$$
\begin{aligned}
&R_{\bar{1}2}(u-v)K_{\bar{1}}^-(u)R_{2\bar{1}}(u+v)K_2^-(v)=K_2^-(v)R_{\bar{1}2}(u+v)K_{\bar{1}}^-(u)R_{2\bar{1}}(u-v),\\
&R_{\bar{1}2}(-u+v)K_{\bar{1}}^+(u)R_{2\bar{1}}(-u-v-12)K_2^+(v)\\
&\qquad=K_2^+(v)R_{\bar{1}2}(-u-v-12)K_{\bar{1}}^+(u)R_{2\bar{1}}(-u+v).
\end{aligned}
$$

By using the fused R-matrices and fused reflection matrices, we construct the fused transfer matrix $\bar{t}(u)$ as

$$\bar{t}(u) = tr_{\bar{0}}\{K^+_{\bar{0}}(u)T_{\bar{0}}(u)K^-_{\bar{0}}(u)\hat{T}_{\bar{0}}(u)\},$$

where

$$T_{\bar{0}}(u) = R_{\bar{0}1}(u-\theta_1)R_{\bar{0}2}(u-\theta_2)\cdots R_{\bar{0}N}(u-\theta_N),$$

$$\hat{T}_{\bar{0}}(u) = R_{N\bar{0}}(u+\theta_N)\cdots R_{2\bar{0}}(u+\theta_2)R_{1\bar{0}}(u+\theta_1).$$

Then using the fusion relations, we have

$$t(\pm\theta_j)t(\pm\theta_j-1) = -\frac{(\pm\theta_j-1)(\pm\theta_j+6)(\pm\theta_j+\frac{5}{2})^2}{(\pm\theta_j+2)(\pm\theta_j+3)}\prod_{i=1}^{N}[(\pm\theta_j-\theta_i-1)$$

$$\times(\pm\theta_j+\theta_i-1)a(\pm\theta_j-\theta_i)a(\pm\theta_j+\theta_i)]\bar{t}(\pm\theta_j-\frac{1}{2}),\ j=1,\cdots,N.$$

We can also prove that the fused transfer matrix $\bar{t}(u)$ satisfies the crossing symmetry

$$\bar{t}(-u-6) = \bar{t}(u).$$

Step 3

The fused R-matrix also has the degenerated points. For example,

$$R_{\bar{1}2}(-\frac{7}{2}) = P_{\bar{1}2}^{(34)} \times S_{34},$$

where S_{34} is an irrelevant constant matrix and $P_{\bar{1}2}^{(34)}$ is a 34-dimensional projector, which allows us to take the fusion again.

Repeated the similar processes, we obtain the next fused R-matrices as

$$P_{\bar{1}2}^{(34)} R_{23}(u) R_{\bar{1}3}(u - \frac{7}{2}) P_{\bar{1}2}^{(34)} = (u+6) R_{\tilde{1}3}(u - \frac{5}{2}),$$
$$P_{2\bar{1}}^{(34)} R_{32}(u) R_{3\bar{1}}(u - \frac{7}{2}) P_{2\bar{1}}^{(34)} = (u+6) Q_{\tilde{1}} R_{3\tilde{1}}(u - \frac{5}{2}) Q_{\tilde{1}}^{-1},$$

where the subscript $\tilde{1}$ denotes the 34-dimensional fused space $\mathbf{V}_{\langle 2\bar{1}\rangle}$ and $Q_{\tilde{1}}$ is a 34×34 matrix defined in the fused space.

The next fused R matrix is a $34^2 \times 7^2$ matrix thus the detailed form is omitted here.

The matrix elements of $R_{\tilde{1}2}(u)$ are the polynomials of u and the maximum degree of these polynomials is 4.

The related next fused reflection matrices are obtained by taking the fusion of reflection matrices with the 34-dimensional projectors as

$$P^{(34)}_{\bar{1}2}K^-_2(u)R_{\bar{1}2}(2u-\frac{7}{2})K^-_{\bar{1}}(u-\frac{7}{2})P^{(34)}_{2\bar{1}} = 4(u+1)K^-_{\tilde{1}}(u-\frac{5}{2})Q^{-1}_{\tilde{1}},$$
$$P^{(34)}_{2\bar{1}}K^+_{\bar{1}}(u-\frac{7}{2})R_{2\bar{1}}(-2u-2\kappa+\frac{7}{2})K^+_2(u)P^{(34)}_{\bar{1}2} = -4(u+6)Q_{\tilde{1}}K^+_{\tilde{1}}(u-\frac{5}{2}),$$

where all the matrix elements of $K^{\mp}_{\tilde{1}}(u)$ are the polynomials of u, and among of them the maximum degree of these polynomials is 3.

Now, we are ready to define the next fused transfer matrix $\tilde{t}(u)$

$$\tilde{t}(u) = tr_{\tilde{0}}\{K^{+}_{\tilde{0}}(u)T_{\tilde{0}}(u)K^{-}_{\tilde{0}}(u)\hat{T}_{\tilde{0}}(u)\}.$$

Computing the quantity $t(u)\bar{t}(u+\delta)$ with the similar steps as before and substituting $u = \pm\theta_j$ and $\delta = -\frac{7}{2}$ into the result, we have

$$t(\pm\theta_j)\bar{t}(\pm\theta_j - \frac{7}{2}) = -\frac{(\pm\theta_j + 1)(\pm\theta_j + 6)}{(\pm\theta_j - \frac{1}{2})(\pm\theta_j - \frac{3}{2})(\pm\theta_j + 3)(\pm\theta_j + 4)}$$

$$\times \prod_{i=1}^{N}(\pm\theta_j - \theta_i + 6)(\pm\theta_j + \theta_i + 6)\tilde{t}(\pm\theta_j - \frac{5}{2}),\ j = 1, \cdots, N.$$

The next fused transfer matrix has the crossing symmetry

$$\tilde{t}(u) = \tilde{t}(-u - 6).$$

Step 4

The fused $R_{\tilde{1}2}(u)$ matrix has two degenerate points.

At the point of $u=-\frac{9}{2}$,

$$R_{\tilde{1}2}(-\frac{9}{2})=P^{(49)}_{\tilde{1}2}\times S_{49},$$

then we get a 49-dimensional projector $P^{(49)}_{\tilde{1}2}$. Direct calculating gives

$$\begin{aligned}
P^{(49)}_{\tilde{1}2}R_{23}(u)R_{\tilde{1}3}(u-\frac{9}{2})P^{(49)}_{\tilde{1}2} &= (u+6)S_{12}R_{13}(u-2)R_{23}(u-5)S^{-1}_{12},\\
P^{(49)}_{2\tilde{1}}R_{32}(u)R_{3\tilde{2}}(u-\frac{9}{2})P^{(49)}_{2\tilde{1}} &= (u+6)\tilde{S}_{12}R_{31}(u-2)R_{32}(u-5)\tilde{S}^{-1}_{12},\\
P^{(49)}_{\tilde{1}2}K^{-}_{2}(u)R_{\tilde{1}2}(2u-\frac{9}{2})K^{-}_{\tilde{1}}(u-\frac{9}{2})P^{(49)}_{2\tilde{1}} &= -2(u+1)(2u+1)(2u-1)\\
&\quad\times S_{12}K^{-}_{1}(u-2)R_{21}(2u-7)K^{-}_{2}(u-5)\tilde{S}^{-1}_{12},\\
P^{(49)}_{2\tilde{1}}K^{+}_{\tilde{1}}(u-\frac{9}{2})R_{2\tilde{1}}(-2u-2\kappa+\frac{9}{2})K^{+}_{2}(u)P^{(49)}_{\tilde{1}2} &= 2(u+6)(2u+5)(2u-3)\\
&\quad\times\tilde{S}_{12}K^{+}_{2}(u-5)R_{12}(-2u-2\kappa+7)K^{+}_{1}(u-2)S^{-1}_{12},
\end{aligned}$$

where S_{12} and $\tilde{S}_{12}$ are the 49×49 irrelevant constant matrices.

Computing the quantity $t(u)\tilde{t}(u+\delta)$ and substituting $u=\pm\theta_j$ and $\delta=-\frac{9}{2}$ in the results, we have

$$t(\pm\theta_j)\tilde{t}(\pm\theta_j-\frac{9}{2})=2^4\frac{(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j+3)(\pm\theta_j+4)}(\pm\theta_j-\frac{1}{2})(\pm\theta_j-\frac{7}{2})(\pm\theta_j+\frac{1}{2})(\pm\theta_j+\frac{5}{2})$$

$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+6)(\pm\theta_j+\theta_i+6)t(\pm\theta_j-2)t(\pm\theta_j-5),\ j=1,\cdots,N.$$

We see that the result is the product of two original transfer matrices.

At the point of $u=-\frac{13}{2}$,

$$R_{\tilde{1}2}(-\frac{13}{2})=P^{(7)}_{\tilde{1}2}\times S_7,$$

we get a 7-dimensional projector $P^{(7)}_{\tilde{1}2}$. Direct calculating gives

$$\begin{aligned}
&P^{(7)}_{\tilde{1}2}R_{23}(u)R_{\tilde{1}3}(u-\frac{13}{2})P^{(7)}_{\tilde{1}2}=(u-4)a(u)S_1R_{13}(u-7)S_1^{-1},\\
&P^{(7)}_{2\tilde{1}}R_{32}(u)R_{3\tilde{2}}(u-\frac{13}{2})P^{(7)}_{2\tilde{1}}=(u-4)a(u)R_{31}(u-7),\\
&P^{(7)}_{\tilde{1}2}K^-_2(u)R_{\tilde{1}2}(2u-\frac{13}{2})K^-_{\tilde{1}}(u-\frac{13}{2})P^{(7)}_{2\tilde{1}}\\
&=4(u-4)(u-1)(2u-11)(2u-1)(2u-5)(2u-3)(2u+1)S_1K^-_1(u-7),\\
&P^{(7)}_{2\tilde{1}}K^+_{\tilde{1}}(u-\frac{13}{2}))R_{2\tilde{1}}(-2u-2\kappa+\frac{13}{2})K^+_2(u)P^{(7)}_{\tilde{1}2}\\
&=-4(u+1)(u+6)(2u-7)(2u-5)(2u+1)(2u+3)(2u+5)K^+_1(u-7)S_1^{-1},
\end{aligned}$$

where S_1 is 7×7 constant matrix.

Then we obtain

$$t(\pm\theta_j)\tilde{t}(\pm\theta_j-\frac{13}{2})=-2^6\frac{(\pm\theta_j-4)(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j-2)(\pm\theta_j+2)(\pm\theta_j+3)}(\pm\theta_j-\frac{11}{2})(\pm\theta_j-\frac{5}{2})$$

$$\times(\pm\theta_j-\frac{3}{2})(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{1}{2})(\pm\theta_j+\frac{5}{2})\prod_{i=1}^{N}[(\pm\theta_j-\theta_i+4)$$

$$\times(\pm\theta_j+\theta_i+4)a(\pm\theta_j-\theta_i)a(\pm\theta_j+\theta_i)]t(\pm\theta_j-7),\ j=1,\cdots,N.$$

We see that no new unknown operators appear and the fusion relations are closed.

Taking the limit of u tends to infinity and using the definitions, we obtain the asymptotic behaviors of the transfer matrices as

$$t(u)|_{u\to\pm\infty} = Au^{6N+2} \times \mathrm{id} + \cdots, \qquad 6N+2 \to 3N+1, (3N+2)$$

$$\bar{t}(u)|_{u\to\pm\infty} = 16[-(\frac{A}{8})^2 + \frac{A}{8} + \frac{3}{4}]u^{4N+2} \times \mathrm{id} + \cdots, \qquad 4N+2 \to 2N+1, (2N+2)$$

$$\tilde{t}(u)|_{u\to\pm\infty} = -128[\frac{3}{2}(\frac{A}{8})^2 + \frac{A}{16} + \frac{3}{8}]u^{8N+6} \times \mathrm{id} + \cdots, \qquad 8N+6 \to 4N+3, (4N+4)$$

where A is a boundary parameters dependent constant. Besides, we know the values of transfer matrices at some special points

$$t(0) = -5\prod_{l=1}^{N}\rho_{12}(\theta_l) \times \mathrm{id},$$

$$t(-1) = -\frac{5}{4}\prod_{l=1}^{N}(\theta_l - 1)(-\theta_l - 1)\bar{t}(-\frac{1}{2}),$$

$$\tilde{t}(-\frac{5}{2}) = -\frac{15}{2}\prod_{l=1}^{N}(\theta_l + 1)(-\theta_l + 1)(\theta_l + 4)(-\theta_l + 4)\bar{t}(-\frac{7}{2}),$$

$$\tilde{t}(-\frac{13}{2}) = 330\prod_{l=1}^{N}(\theta_l - 4)(-\theta_l - 4)t(-7),$$

$$\tilde{t}(-1) = 0.$$

Above $9N+8$ functional relations allow us completely to determine the eigenvalues, which can be given in terms of some inhomogeneous $T-Q$ relations as

$$\Lambda(u) = \sum_{i=1}^{7} Z_i(u) + \sum_{k=1}^{2} f_k(u),$$

$$\bar{\Lambda}(u-\frac{1}{2}) = -\frac{(u+2)(u+3)}{(u-1)(u+6)(u+\frac{5}{2})(u+\frac{5}{2})}\prod_{i=1}^{N}[(u+\theta_i-1)(u-\theta_i-1)a(u-\theta_i)]^{-1}$$

$$\times a^{-1}(u+\theta_i)\{Z_1(u)[\sum_{i=2}^{7} Z_i(u-1)+f_1(u-1)+f_2(u-1)]+[\sum_{i=2}^{6} Z_i(u)+f_1(u)$$

$$+f_2(u)]Z_7(u-1)+[Z_2(u)+f_1(u)+Z_3(u)][Z_5(u-1)+f_2(u-1)+Z_6(u-1)]\},$$

$$\tilde{\Lambda}(u-\frac{5}{2}) = \frac{u(u-1)(u+4)(u-\frac{1}{2})(u-\frac{3}{2})}{(u+1)(u+6)(u-4)(u-\frac{1}{2})^2}\prod_{i=1}^{N}[(u+\theta_i-4)(u-\theta_i-4)]^{-1}$$

$$\times[(u+\theta_i+6)(u-\theta_i+6)a(u+\theta_i-3)a(u-\theta_i-3)]^{-1}$$

$$\times\{[\sum_{i=1}^{4} Z_i(u)+f_1(u)][\sum_{k=1}^{6} Z_k(u-3)+f_1(u-3)+f_2(u-3)]Z_7(u-4)$$

$$+Z_1(u)Z_1(u-3)[\sum_{k=4}^{6} Z_k(u-4)+f_2(u-4)]+Z_5(u)Z_6(u-3)Z_7(u-4)$$

$$+Z_1(u)[Z_2(u-3)+f_1(u-3)+Z_3(u-3)][Z_5(u-4)+f_2(u-4)+Z_6(u-4)]$$

$$+Z_1(u)Z_3(u-3)[Z_5(u-4)+f_2(u-4)+Z_6(u-4)]\}.$$

All the eigenvalues are the polynomials of u, thus the residues of right hand sides of inhomogeneous $T-Q$ relations should be zero, which gives rise to the Bethe ansatz equations (BAEs)

$$\frac{Q^{(1)}(i\mu_k^{(1)}+\frac{1}{2})Q^{(2)}(i\mu_k^{(1)}-\frac{7}{2})}{Q^{(1)}(i\mu_k^{(1)}-\frac{3}{2})Q^{(2)}(i\mu_k^{(1)}-\frac{1}{2})}=-\frac{(i\mu_k^{(1)}+\frac{1}{2})}{(i\mu_k^{(1)}-\frac{1}{2})}\frac{\prod_{j=1}^{N}(i\mu_k^{(1)}-\theta_j+\frac{1}{2})(i\mu_k^{(1)}+\theta_j+\frac{1}{2})}{\prod_{j=1}^{N}(i\mu_k^{(1)}-\theta_j-\frac{1}{2})(i\mu_k^{(1)}+\theta_j-\frac{1}{2})},$$
$$k=1,2,\cdots,L_1,$$

$$\frac{(i\mu_l^{(2)}-\frac{3}{2})}{i\mu_l^{(2)}}\frac{Q^{(2)}(i\mu_l^{(2)}-5)}{Q^{(1)}(i\mu_l^{(2)}-2)}+\frac{(i\mu_l^{(2)}+\frac{3}{2})}{i\mu_l^{(2)}}\frac{Q^{(2)}(i\mu_l^{(2)}+1)}{Q^{(1)}(i\mu_l^{(2)}+1)}$$
$$=-x(i\mu_l^{(2)}-\frac{3}{2})(i\mu_l^{(2)}+\frac{3}{2}),\quad l=1,2,\cdots,L_2,$$

where

$$Q^{(1)}(u)=\prod_{k=1}^{L_1}(iu+\mu_k^{(1)}+\frac{i}{2})(iu-\mu_k^{(1)}+\frac{i}{2}),\quad Q^{(2)}(u)=\prod_{k=1}^{L_2}(iu+\mu_k^{(2)}+2i)(iu-\mu_k^{(2)}+2i),$$

From the asymptotic behaviors of $\Lambda(u)$, $\bar{\Lambda}(u)$ and $\tilde{\Lambda}(u)$, we obtain the constraint between the integers L_1 and L_2 i.e., $L_2=L_1+1$.

Results for the periodic boundary condition

$$t^{(p)}(u) = tr_0 T_0(u), \quad \bar{t}^{(p)}(u) = tr_{\bar{0}} T_{\bar{0}}(u), \quad \tilde{t}^{(p)}(u) = tr_{\tilde{0}} T_{\tilde{0}}(u).$$

Closed operators product identities

$$\begin{aligned}
t^{(p)}(\theta_j)\, t^{(p)}(\theta_j - 6) &= \prod_{i=1}^{N} a(\theta_j - \theta_i)e(\theta_j - \theta_i - 6) \times \mathrm{id}, \\
t^{(p)}(\theta_j)\, t^{(p)}(\theta_j - 4) &= \prod_{i=1}^{N} (\theta_j - \theta_i + 1)(\theta_j - \theta_i - 4)(\theta_j - \theta_i - 6)\, t^{(p)}(\theta_j - 2), \\
t^{(p)}(\theta_j)\, t^{(p)}(\theta_j - 1) &= \prod_{i=1}^{N} (\theta_j - \theta_i - 1)a(\theta_j - \theta_i)\, \bar{t}^{(p)}(\theta_j - \frac{1}{2}), \\
t^{(p)}(\theta_j)\, \bar{t}^{(p)}(\theta_j - \frac{11}{2}) &= \prod_{i=1}^{N} (\theta_j - \theta_i + 4)(\theta_j - \theta_i + 6)\, t^{(p)}(\theta_j - 5), \\
t^{(p)}(\theta_j)\, \bar{t}^{(p)}(\theta_j - \frac{7}{2}) &= \prod_{i=1}^{N} (\theta_j - \theta_i + 6)\, \tilde{t}^{(p)}(\theta_j - \frac{5}{2}), \\
t^{(p)}(\theta_j)\, \tilde{t}^{(p)}(\theta_j - \frac{7}{2}) &= \prod_{i=1}^{N} (\theta_j - \theta_i - 1)(\theta_j - \theta_i - 4)a(\theta_j - \theta_i)\, \bar{t}^{(p)}(\theta_j - \frac{5}{2}), \\
t^{(p)}(\theta_j)\, \tilde{t}^{(p)}(\theta_j - \frac{9}{2}) &= \prod_{i=1}^{N} (\theta_j - \theta_i + 6)\, t^{(p)}(\theta_j - 2)t^{(p)}(\theta_j - 5), \\
t^{(p)}(\theta_j)\, \tilde{t}^{(p)}(\theta_j - \frac{13}{2}) &= \prod_{i=1}^{N} (\theta_j - \theta_i - 4)a(\theta_j - \theta_i)\, t^{(p)}(\theta_j - 7), \\
t^{(p)}(\theta_j)\, \tilde{t}^{(p)}(\theta_j - \frac{3}{2}) &= \prod_{i=1}^{N} (\theta_j - \theta_i - 1)(\theta_j - \theta_i - 6)\, t^{(p)}(\theta_j - 2)\bar{t}^{(p)}(\theta_j - \frac{1}{2}).
\end{aligned} \tag{15}$$

Moreover, the asymptotic behaviors of transfer matrices become

$$
\begin{aligned}
t^{(p)}(u)|_{u\to\pm\infty} &= 7u^{3N} \times \mathrm{id} + \cdots \,, \\
\bar{t}^{(p)}(u)|_{u\to\pm\infty} &= 15u^{2N} \times \mathrm{id} + \cdots \,, \\
\tilde{t}^{(p)}(u)|_{u\to\pm\infty} &= 34u^{4N} \times \mathrm{id} + \cdots \,.
\end{aligned} \tag{16}
$$

From the definitions, we know that the transfer matrices $t^{(p)}(u)$, $\bar{t}^{(p)}(u)$ and $\tilde{t}^{(p)}(u)$ are the polynomials of u with degrees $3N$, $2N$ and $4N$, respectively. Thus their eigenvalues can be determined by $9N+3$ independent conditions. The constraints (15)-(16) give us sufficient information to obtain these eigenvalues.

Denote the eigenvalues of $t^{(p)}(u)$, $\bar{t}^{(p)}(u)$ and $\tilde{t}^{(p)}(u)$ as $\Lambda^{(p)}(u)$, $\bar{\Lambda}^{(p)}(u)$ and $\tilde{\Lambda}^{(p)}(u)$, respectively. Then we have

$$\Lambda^{(p)}(u) = \sum_{j=1}^{7} Z_j^{(p)}(u),$$

$$\Lambda_2^{(p)}(u-\frac{1}{2}) = \prod_{i=1}^{N}((u-\theta_i-1)a(u-\theta_i))^{-1}\{Z_1^{(p)}(u)[\sum_{j=2}^{7} Z_j^{(p)}(u-1)]$$

$$+[\sum_{j=2}^{6} Z_j^{(p)}(u)]Z_7^{(p)}(u-1)+[Z_2^{(p)}(u)+Z_3^{(p)}(u)][Z_5^{(p)}(u-1)+Z_6^{(p)}(u-1)]\},$$

$$\Lambda_3^{(p)}(u-\frac{5}{2}) = \prod_{i=1}^{N}((u-\theta_i-4)(u-\theta_i+6)a(u-\theta_i-3))^{-1}$$

$$\times\{(\sum_{j=1}^{4} Z_j^{(p)}(u))(\sum_{k=1}^{6} Z_j^{(p)}(u-3))Z_7^{(p)}(u-4)+Z_1^{(p)}(u)Z_1^{(p)}(u-3)[\sum_{j=4}^{6} Z_j^{(p)}(u-4)]$$

$$+Z_1^{(p)}(u)[Z_2^{(p)}(u-3)+Z_3^{(p)}(u-3)][Z_5^{(p)}(u-4)+Z_6^{(p)}(u-4)]+Z_2^{(p)}(u)Z_3^{(p)}(u-3)$$

$$\times[Z_5^{(p)}(u-4)+Z_6^{(p)}(u-4)]+Z_5^{(p)}(u)Z_6^{(p)}(u-3)Z_7^{(p)}(u-4)\}.$$

The regularity of the expressions of the eigenvalues requires that the Bethe roots should satisfy the BAEs

$$\frac{Q_p^{(1)}(i\mu_k^{(1)}+\frac{1}{2})Q_p^{(2)}(i\mu_k^{(1)}-\frac{7}{2})}{Q_p^{(1)}(i\mu_k^{(1)}-\frac{3}{2})Q_p^{(2)}(i\mu_k^{(1)}-\frac{1}{2})}=-\prod_{j=1}^{N}\frac{i\mu_k^{(1)}+\frac{1}{2}-\theta_j}{i\mu_k^{(1)}-\frac{1}{2}-\theta_j},\quad k=1,\cdots,L_1,$$

$$\frac{Q_p^{(1)}(i\mu_l^{(2)}-2)Q_p^{(2)}(i\mu_l^{(2)}+1)}{Q_p^{(1)}(i\mu_l^{(2)}+1)Q_p^{(2)}(i\mu_l^{(2)}-5)}=-1,\quad l=1,\cdots,L_2,$$

where

$$Q_p^{(1)}(u)=\prod_{k=1}^{L_1}(iu+\mu_k^{(1)}+i\frac{1}{2}),\quad Q_p^{(2)}(u)=\prod_{k=1}^{L_2}(iu+\mu_k^{(2)}+2i).$$

We have applied ODBA to the typical quantum integrable models, including the models associated with A_n, B_n, C_n and D_n Lie algebras.

We obtain the exact solution of $D_2^{(2)}$ spin chain and G_2 model with unparallel boundary fields.

This method is a universal.

Other things we can do:

In fact, besides the eigenvalues, we have obtained the eigenstates, exact physical quantities in the thermodynamic limit such as ground state energy density, elementary excitations and surface energy of the XXZ spin chain with unparallel boundary fields. All these results can be used to study the $D_2^{(2)}$ spin chain directly.

For example:

Eigenstates of XXZ spin chain with non-diagonal boundary reflection

The bases of Hilbert space is (In the following, the number of sites is N)

$$|\theta_{p_1},\cdots,\theta_{p_n};m\rangle=\mathcal{A}_m(\theta_{p_1})\cdots\mathcal{A}_m(\theta_{p_n})|m\rangle,$$

where $p_j\in\{1,\cdots,N\}$, $p_1<p_2<\cdots<p_n$,

$$\begin{aligned}|m\rangle&=\otimes_{n=1}^{N}|m\rangle_n,\\ |m\rangle_n&=e^{-[\theta_n+(m+N-n+1+\alpha)\eta]}|\uparrow\rangle_n+|\downarrow\rangle_n.\end{aligned}$$

Conjugate bases

$$\langle\theta_{p_1}\cdots\theta_{p_n};m|=\langle m|\bar{\mathcal{D}}_m(-\theta_{p_1})\cdots\bar{\mathcal{D}}_m(-\theta_{p_n}),$$

where $p_j\in\{1,\cdots,N\}$, $\theta_{p_1}<\theta_{p_2}\cdots<\theta_{p_n}$,

$$\begin{aligned}\langle m|&=\otimes_{n=1}^{N}\ {}_n\langle m|,\\ {}_n\langle m|&=\frac{e^{\theta_n+\alpha\eta}}{2\sinh(m+n-N-1)\eta}\left\{{}_n\langle\uparrow|-e^{-[\theta_n+(\alpha+m+n-N-1)\eta]}\ {}_n\langle\downarrow|\right\}.\end{aligned}$$

Expanding an eigenstate $|\Psi\rangle$ of the transfer matrix $\tilde{t}(u)$ with the bases, then the coefficients are

$$F_n(\theta_{p_1}, \cdots, \theta_{p_n}; m_0) = \langle \theta_{p_1}, \cdots, \theta_{p_n}; m_0|\Psi\rangle, \quad n = 0, \cdots, N.$$

In order to calculate the efficients, we consider the quantity $\langle \theta_{p_1}, \cdots, \theta_{p_n}; m_0|\tilde{t}(-\theta_{p_{n+1}})|\Psi\rangle$, where $p_{n+1} \neq p_1, \cdots, p_n$.

By acting $\tilde{t}(-\theta_{p_{n+1}})$ to left and to right alternately, we obtain

$$F_n(\theta_{p_1}, \cdots, \theta_{p_n}; m_0) = \prod_{j=1}^{n} \left[\bar{K}_{22}^{+}(m_0| - \theta_{p_j}) - \frac{\sinh((m_0-1)\eta + 2\theta_{p_j}) \sinh\eta}{\sinh(m_0-1)\eta \sinh(2\theta_{p_j} - \eta)} \right.$$
$$\left. \times \bar{K}_{11}^{+}(m_0| - \theta_{p_j}) \right]^{-1} \tilde{\Lambda}(-\theta_{p_j}) F_0(m_0),$$

where $F_0(m_0) = \langle m_0|\Psi\rangle$ is an overall scalar factor.

Therefore, we have retrieved the eigenstates by using the obtained eigenvalues.

Exact physical quantities in the thermodynamic limit

Difficulty: The Bethe ansatz equations (14) are not the from of product. Thus the thermodynamic Bethe ansatz can not be applied.

$$\ln\prod \;\Rightarrow\; \sum \;\Rightarrow \int$$
$$\ln(\prod+\prod) \;\not\Rightarrow\; \sum$$

- Degenerate points, at which the inhomogeneous term in the $T-Q$ relations is zero
- $t-W$ scheme

Ground state energy density, elementary excitations, surface energy, free energy at finite temperature

$t - W$ relation

By using the fusion, we have the following $t - W$ relation

$$\tilde{t}(u)\tilde{t}(u-\eta) = \Delta_q(u) \times \mathbf{id} + \bar{d}(u)\mathbf{W}(u), \tag{17}$$

where $\Delta_q(u)$ is the quantum determinant.

Because $\tilde{t}(u)$ and $\mathbf{W}(u)$ commutate with each other,

$$[\mathbf{W}(u), t(u)] = 0.$$

They have common eigenstates.

Acting (17) on a common eigenstate $|\Psi\rangle$ we have

$$\tilde{\Lambda}(u)\tilde{\Lambda}(u-\eta) = \Delta_q(u) + \bar{d}(u)W(u), \tag{18}$$

where $W(u)$ is the eigenvalue of $\mathbf{W}(u)$.

Because $\tilde{\Lambda}(u)$ is a trigonometric polynomial of u with degree $2N+4$ and satisfies the crossing symmetry trigonometric $\tilde{\Lambda}(u) = \tilde{\Lambda}(-u-\eta)$ and $\tilde{\Lambda}(u+i\pi) = \tilde{\Lambda}(u)$, the eigenvalue $\tilde{\Lambda}(u)$ can be characterized by its roots $\{z_j\}$ as

$$\tilde{\Lambda}(u) = \bar{\Lambda}_0 \prod_{j=1}^{N+2} \sinh(u - z_j + \frac{\eta}{2}) \sinh(u + z_j + \frac{\eta}{2}), \tag{19}$$

where $\bar{\Lambda}_0 = -8\cos(\theta_- - \theta_+)\sinh^{-2N}\eta$ is determined by the asymptotic behavior of $\tilde{t}(u)$ when $u \to \infty$. The $\tilde{\Lambda}(u)$ has $2N+2$ roots, $z_j - \frac{\eta}{2}$ and $-z_j - \frac{\eta}{2}$.

$W(u)$ is a trigonometric polynomial of u with degree $2N+4$, and we put

$$W(u) = \bar{W}_0 \prod_{l=1}^{2N+4} \sinh(u - w_l).$$

An important fact is that (18) is a degree $4N+8$ polynomial equation and thus gives $4N+9$ independent equations for the coefficients to determine the $N+2$ z_j and $2N+4$ w_l completely.

Bethe ansatz equations with product form

If $u = z_j - \frac{\eta}{2}$, then $\tilde{\Lambda}(u) = 0$. From $t - W$ relation, $\tilde{\Lambda}(u)\tilde{\Lambda}(u-\eta) = \Delta_q(u) + \bar{d}(u)W(u)$, we have

$$\Delta_q(z_j - \frac{\eta}{2}) = -\bar{d}(z_j - \frac{\eta}{2})W(z_j - \frac{\eta}{2}).$$

If $u = w_j$, then $W(u) = 0$. From $t - W$ relation, we have

$$\tilde{\Lambda}(w_l)\tilde{\Lambda}(w_l - \eta) = \Delta_q(w_l).$$

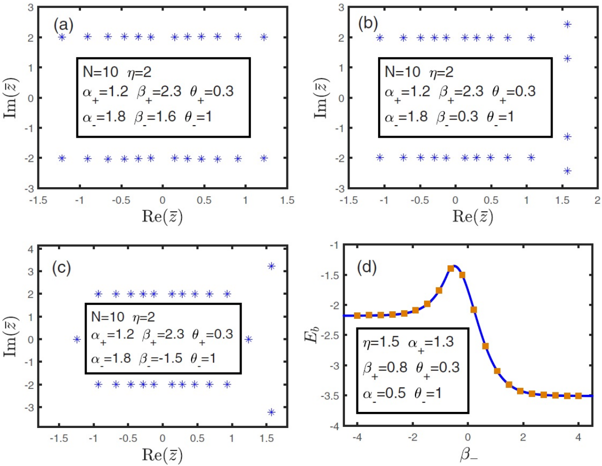
Alternating form

The Bethe ansatz equations for the zero roots with product form can also be obtained as follows. At the inhomogeneous points, for a eigenvalue $\tilde{\Lambda}(u)$, the $t-W$ relations reduce as

$$\begin{aligned} &\tilde{\Lambda}(\theta_j)\tilde{\Lambda}(\theta_j-\eta)=a(\theta_j)a(-\theta_j), \quad j=1,\cdots,N, \\ &\tilde{\Lambda}(0)=a(0), \quad \tilde{\Lambda}\Big(\frac{i\pi}{2}\Big)=a\Big(\frac{i\pi}{2}\Big), \end{aligned} \tag{20}$$

Substituting the parametrization (19) into above equations, we can obtain sufficient conditions to determine the values of zero roots $\{z_j|j=1,\cdots,N+2\}$ completely for a given set of inhomogeneity parameters.

By choosing a proper set of inhomogeneity parameters, we find that the root distributions has ceratin patterns in the thermodynamic limit.



Taking the logarithm of Eq.(20), then the product becomes summation. In the thermodynamic limit, the summation becomes integration. By choosing the suitable distribution of inhomogeneous parameters, we obtain the density of zero roots

$$\tilde{\rho}(k) = [2N\tilde{b}_2\tilde{\sigma}(k) + [1+(-1)^k](\tilde{b}_2 - \tilde{b}_1) + \tilde{b}_{\frac{2\alpha_+}{\eta}} + \tilde{b}_{\frac{2\alpha_-}{\eta}} + (-1)^k(\tilde{b}_{\frac{2\beta_+}{\eta}} + \tilde{b}_{\frac{2\beta_-}{\eta}})]/[N(\tilde{b}_1 + \tilde{b}_3)].$$

Based on it, we obtain the surface energy as

$$E_{b1} = e_b(\alpha_+, \beta_+) + e_b(\alpha_-, \beta_-) + e_{b0}.$$

Here $e_b(\alpha, \beta)$ indicates the contribution of one boundary field and e_{b0} is the surface energy induced by the free open boundary.

Thank you for your attention!