Numerical methods and analytic results for one-dimensional strongly interacting spinor gases

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Outline

- Trapped one-dimensional spinor gases
  - Eigenstates in the strongly interacting regime
  - Spin-charge factorization of the correlators
- Efficient numerical evaluation of multidimensional integrals
- Numerical results
- Determinant formulae for spin incoherent correlators
- Conclusions

Trapped one-dimensional spinor gases

1D spinor gases with κ components and strong repulsive contact interactions

$$H = \int dz \, \frac{\hbar^2}{2m} \left(\partial_z \boldsymbol{\Psi}^\dagger \partial_z \boldsymbol{\Psi} \right) + \frac{g}{2} : \left(\boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \right)^2 : + V(z) \left(\boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \right) - \boldsymbol{\Psi}^\dagger \boldsymbol{\mu} \boldsymbol{\Psi} \, ,$$

- $\boldsymbol{\Psi} = (\Psi_1(z), \cdots, \Psi_\kappa(z))^T$, $\boldsymbol{\Psi}^\dagger = \left(\Psi_1^\dagger(z), \cdots, \Psi_\kappa^\dagger(z) \right)$, $: \;:$ normal ordering
- m is the mass of the particles, $g > 0$ coupling strength
- $\boldsymbol{\mu}$ matrix with $\mu_1, \cdots, \mu_\kappa$ on the diagonal and zero otherwise.
- Bosonic ($\epsilon = 1$) or fermionic ($\epsilon = -1$) fields:

$$\Psi_\sigma(x) \Psi_{\sigma'}^\dagger(y) - \epsilon \Psi_{\sigma'}^\dagger(y) \Psi_\sigma(x) = \delta_{\sigma,\sigma'} \delta(x - y) \, , \quad \sigma, \sigma' \in \{1, \cdots, \kappa\}$$

- External potential: $V(z) = m\omega^2 z^2/2$; Dirichlet boundary conditions $V(z) = 0$ for $z \in [-L/2, L/2]$ and $V(z) = \infty$ for $z \notin [-L/2, L/2]$.

When $V(z) = 0$ the system is integrable: $\kappa = 1$ Lieb-Liniger model [Lieb, Liniger, 1963]; $\kappa = 2$ Gaudin-Yang model [Gaudin 1967; Yang 1967]; $\kappa = 3, 4, \cdots$ Sutherland model [Sutherland 1968]

- Experimental realization $\kappa = 1, \cdots, 6$ [Pagano et al. 2014]; $\kappa = 2$ Gaudin-Yang model R. Hulet group (Rice Univ.): dynamical structure factor [Yang et al. 2018]; spin-charge separation [Senaratne et al. 2022]; spin-incoherent regime [Cavazos-Cavazos et al. 2022].

Groundstate manifold of states for g large but finite

- $g=\infty$ Large degeneracies; g large but finite: degeneracy lifted, spin-charge factorization of the wavefunction [Ogata and Shiba 1990; Izergin and Pronko 1998, Deuretzbacher et al. 2008, Guan et al. 2009; $\cdots$ Osterloh, Polo, Chetcuti, Amico 2023]

Groundstate manifold wavefunctions [$\boldsymbol{z}=(z_1,\cdots,z_N)$]

$$\psi^{\sigma_1\cdots\sigma_N}(\boldsymbol{z})=\left[\sum_{P\in S_N}(-\epsilon)^P P\theta(\boldsymbol{z})P\chi(\sigma_1,\cdots,\sigma_N)\right]\psi_F(\boldsymbol{z}|\boldsymbol{q}^0)$$

$\theta(\boldsymbol{z})\equiv\theta(z_1<\cdots<z_N)$ generalized Heaviside function, $P\theta(\boldsymbol{z})=\theta(z_{P_1}<\cdots<z_{P_N})$ and $P\chi(\sigma_1,\cdots,\sigma_N)=\chi(\sigma_{P_1},\cdots,\sigma_{P_N})$.

Charge degrees of freedom: Slater determinant formed from the lowest N orbitals of a system of spinless fermions subjected to the same potential $V(z)$

$$\psi_F(\boldsymbol{z}|\boldsymbol{q}^0)=\frac{1}{\sqrt{N!}}\det_N\left[\phi_{q_j^0}(z_i)\right]_{i,j=1,\cdots,N}\,,\quad \boldsymbol{q}^0=(1,\cdots,N)$$

Spin sector: $\chi(\sigma_1,\cdots,\sigma_N)$ is an eigenfunction of a spin chain [Deuretzbacher et al. 2014, Volosniev et al. 2014, 2015, Levinsen et al. 2015, Yang, Guan and Pu 2015, Yang and Cui 2016]

$$H_{sc}^0=E_F(\boldsymbol{q}^0)-\frac{1}{g}\sum_{i=1}^{N-1}J_i^0\left(1+\epsilon\hat{P}_{i,i+1}\right)$$

$E_F(\boldsymbol{q}^0)=\sum_{j=1}^N\varepsilon(q_j^0)$ is the groundstate energy of the spinless fermionic system, $\hat{P}_{i,i+1}$ permutes the spins on positions i and $i+1$, J_i^0 position dependent local exchange coefficients

Local exchange coefficients

Spin sector described by

$$H_{sc}^0 = E_F(\boldsymbol{q}^0) - \frac{1}{g}\sum_{i=1}^{N-1} J_i^0 \left(1 + \epsilon \hat{P}_{i,i+1}\right)$$

J_i^0 are the (position variable) local exchange coefficients position which reflect the inhomogeneity of the trapping potential [Deuretzbacher et al. 2014, Volosniev et al. 2014]

Local exchange coefficients

$$J_i^0 = N! \int dz_1 \cdots dz_N \, \delta(z_i - z_{i+1}) \, \theta(z_1 < \cdots < z_N) \left| \frac{\partial \psi_F(\boldsymbol{q}^0)}{\partial z_i} \right|^2 \quad i = 1, \cdots, N-1$$

N-1 dimensional integral

- $N \leq 16$ Monte-Carlo integration [Deuretzbacher et al. 2014; Volosniev et al. 2014; Levinsen et al. 2015; Yang, Guan and Pu 2015; Jen and Yip 2016, 2017,...]
- $N \leq 30 \sim 35$ CONAN [Loft, Kristensen, Thomsen, Volosniev, and Zinner 2016]
- $N \leq 60$ [Deuretzbacher, Becker, and Santos 2016]

Require arbitrary precision subroutines (for $N = 30 \sim 35 \rightarrow 2048$ bits of precision), large evaluation times $N = 30$, 1 hour, $N = 60$ approx. 10 days!

Excited manifolds

Excited manifolds wavefunctions [Yang and Pu 2016]

$$\psi^{\sigma_1\cdots\sigma_N}(\boldsymbol{z}) = \left[\sum_{P\in S_N} (-\epsilon)^P P\theta(\boldsymbol{z})P\chi(\sigma_1,\cdots,\sigma_N)\right]\psi_F(\boldsymbol{z}|\boldsymbol{q}^k)$$

Charge degrees of freedom: $\psi_F(\boldsymbol{z}|\boldsymbol{q}^k)$ excited states of the dual system of spinless fermions (first excited state $\boldsymbol{q}^1 = (1,\cdots,N-1,N+1)$)
Spin sector

$$H_{sc}^k = E_F(\boldsymbol{q}^k) - \frac{1}{g}\sum_{i=1}^{N-1} J_i^k\left(1+\epsilon\hat{P}_{i,i+1}\right)$$

Local exchange coefficients for the excited spin Hamiltonians

$$J_i^k = N!\int dz_1\cdots dz_N\,\delta(z_i-z_{i+1})\,\theta(z_1<\cdots<z_N)\left|\frac{\partial\psi_F(\boldsymbol{q}^k)}{\partial z_i}\right|^2 \quad i=1,\cdots,N-1$$

Separation of energy scales

Harmonic trapping ($\omega = m = 1$), single particle orbitals Hermite functions

- Groundstate $\psi_F = \psi_F(z_1, \cdots, z_N|\boldsymbol{q}_0)$ with $\boldsymbol{q}_0 = (0, 1, \cdots, N-1)$
- First excited state $\psi_F(z_1, \cdots, z_N|\boldsymbol{q}_1)$ with $\boldsymbol{q}_1 = (0, 1, \cdots, N-2, N)$

$$E_F(\boldsymbol{q}^0) = \sum_{j=0}^{N-1}(j + 1/2) = N^2/2\,, \quad E_F(\boldsymbol{q}^1) = E_F(\boldsymbol{q}^0) + 1$$

For a given sector $\boldsymbol{N} = [N_1, \cdots, N_\kappa]$ and $\boldsymbol{q}$ there are $N!/[N_1! \cdots N_\kappa!]$ spin states $|\Phi_{\boldsymbol{N},\boldsymbol{q},n}\rangle$ with $n = 1, \cdots, N!/[N_1! \cdots N_\kappa!]$.
$E(\boldsymbol{N}, \boldsymbol{q}^{0,1}, n)$ energies of $|\Phi_{\boldsymbol{N},\boldsymbol{q}^{0,1},n}\rangle$ ($E_{spin}(\boldsymbol{N}, \boldsymbol{q}^{0,1}, n)$ energies spin states)

$$E(\boldsymbol{N}, \boldsymbol{q}^0, n) = E_F(\boldsymbol{q}^0) - \frac{1}{g}E_{spin}(\boldsymbol{N}, \boldsymbol{q}^0, n)\,, \ |E_{spin}(\boldsymbol{N}, \boldsymbol{q}^0, n)| \in \left[0, 2\sum J_i^0\right]\,,$$

$$E(\boldsymbol{N}, \boldsymbol{q}^1, n) = E_F(\boldsymbol{q}^1) - \frac{1}{g}E_{spin}(\boldsymbol{N}, \boldsymbol{q}^1, n)\,, \ |E_{spin}(\boldsymbol{N}, \boldsymbol{q}^1, n)| \in \left[0, 2\sum J_i^1\right]\,,$$

Strong interaction regime: energy scale of the spin sector $E_{spin} \sim 1/g$; energy scale of the charge sector $E_{charge} \equiv E(\boldsymbol{N}, \boldsymbol{q}^1, n) - E(\boldsymbol{N}, \boldsymbol{q}^0, n) \sim E_F(\boldsymbol{q}^1) - E_F(\boldsymbol{q}^0) = 1$

$$\boxed{E_{spin} \ll E_{charge}}$$

Zero temperature correlators: I

Large but finite $g \rightarrow$ unique groundstate $|GS\rangle$

$$\rho_\sigma(x,y) = \langle GS|\Psi^\dagger_\sigma(x)\Psi_\sigma(y)|GS\rangle\,,\ \sigma = \{1,\cdots,\kappa\}$$

Spin charge factorization [Yang, Guan and Pu 2015]

$$\rho_\sigma(x,y) = \sum_{d_1=1}^{N}\sum_{d_2=d_1}^{N}(-\epsilon)^{d_2+d_1} S_\sigma(d_1,d_2)\rho_{d_1,d_2}(x,y)\,,$$

One-body density matrix elements

$$\rho_{d_1,d_2}(x,y) = N!\int_{\Gamma_{d_1,d_2}(x,y)}\prod_{\substack{k=1\\k\neq d_1}}^{N} dz_k\,\overline{\psi}_F(z_1,\cdots,z_{d_1-1},x,z_{d_1+1}\cdots,z_N|\boldsymbol{q_0})$$
$$\times\,\psi_F(z_1,\cdots,z_{d_1-1},y,z_{d_1+1}\cdots,z_N|\boldsymbol{q_0})$$

$\Gamma_{d_1,d_2}(x,y) = L_- \leq z_1 < \cdots < z_{d_1-1} < x < z_{d_1+1} < \cdots < z_{d_2} < y < z_{d_2+1} < \cdots < z_N \leq L_+$

Spin functions [Ogata and Shiba 1990]

$$S_\sigma(d_1,d_2) = \langle\chi|P^{(d_1)}_\sigma(d_1\cdots d_2)|\chi\rangle$$

$P^{(d_1)}_\sigma = |\sigma\rangle_{d_1}\langle\sigma|_{d_1}$; $(d_1\cdots d_2)$ is the permutation that cyclically permutes the spins between the positions d_1 and d_2

Zero temperature correlators: II

Momentum distributions

$$n_\sigma(k) = \frac{1}{2\pi} \int\int e^{-ik(x-y)} \rho_\sigma(x,y)\, dxdy$$

Densities

$$\rho_\sigma(x) \equiv \rho_\sigma(x,x) = \sum_{d=1}^{N} S_\sigma(d) \rho_d(x)$$

Single particle densities

$$\rho_d(x) = N! \int_{\Gamma_d(x)} \prod_{\substack{k=1 \\ k\neq d}}^{N} dz_k \left| \psi_F(z_1, \cdots, z_{d-1}, x, z_{d+1} \cdots, z_N | \boldsymbol{q}^0) \right|^2$$

$\Gamma_d = L_- \leq z_1 < \cdots < z_d < x < z_{d+1} < \cdots < z_N \leq L_+$

Spin function $S_\sigma(d) = \langle \chi | P_\sigma^{(d)} | \chi \rangle$

Low-temperature correlators

Temperatures smaller than E_{charge} $[E(\boldsymbol{N}, \boldsymbol{q}^1, n) - E(\boldsymbol{N}, \boldsymbol{q}^0, n) \sim E_F(\boldsymbol{q}^1) - E_F(\boldsymbol{q}^0)]$.
$\boldsymbol{N} = [N_1, \cdots, N_\kappa]$ sector

Canonical ensemble (only the first manifold of states contribute)

$$\rho_\sigma^T(x,y) = \sum_{n=1}^{N!/[N_1!\cdots N_\kappa!]} \frac{e^{-E(\boldsymbol{N},\boldsymbol{q}^0,n)/T}}{Z} \langle \Phi_{\boldsymbol{N},\boldsymbol{q}^0,n} | \Psi_\sigma^\dagger(x)\Psi_\sigma(y) | \Phi_{\boldsymbol{N},\boldsymbol{q}^0,n}\rangle$$

with $Z = \sum_{n=1}^{N!/[N_1!\cdots N_\kappa!]} e^{-E(\boldsymbol{N},\boldsymbol{q}^0,n)/T}$. In the limit $T \to 0$ Luttinger Liquid (LL) correlator.

Temperatures much larger than the spin energy $|E_{spin}(\boldsymbol{N}, \boldsymbol{q}^0, n)|/g \ll T \ll E_{charge}$

$$e^{-E(\boldsymbol{N},\boldsymbol{q}^0,n)/T} = e^{-E_F(\boldsymbol{q}^0)/T} e^{E_{spin}(\boldsymbol{N},\boldsymbol{q}^0,n)|/gT} \sim e^{-E_F(\boldsymbol{q}^0)/T} \cdot 1$$

$$\rho_\sigma^{SILL}(x,y) = \sum_{n=1}^{N!/[N_1!\cdots N_\kappa!]} \frac{1}{Z} \langle \Phi_{\boldsymbol{N},\boldsymbol{q}^0,n} | \Psi_\sigma^\dagger(x)\Psi_\sigma(y) | \Phi_{\boldsymbol{N},\boldsymbol{q}^0,n}\rangle$$

with $Z = N!/[N_1!\cdots N_\kappa!]$. Spin-incoherent Luttinger Liquid (SILL) correlator [Berkovich and Lowenstein 1987; Berkovich 1991; Cheianov and Zvonarev 2004, Fiete and Balents 2004; Matveev 2004]

Multidimensional integrals for charge functions

Single particle densities

$$\rho_d(x) = N! \int_{\Gamma_d(x)} \prod_{\substack{k=1 \\ k\neq d}}^{N} dz_k \left| \psi_F(z_1, \cdots, z_{d-1}, x, z_{d+1} \cdots, z_N|\boldsymbol{q}^0) \right|^2$$

$\Gamma_d = L_- \leq z_1 < \cdots < z_d < x < z_{d+1} < \cdots < z_N \leq L_+$

Local exchange coefficients

$$J_i^0 = N! \int dz_1 \cdots dz_N \, \delta(z_i - z_{i+1}) \, \theta(z_1 < \cdots < z_N) \left| \frac{\partial \psi_F(\boldsymbol{q}^0)}{\partial z_i} \right|^2$$

One-body density matrix elements

$$\rho_{d_1,d_2}(x,y) = N! \int_{\Gamma_{d_1,d_2}(x,y)} \prod_{\substack{k=1 \\ k\neq d_1}}^{N} dz_k \, \overline{\psi}_F(z_1, \cdots, z_{d_1-1}, x, z_{d_1+1} \cdots, z_N|\boldsymbol{q_0}) \times \psi_F(z_1, \cdots, z_{d_1-1}, y, z_{d_1+1} \cdots, z_N|\boldsymbol{q_0})$$

$\Gamma_{d_1,d_2}(x,y) = L_- \leq z_1 < \cdots < z_{d_1-1} < x < z_{d_1+1} < \cdots < z_{d_2} < y < z_{d_2+1} < \cdots < z_N \leq L_+$

Efficient computation of single particle densities

"Phase trick" [Izergin and Pronko 1998; Imambekov and Demler 2006; O.I.P. 2022]

$$\rho_d(x) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-i(d-1)\alpha} \underbrace{\left[\det_N \left(e^{i\alpha}\mathrm{M}^0 + \mathrm{M}^1 + \mathrm{M}^r\right) - \det_N \left(e^{i\alpha}\mathrm{M}^0 + \mathrm{M}^1\right)\right]}_{f(\alpha|x)}(x)$$

$\mathrm{M}^{0,1,r}(x)$ $N \times N$ matrices (partial overlaps of single particle orbitals)

$$[\mathrm{M}^0(x)]_{a,b} = \int_{L_-}^{x} \bar{\phi}_a(z)\phi_b(z)\, dz\,,\ [\mathrm{M}^1(x)]_{a,b} = \int_{x}^{L_+} \bar{\phi}_a(z)\phi_b(z)\, dz\,,\ [\mathrm{M}^r(x)]_{a,b} = \bar{\phi}_a(x)\phi_b(x)\,.$$

- $f(\alpha|x)$ is a polynomial of order $N-1$ in $e^{i\alpha}$: $f(\alpha|x) = \sum_{n=0}^{N-1} a_n(x) e^{i\alpha n}$ with $a_n(x) = \rho_{n+1}(x)$
- Evaluate $f_k \equiv f\left(\frac{2\pi k}{N}|x\right) = \sum_{n=0}^{N-1} a_n(x) e^{i\frac{2\pi k}{N}n}$ for $k = 0, 1, \cdots, N-1$
- The single particle densities $\to$ inverse Discrete Fourier Transform of the vector $\boldsymbol{f}$: $a_n(x) \equiv \rho_{n+1}(x) = \frac{1}{N}\sum_{k=0}^{N-1} f_k(x) e^{-i\frac{2\pi n}{N}k}$

Efficient computation of the local exchange coefficients

Can be written as integrals over quantities similar with the single particle densities

$$J_d^0 = \int_{L_-}^{L_+} I_d(\xi)\, d\xi\,, \quad d = 1, \cdots, N-1$$

$$I_d(\xi) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-i(d-1)\alpha} \Big[\det_N \left(e^{i\alpha} \mathrm{M}^0 + \mathrm{M}^1 + \mathrm{M}^r + \mathrm{M}^d \right) - \det_N \left(e^{i\alpha} \mathrm{M}^0 + \mathrm{M}^1 + \mathrm{M}^d \right)$$
$$- \det_N \left(e^{i\alpha} \mathrm{M}^0 + \mathrm{M}^1 + \mathrm{M}^r \right) - \det_N \left(e^{i\alpha} \mathrm{M}^0 + \mathrm{M}^1 \right) \Big] (\xi)$$

$\mathrm{M}^{0,1,r,d}(x)$ $N \times N$ matrices

$$[\mathrm{M}^0(x)]_{a,b} = \int_{L_-}^{x} \overline{\phi}_a(z)\phi_b(z)\, dz\,, \quad [\mathrm{M}^1(x)]_{a,b} = \int_{x}^{L_+} \overline{\phi}_a(z)\phi_b(z)\, dz\,,$$
$$[\mathrm{M}^r(x)]_{a,b} = \overline{\phi}_a(x)\phi_b(x)\,, \qquad [\mathrm{M}^d(\xi)]_{a,b} = \overline{\phi'}_a(\xi)\phi'_b(\xi)$$

- The integrand $g(\alpha|\xi)$ is a polynomial of order $N-2$ in $e^{i\alpha}$ i.e $g(\alpha,\xi) = \sum_{n=0}^{N-2} b_n(\xi) e^{in\alpha}$ with $b_n(\xi) = I_{n+1}(\xi)$.
- $\boldsymbol{g}(\xi) = (g_0(\xi), \cdots, g_{N-1}(\xi))$ with elements $g_k(\xi) \equiv g(\frac{2\pi k}{N}|\xi) = \sum_{n=0}^{N-1} b_n(\xi) e^{i\frac{2\pi k}{N} n}$
- $I_n(\xi) \rightarrow$ inverse Discrete Fourier Transform of $\boldsymbol{g}(\xi)$
- Quadrature for the integral $J_d^0 = \int_{L_-}^{L_+} I_d(\xi)\, d\xi \sim \sum_{j=1}^{M} I_d(\xi_j) w_j$

Efficient computation of the one-body density matrix elements

Two phases

$$\rho_{d_1,d_2}(x,y) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-i(d_1-1)\alpha} \int_0^{2\pi} \frac{d\beta}{2\pi} e^{-i(d_2-d_1)\beta} \underbrace{\left[\det_N\left(e^{i\alpha}\mathrm{M}^0 + e^{i\beta}\mathrm{M}^2 + \mathrm{M}^1 + \mathrm{M}^n\right) - \det_N\left(e^{i\alpha}\mathrm{M}^0 + e^{i\beta}\mathrm{M}^2 + \mathrm{M}^1\right)\right](x,y)}_{h(\alpha,\beta|x,y)}$$

$\mathrm{M}^{0,1,2,n}(x)$ $N \times N$ matrices

$$[\mathrm{M}^0(x)]_{a,b} = \int_{L_-}^{x} \bar{\phi}_a(z)\phi_b(z)\,dz\,, \quad [\mathrm{M}^1(x)]_{a,b} = \int_{x}^{L_+} \bar{\phi}_a(z)\phi_b(z)\,dz\,,$$

$$[\mathrm{M}^2(x,y)]_{a,b} = \int_{x}^{y} \bar{\phi}_a(z)\phi_b(z)\,dz\,, \quad [\mathrm{M}^n(x,y)]_{a,b} = \bar{\phi}_a(x)\phi_b(y)$$

- $h(\alpha,\beta|x,y) = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} c_{n_1,n_2}(x,y)e^{i\alpha n_1}e^{i\beta n_2}$, with $c_{n_1,n_2}(x,y) = 0$ if $n_1+n_2 > N-1$.
- Compute $\boldsymbol{h}_{k_1,k_2}(x,y) = h\left(\frac{2\pi k_1}{N}, \frac{2\pi k_2}{N}|x,y\right)$ with $k_1,k_2 = 0,\cdots,N-1$
- c_{n_1,n_2} coefficients, or, equivalently, the one-body density matrix elements can be obtained via a 2D Discrete Fourier $c_{n_1,n_2}(x,y) = \frac{1}{N^2}\sum_{k_1=0}^{N-1}\left[e^{-i\frac{2\pi n_1}{N}k_1}\sum_{k_2=0}^{N-1} e^{-i\frac{2\pi n_2}{N}k_2}\boldsymbol{h}_{k_1,k_2}(x,y)\right]$

Comparison with other approaches

- Our method: exact, extremely simple, polynomial complexity $O(N^5)$, numerically stable, no arbitrary precision subroutines

- [Deuretzbacher, Becker, and Santos 2016] MATHEMATICA code $N \leq 60$ for local exchange coefficients

Evaluation times (in seconds) of the local exchange coefficients J_i , single particle densities $\rho_d(x)$ and one-body density matrix elements $\rho_{d_1,d_2}(x,y)$

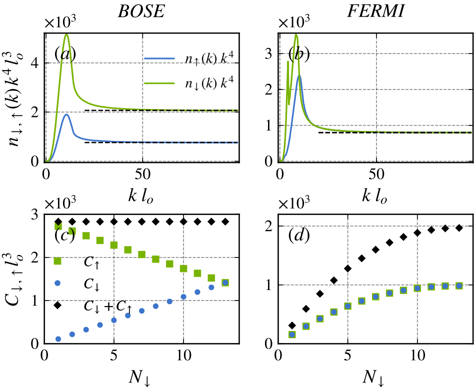
	J_i		$\rho_d(x)$		$\rho_{d_1,d_2}(x,y)$	
N	OIP 2024	Deuretzbacher	OIP 2024	Deuretzbacher	OIP 2024	Deuretzbacher
5	0.07365	1.07895	0.00035	0.05158	0.00139	0.16181
10	0.08448	3.01429	0.00056	0.12947	0.00261	2.08619
15	0.09863	12.7829	0.00094	0.31709	0.00608	14.4611
20	0.11728	364.804	0.00150	4.24389	0.01523	742.804
30	0.28917	3327.92	0.00410	35.6766	0.05876	–
60	1.55943	853895	0.01854	–	0.77707	–
120	28.8309	–	0.12357	–	12.3981	–

- [Loft, Kristensen, Thomsen, Volosniev, and Zinner 2016] CONAN code: Local exchange coefficients for $N \leq 35$. Evaluation time N=10 ($\sim$10 seconds); N=20 ($\sim$10 min); N=30 ($\sim$1 hour)
- [Yang and Pu 2017] One-body density matrix elements: $O(N^7)$ complexity

Tan contacts

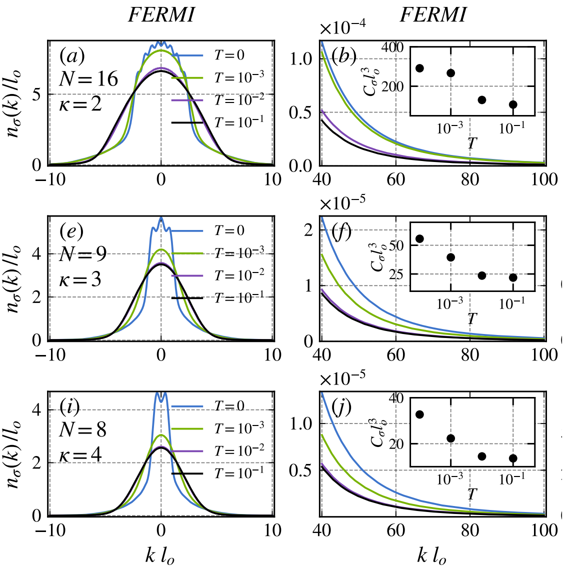
- Momentum distributions have wide tails $\lim_{k\to\infty} n_\sigma(k) \sim C_\sigma/k^4$. C_σ Tan contacts [Tan 2008; Barth an Zwerger 2011; O.I.P and Klümper 2017]
- Bosons: $C = C_\downarrow + C_\uparrow$ is independent of imbalance but the individual contacts increase and decrease linearly as a function of $N_\downarrow$
- Fermions: $C_\uparrow = C_\downarrow$ independent of spin imbalance. Monotonically increasing, but not linearly, function of $N_\downarrow$ with the maximum attained for the balanced system

Harmonically trapped fermions and bosons with $[N_\uparrow, N_\downarrow] = [7, 19]$ and $m = \omega = 1, g = 100$.



Momentum reconstruction

- Temperature dependence of the momentum distribution for temperatures $E_{spin} \ll T \ll E_{charge}$
- Transition from the LL/ferromagnetic liquid phase to the SILL regime
- As the temperature is increased $\rightarrow$ momentum reconstruction in which the number of particles at high momenta decreases: monotonically decreasing contacts ($\lim_{k\to\infty} n_\sigma(k) \sim C_\sigma/k^4$). Present in both bosonic and fermionic systems with any number of components
- Predicted for $\kappa = 2$ [Cheianov, Smith and Zvonarev 2005, O.I.P., Klümper and Foerster 2018; Capuzzi and Vignolo 2020]
- Amplitude of the reconstruction to decrease as the number of components becomes large

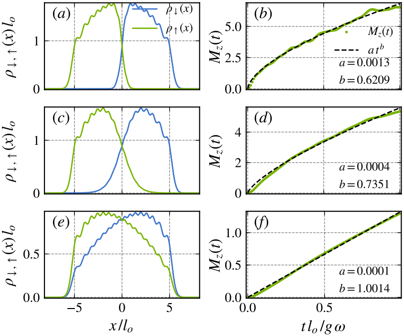


Dynamics from a spin segregated state

- Balanced fermionic system ($N = 16$): initially prepared in an thermal state of the Hamiltonian with a strong gradient G $[-Gz\mathbf{\Psi}^\dagger\sigma^z\mathbf{\Psi}]$ $\rightarrow$ spin segregation
- Integrated magnetization

$$M_z(t) = \int_0^\infty \left[\rho_\downarrow(z,t) - \rho_\uparrow(z,t)\right] - \left[\rho_\downarrow(z,0) - \rho_\uparrow(z,0)\right]\, dz$$

- At $t = 0$ we quench the gradient to zero and let the system evolve [Yang, Guan and Pu, 2016; Yang and Pu 2016; Pecci, Vignolo, and Minguzzi 2022]
- $T = 0$ magnetization presents superdiffusive behaviour $M_z(t) \sim t^{0.62}$
- $T = 0.1\omega$ (spin sector completely excited) $M_z(t) \sim t^{1.001}$
- [Moca et al. 2023] Homogeneous Hubbard model after a quench from a thermal state at *infinite temperature* weak imbalance in the magnetization $\rightarrow$ spin current Kardar-Parisi-Zhang scaling $M_z(t) \sim t^{2/3}$



First row $T = 0$, second row $T = 0.01\omega$ and third row $T = 0.1\omega$.

Correlators in the spin-incoherent regime

Spin-incoherent regime [Berkovich and Lowenstein 1987; Berkovich 1991; Cheianov and Zvonarev 2004, Fiete and Balents 2004; Matveev 2004]

- $g = \infty$ (impenetrable particles) All states in a manifold described by $\boldsymbol{q}$ are degenerate
- Or large but finite g if the thermal energy is much larger than the energy of the spin sector $E_{spin}/g \ll T$.

In both cases $e^{-E_{spin}(\boldsymbol{N},\boldsymbol{q},n)/gT} \sim 1$
Grandcanonical ensemble

$$\rho_\sigma^{SILL}(x,y) = \frac{1}{Z}\sum_{N=0}^{\infty}\sum_{q_1<\cdots<q_N}\sum_{N_1=0}^{N}\sum_{N_2=0}^{N-N_1}\cdots\sum_{N_{\kappa-1}=0}^{N-(N_1+\cdots+N_{\kappa-2})}\sum_{n=1}^{N!/[N_1!\cdots N_\kappa!]} \times e^{-\sum_{j=1}^{N}\varepsilon(q_j)/T+\sum_{\sigma'=1}^{\kappa}\mu_{\sigma'}N_{\sigma'}/T}\langle\Phi_{\boldsymbol{N},\boldsymbol{q},n}|\Psi_\sigma^\dagger(x)\Psi_\sigma(y)|\Phi_{\boldsymbol{N},\boldsymbol{q},n}\rangle$$

Z grandcanonical partition function

Determinant representation for trapped SILL correlators

$$\rho_\sigma(x,y) = \det\left[\mathbf{1} - (1+\epsilon f(\sigma))\,\mathrm{V}_T + f(\sigma)\mathrm{R}_T\right] - \det\left[\mathbf{1} - (1+\epsilon f(\sigma))\,\mathrm{V}_T\right]\ ,$$

V_T and R_T infinite matrices ($a, b = 1, 2, \cdots$)

$$[\mathrm{V}_T(x,y)]_{a,b} = (\vartheta(a)\vartheta(b))^{1/2}\int_x^y \overline{\phi}_a(z)\phi_b(z)\,dz$$

$$[\mathrm{R}_T(x,y)]_{a,b} = (\vartheta(a)\vartheta(b))^{1/2}\,\overline{\phi}_a(x)\phi_b(y)$$

- $\phi_a(z)$ and $\varepsilon(a)$ are the single particle wavefunctions and their respective energies
- $\vartheta(a)$ is a generalized Fermi function

$$\text{Fermi function } \vartheta(a) = \frac{\left(e^{\mu_1/T} + \cdots + e^{\mu_\kappa/T}\right) e^{-\varepsilon(a)/T}}{1 + \left(e^{\mu_1/T} + \cdots + e^{\mu_\kappa/T}\right) e^{-\varepsilon(a)/T}}$$

-

$$f(\sigma) = \left(\frac{e^{\mu_\sigma/T}}{e^{\mu_1/T} + \cdots + e^{\mu_\kappa/T}}\right)\ ,\quad \sigma = \{1, \cdots, \kappa\}$$

- Nonequilibrium (quenches of the trapping potential) $\phi_a(z) \to \phi_a(z,t)$

Generalization of: $\kappa = 1$ [Pezer and Buljan 2007; del Campo 2008; Atas et al. 2017; O.I.P. 2020]; $\kappa = 2$ [O.I.P. 2023];

Determinant representation for $T = 0$ homogeneous SILL correlators

- Zero temperature and equal chemical potentials
- The single particle orbitals (PBC) $\phi_a(z) = e^{ik_a z}/\sqrt{L}$ with $k_a = 2\pi a/L\,,\, a = 0, \pm 1, \cdots$.
- Arbitrary statistics $\epsilon = -e^{i\pi\varphi}$ with $\varphi \in [0,1]$. The fermionic (bosonic) case is recovered for $\varphi = 0$ ($\varphi = 1$).

Fredholm determinant representation

$$\rho_\sigma^h(x,y) = \det\left[\mathbf{1} - \left(1 - \frac{e^{i\pi\varphi}}{\kappa}\right)\hat{\mathrm{v}} + \frac{1}{\kappa}\hat{\mathrm{r}}\right] - \det\left[\mathbf{1} - \left(1 - \frac{e^{i\pi\varphi}}{\kappa}\right)\hat{\mathrm{v}}\right]$$

$\hat{\mathrm{v}}$ and $\hat{\mathrm{r}}$ integral operators $(\hat{\mathrm{v}}\phi)(k) = \int_{-k_F}^{k_F} \mathrm{v}(k,k')\phi(k')\, dk'$ ($k_F = \pi D$)

$$\mathrm{v}(k,k') = \frac{\sin\left[(k-k')(y-x)/2\right]}{\pi(k-k')}$$

$$\mathrm{r}(k,k') = \frac{e^{i(k+k')(y-x)/2}}{2\pi}$$

- $\kappa = 1$ Bosons [Schultz 1963; Lenard 1966; Korepin and Slavnov 1990]; Anyons [O.I.P., Korepin and Averin 2008]; Fermions $\sin[k_F(y-x)]/\pi(y-x)$
- $\kappa = 2$ Bosons and fermions [Izergin and Pronko 1998]; Anyons [O.I.P. 2019]

Large distance asymptotics for SILL correlators

Large distance asymptotics of the static correlators: [Cheianov and Zvonarev 2004; O.I.P 2019]
Main ingredient: asymptotics of the generalized sine-kernel [Kitanine, Kozlowski, Maillet, Slavnov and Terras 2009]

$$\xi = -\left(1 - \frac{e^{i\pi\varphi}}{\kappa}\right)\,, \quad \nu = -i\frac{\ln\kappa}{2\pi} - \frac{\varphi}{2}$$

Asymptotics

$$\rho_\sigma^h(x,0) = \frac{1}{\kappa}\frac{\pi e^{\mathcal{C}(\nu)}}{\xi\sin(\pi\nu)}\frac{e^{-2ik_F\nu x}}{x^{2\nu^2+1}}\left[\frac{(2k_F x)^{-2\nu}}{\Gamma^2(-\nu)}e^{-ik_F x} - \frac{(2k_F x)^{2\nu}}{\Gamma^2(\nu)}e^{ik_F x}\right]\,, \quad x\to\infty$$

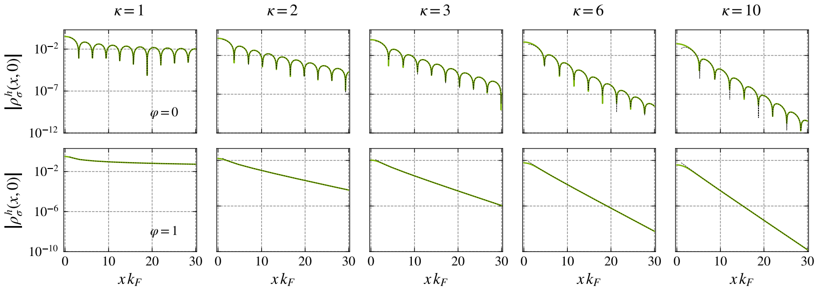
with $\mathcal{C}(\nu) = -2\nu^2\left[1+\ln(2k_F)\right] + 2\nu\ln\left[\frac{\Gamma(\nu)}{\Gamma(-\nu)}\right] - 2\int_0^\nu \ln\left[\frac{\Gamma(t)}{\Gamma(-t)}\right]\,dt\,,$

- Main feature: $Re|e^{-2ik_F\nu x}| = e^{-xk_F\ln\kappa/\pi}$ exponential decay even at $T=0$
- For $\varphi\sim 1$ (bosons) the first term is dominant; when $\varphi\sim 0$ (fermions) both terms are important

- $\kappa = 1$ Bosons [Vaidya and Tracy 1979; Jimbo, Miwa, Mori and Sato 1980; Gangardt 2004]; Anyons [Calabrese and Mintchev 2008]; Fermions $\sin[k_F(y-x)]/\pi(y-x)$
- $\kappa = 2$ Fermions [Berkovich and Lowenstein 1987, Berkovich 1991; Cheianov and Zvonarev 2004; Fiete and Balents 2004]; Bosons [Cheianov, Smith and Zvonarev 2005; O.I.P. 2019]; Anyons [O.I.P. 2019]

Numerical evidence

Plots of the absolute value of the correlator $|\rho^h_\sigma(x,0)|$ (green continuous line) computed from the Fredholm determinant representation and the absolute value of the asymptotics (black dashed line) for fermionic (first row) and bosonic (second row) systems with $\kappa = \{1, 2, 3, 6, 10\}$ and $k_F = \pi D = 1$.



Conclusions

- New efficient method of computing the multidimensional integrals describing the charge sector of strongly interacting spinor gases
- Full mapping of the momentum reconstruction LL/ferromagnetic liquid $\rightarrow$ spin-incoherent regime
- Determinant formulae for spin incoherent correlators

Possible extensions

- Spinor gases with different inter- and intra-particle repulsion strengths, Bose-Fermi mixtures
- Efficient numerical evaluation for the full counting statistics for large but finite repulsion
- Determinant representation for the full counting statistics in the spin-incoherent regime