Ultracold gases in 1D:
Dipolar-enhanced stability, BAF mapping and anyon construction

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Workshop on Mathematical physics for quantum science
2024/11/4, Hangzhou

Outline

- Dipolar-enhanced stability of sTG gas

  Yu Chen, XC, PRL 131, 203002 (2023)

- Generalized BAF mapping and anyon construction

  Haitian Wang, Yu Chen, XC, arxiv: 2410.21632

Yu Chen (陈豫)

Haitian Wang (王海天)

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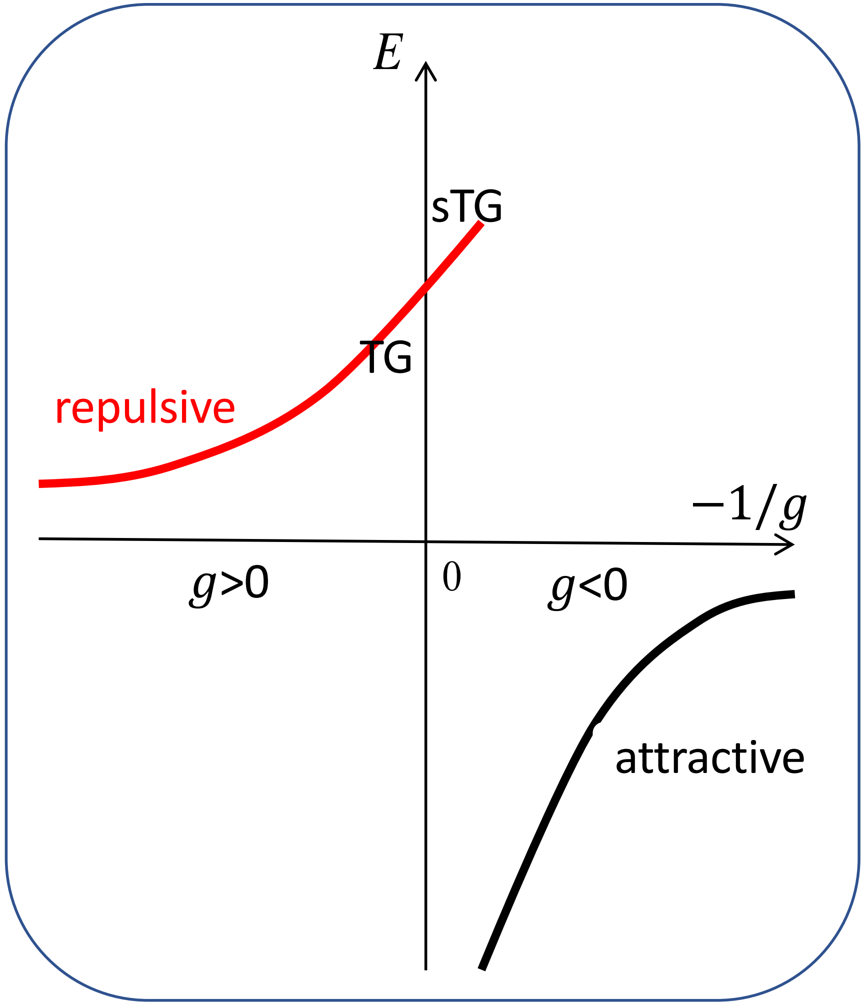
Strongly repulsive atomic gas with short-range interaction:

$$\boldsymbol{g\delta(x)}$$

How repulsive the system could be?

- Hard-core: $g = +\infty$ ←→ Pauli exclusion
- *Beyond* hard-core?

Repulsive *Bose* gas in 1D: TG and sTG



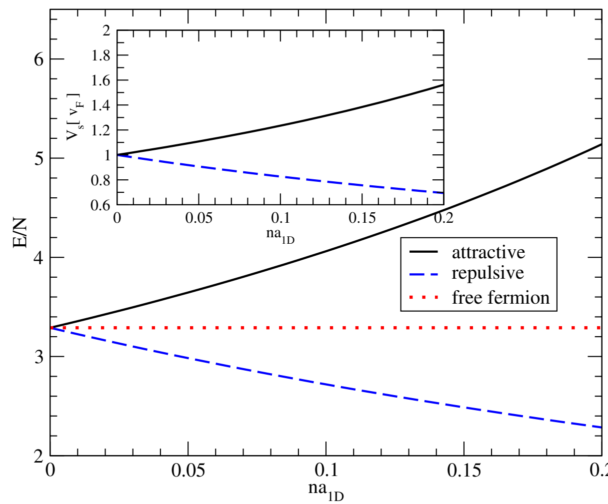
- **Tonks-Girardeau (TG):** $g \to +\infty$

$$\left|Det[\phi_i(x_j)]\right| \Longleftrightarrow Det[\phi_i(x_j)]$$

Hard-core bosons ⟺ free fermions

M. Girardeau, J. Math. Phys. 1, 516 (1960)

- **Super-TG (sTG):** $g \to -\infty$



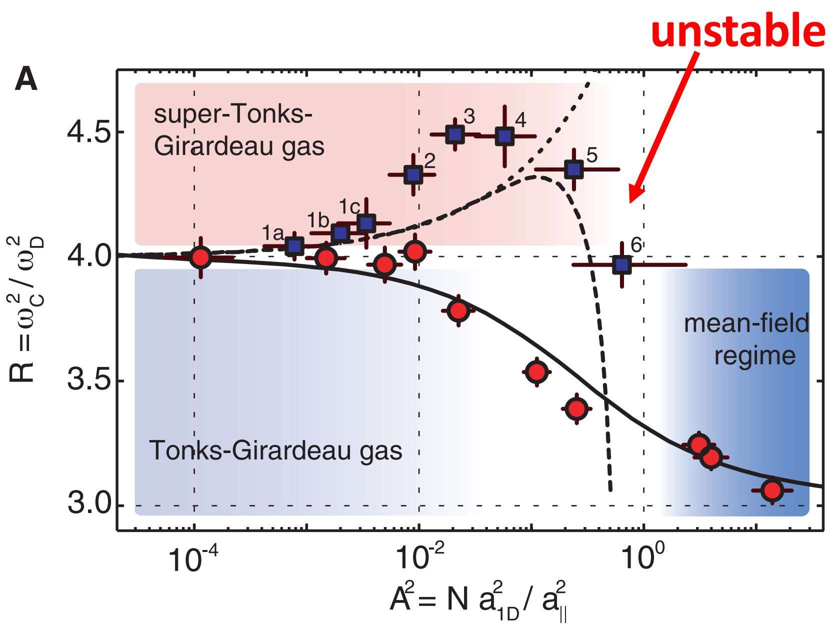
Astrakharchik, Boronat, Casulleras, Giorgini, PRL 95, 190407 (2005)
Batchelor, Bortz, Guan, Oelkers, J. Stat. Mech. (2005) L10001

Expt on TG and sTG Bose gas:

Realization of an Excited, Strongly Correlated Quantum Gas Phase

Elmar Haller,[1] Mattias Gustavsson,[1] Manfred J. Mark,[1] Johann G. Danzl,[1] Russell Hart,[1] Guido Pupillo,[2,3] Hanns-Christoph Nägerl[1*]

Science **325**, 1224 (2009)



Instability in sTG regime:

for a_{3D}. Our results must be connected to the fact that the energy spectrum of the system changes dramatically across the CIR, from the TG to the sTG regime (*19*). The system acquires a deeply lying ground state together with a family of lower lying many-body excited states, potentially opening up new decay channels. Also, the CIR strongly

Fermionic TG and sTG:

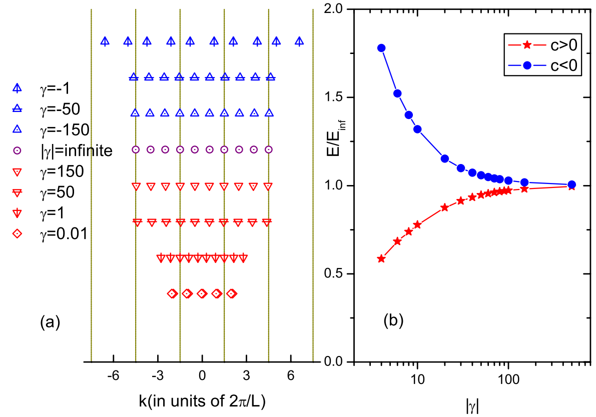
- Construction of TG ground state:

$$\Psi = \psi_A \psi_S$$

$$\psi_S = \left\{ \sum_{\alpha=1}^{N!/(n!m!)} P_\alpha \right\} \{ Q_1 (\mathcal{Y}_1^{[n,m]} Z_1) \} = \sum_{\alpha=1}^{N!/(n!m!)} \{ \mathcal{Y}_\alpha^{[n,m]} Q_\alpha \} Z_\alpha .$$

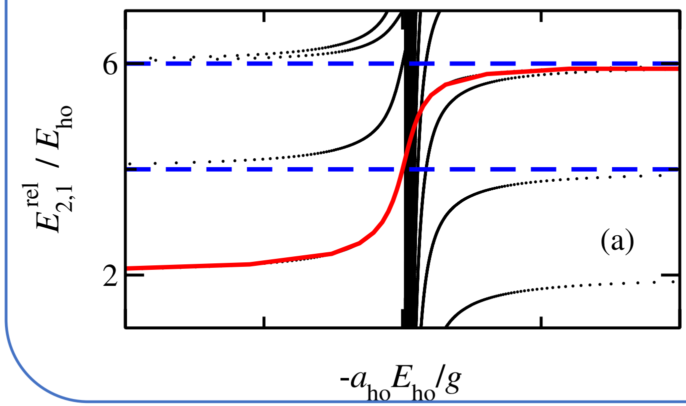
Guan, Chen, Wang, Ma, PRL 102, 160402 (2009)

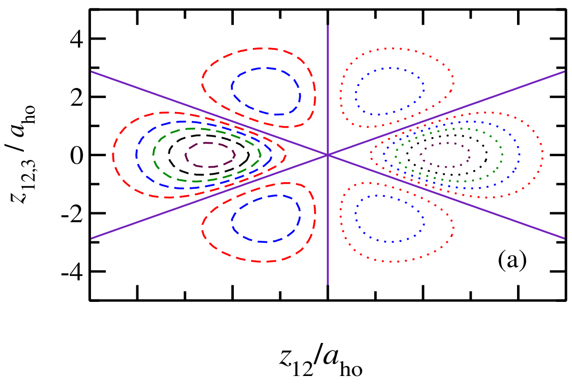
- BA solution of Fermionic sTG:



Guan, Chen, PRL 105, 175301 (2010)

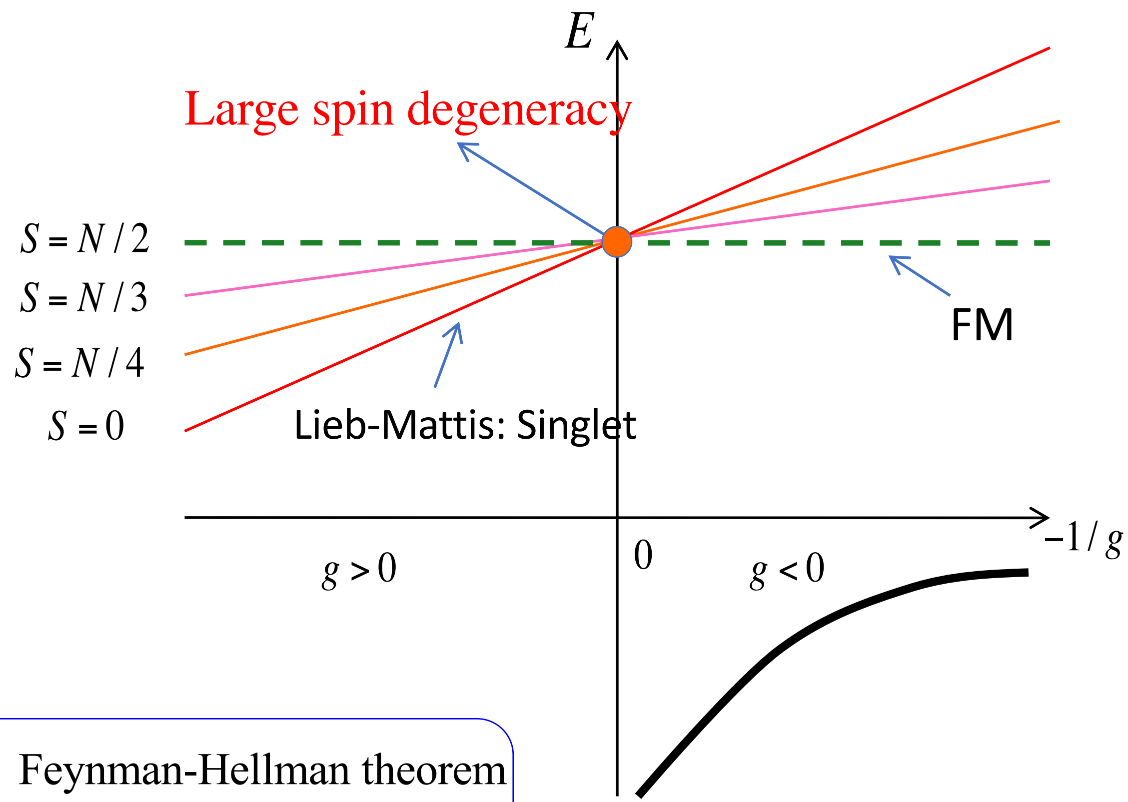
- Adiabatic evolution and spin texture:





Gharashi, Blume, PRL 111, 045302 (2013)

Specialty of fermionic TG&sTG: large spin degeneracy and FM transition

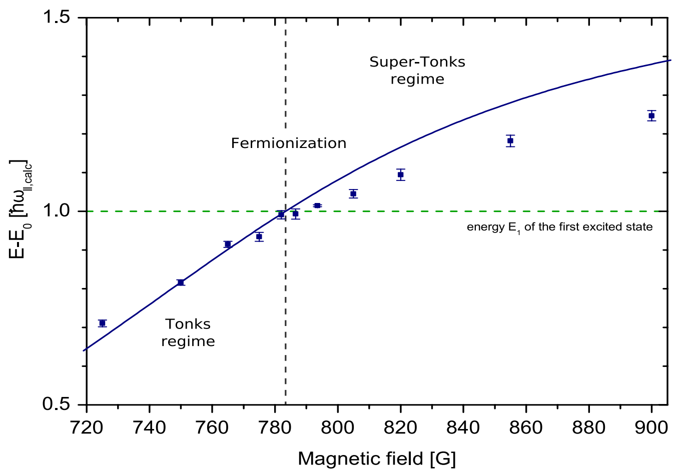


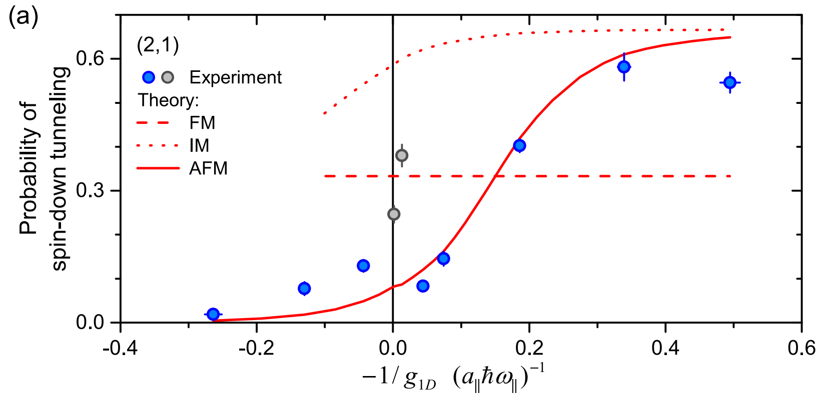
✧ Feynman-Hellman theorem

$$\frac{dE}{d(-1/g)} = g^2 \langle \psi^+ \psi^+ \psi \psi \rangle \geq 0$$

Cui, Ho, PRL 110, 165302 (2013); PRA 89, 023611 (2014)

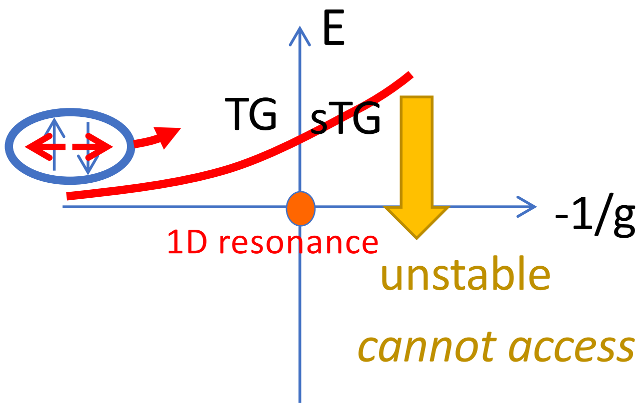
Expts on fermionic TG and sTG: (Heidelberg group)





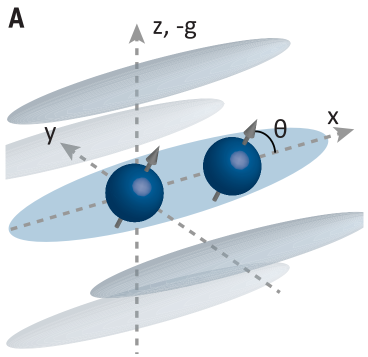
- Two fermions from TG to sTG
  PRL 108, 075303 (2012)

- AFM spin chain (tunnelling)
  PRL 115, 215301 (2015)



cannot access more repulsive regime!

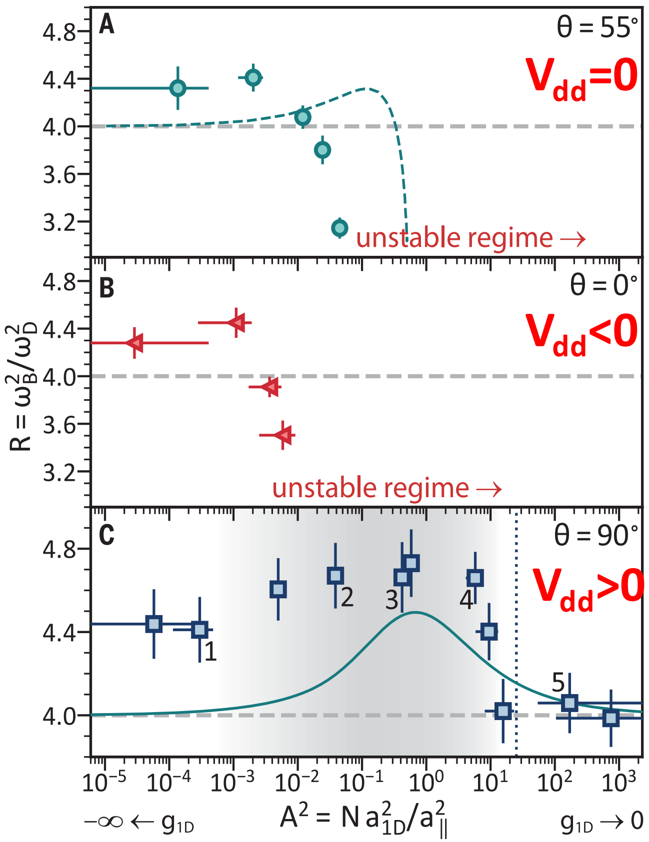
A surprise: **Ultrastable** sTG gas with a weak dipole repulsion



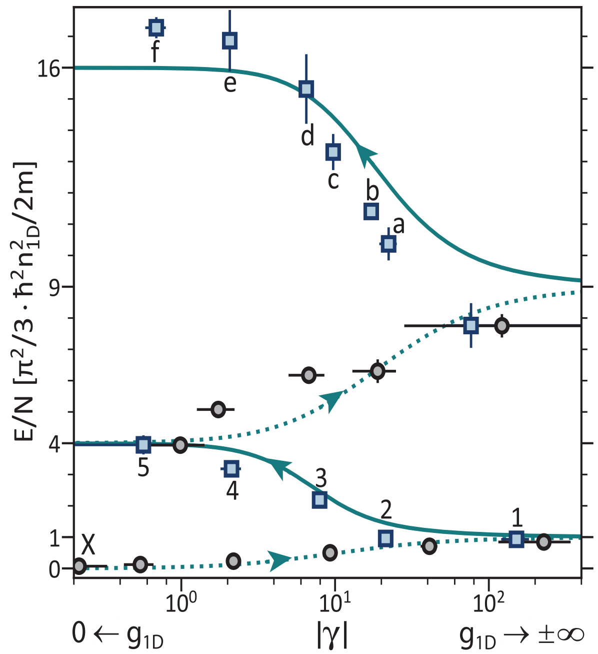
$\theta = 0$: attractive

$\theta = 90$: repulsive

$$V_{dd}(r) = \frac{d^2}{4\pi\varepsilon_0 r^3}(1 - 3cos^2\theta)$$



Puzzle1: stability depending on the sign of dipole interaction



Puzzle2: weak dipole --- sTG spectrum unchanged

Stanford group: Science 371, 296 (2021)

To explain:

- Why sTG stability depends on the sign of V_{dd}
- Why a weak V_{dd}, which barely affects the spectrum, significantly changes the stability

Our work: Exact solutions of three bosons/fermions in a harmonic trap

$$H = \sum_i \left(-\frac{1}{2m}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2} m\omega^2 x_i^2 \right) + \sum_{\langle i,j \rangle} \left(g\delta(x_i - x_j) + V_{dd}(x_i - x_j) \right)$$

$=D/|x|^3$ for $|x|<r_c$ and $=0$ otherwise

$$\Psi(r,\rho) = \sum_{mn} c_{mn} \phi_m(r) \phi_n(\rho)$$

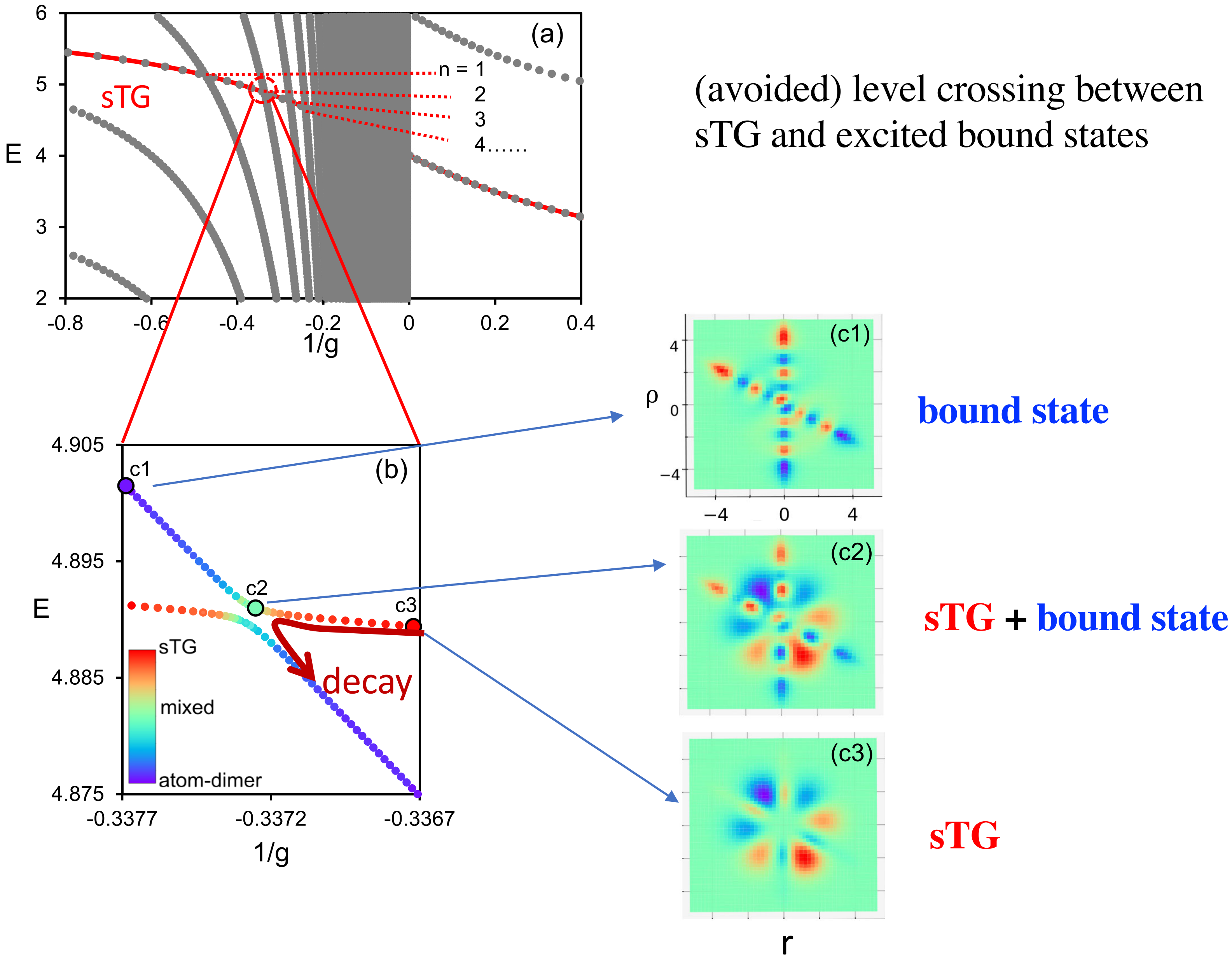
$$(E - \epsilon_m - \epsilon_n) c_{mn} = g \sum_{ij} c_{ij} \phi_i(0) \left(\phi_m(0) \delta_{j,n} - A_{mn,j} \right) + D \sum_{ij} c_{ij} \left(B_{m,i} \delta_{j,n} - B^+_{mn,ij} + B^-_{mn,ij} \right)$$

1+2 fermions

Yu Chen, XC, PRL 131, 203002 (2023)

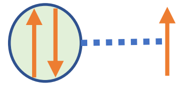
Understanding the loss mechanism of sTG (V_{dd}=0):

(avoided) level crossing between sTG and excited bound states



Analytical description of sTG and bound states:

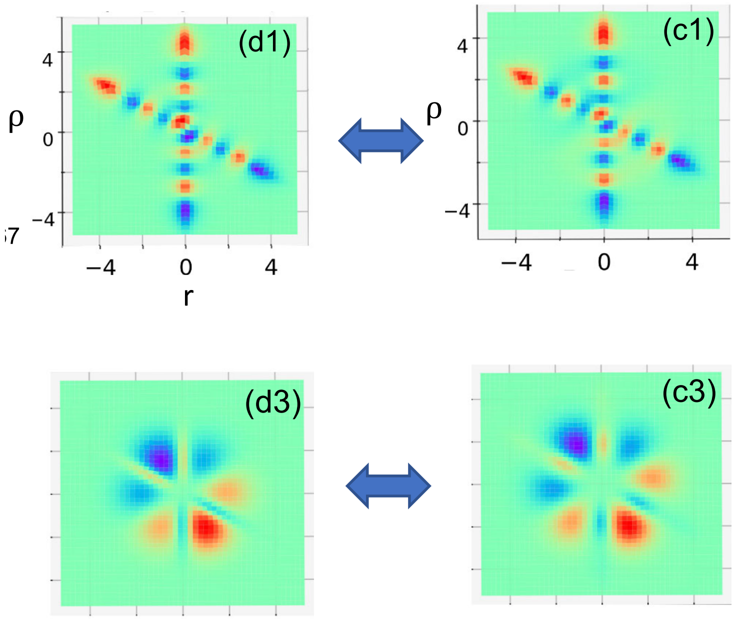
- sTG: $\Psi_{sTG} = \Psi_0 - \frac{1}{g}\Psi_1;$ $\quad E_{sTG} = E_0 - \frac{3J}{2g}$ ⟸ $H_{\text{eff}} = \sum_l \frac{J_l}{g}\left(\mathbf{s}_l \cdot \mathbf{s}_{l+1} - \frac{1}{4}\right)$
- atom-dimer: $\psi_{ad}^{(m)} = \Phi_d(r)\phi_m(\rho) - (x_2 \leftrightarrow x_3),$ $\quad E_{ad}^{(m)} = E_d + \epsilon_m$ ⟸



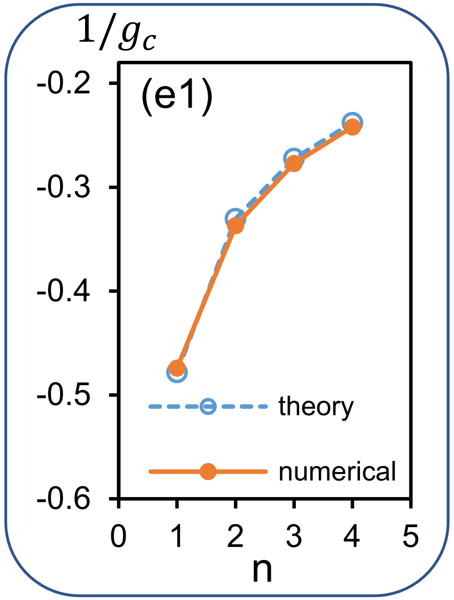
analytical | exact

bound state

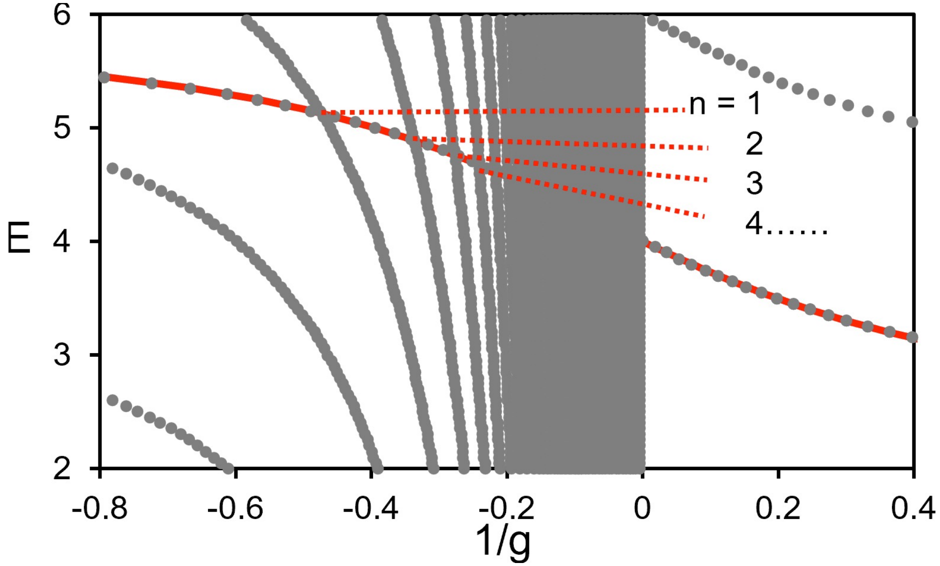
sTG



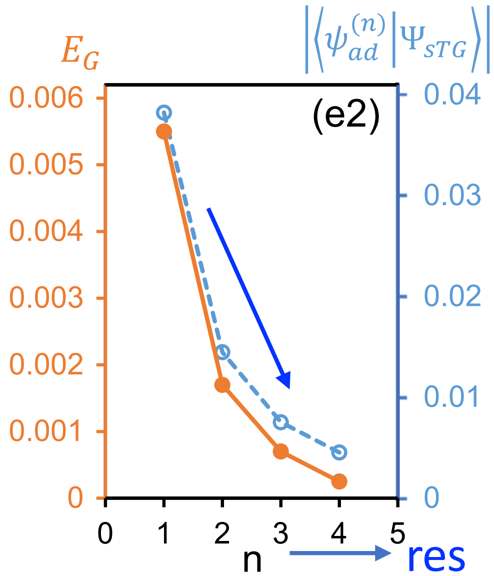
Location of level crossing



Possibility of sTG decaying to bound states:



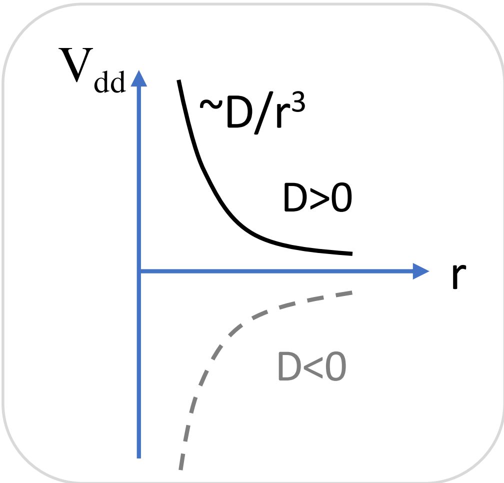
Energy gap vs. inter-branch coupling

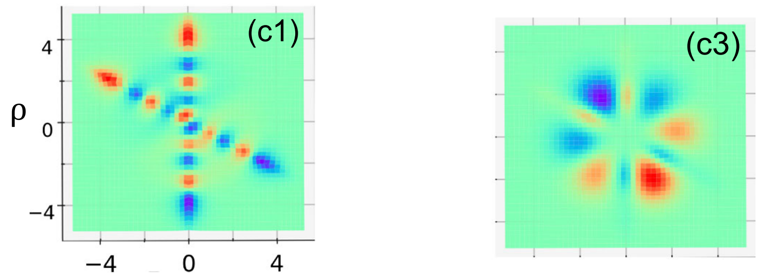


Index of level crossing

**Closer to resonance, smaller inter-branch coupling (reduced energy gap) → more stable sTG

Effect of dipolar interaction ($V_{dd} \neq 0$) :





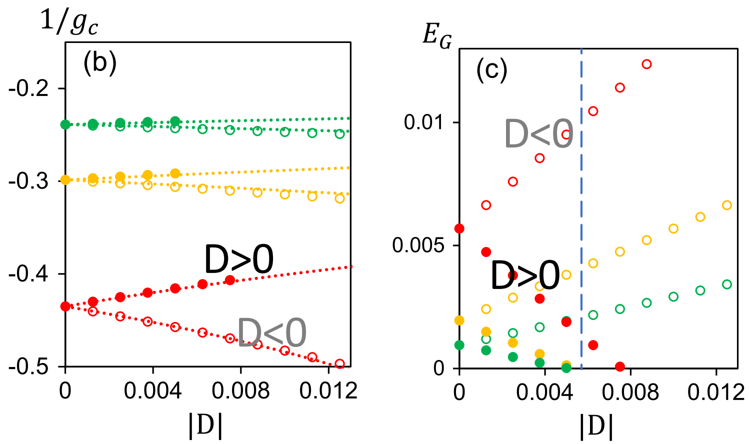
Localized bound state

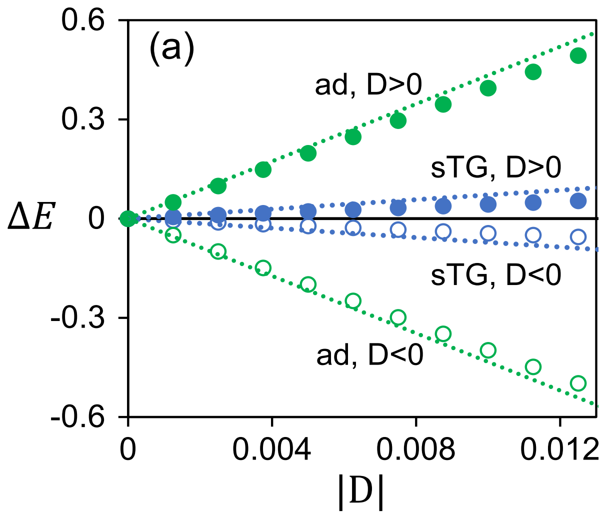
Extended sTG

$$\langle V_{dd} \rangle_{ad} \gg \langle V_{dd} \rangle_{sTG}$$

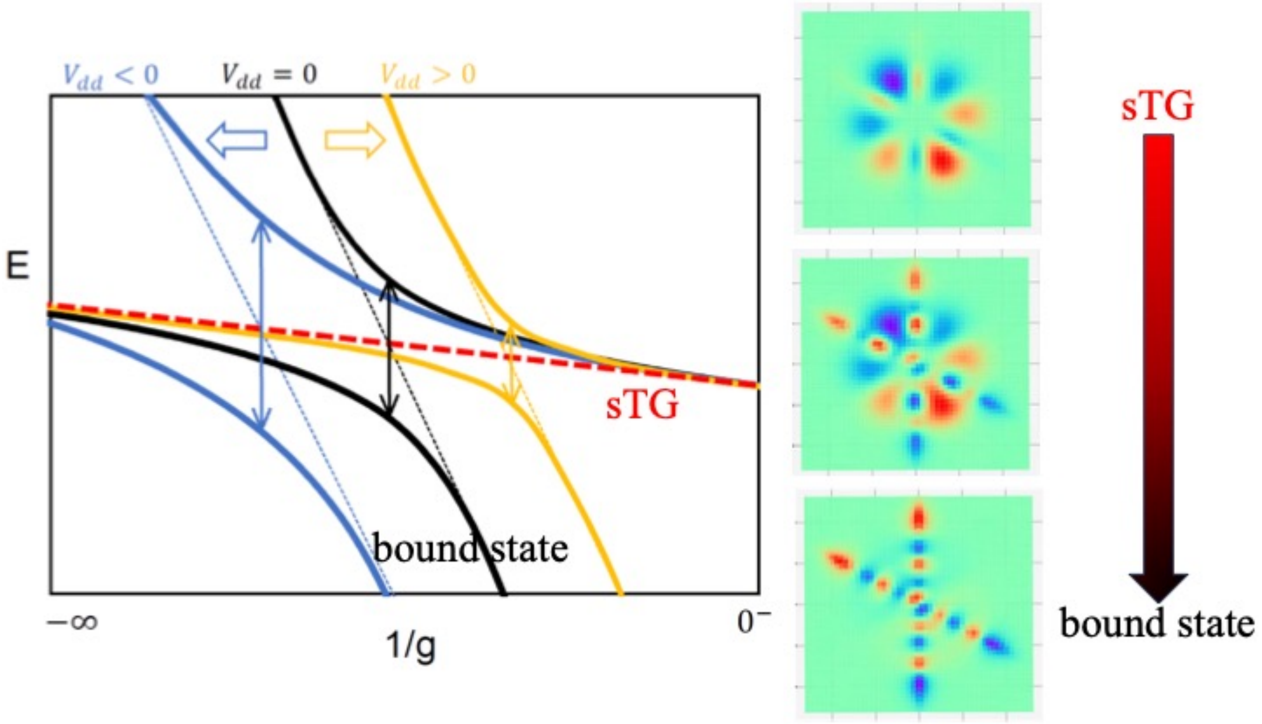
Distinct energy responses to V_{dd}!

New position and width for level-crossing:





sTG stability changed by a weak dipole force:



In the presence of dipole force:

Extended sTG vs. localized bound state → distinct spectral response to V_{dd} → shift of level crossing → change of inter-branch coupling → enhanced/reduced sTG stability

** tune the stability of target state by manipulating its decay channel under designed potentials

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  Yu Chen, XC, PRL 131, 203002 (2023)

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Exchange statistics of quantum particles:

$$\psi(r_1, r_2) = e^{i\alpha}\psi(r_2, r_1)$$

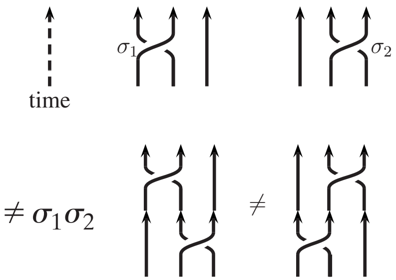
Boson	Anyon	Fermion
$\alpha = 0$	$0 < \alpha < \pi$	$\alpha = \pi$

Fractional statistics

Laughlin's wf: (PRL1983)

$$\psi_{g.s.}(\{z_i\},\{\bar{z}_i\}) = \prod_{i<j}(z_i - z_j)^3 \prod_i \exp -|z_i|^2/4l_h^2$$

Fractional quantum Hall effect



$\sigma_2\sigma_1 \neq \sigma_1\sigma_2$

Topological quantum computation

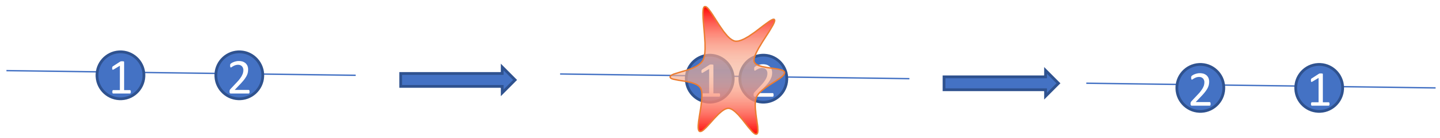
Statistics & Interaction

$$\psi(-r) = e^{i\alpha}\psi(r), \quad r \equiv r_2 - r_1$$

Boson	**Anyon**	Fermion
$\alpha = 0$	$\mathbf{0 < \alpha < \pi}$	$\alpha = \pi$
↓	↓	↓
$U_s \sim \delta(r)$	**?**	$U_p \sim \nabla_r \delta(r) \nabla_r$
S-wave interaction		P-wave interaction

Statistics & Interaction in 1D

They are even more correlated because particles can only exchange via collision



Boson	Anyon	Fermion
$\alpha = 0$	$\mathbf{0 < \alpha < \pi}$	$\alpha = \pi$
$U_s \sim \delta(x)$		$U_p \sim \partial_x \delta(x) \partial_x$

Exact mapping

Bose-Fermi mapping/duality

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 1, NUMBER 6 NOVEMBER-DECEMBER, 1960

Relationship between Systems of Impenetrable Bosons and Fermions in One Dimension*†

M. GIRARDEAU‡
Brandeis University, Waltham, Massachusetts

(Received March 3, 1960)

Hard-core bosons $\Longleftrightarrow$ free fermions

$$\left|Det[\phi_i(x_j)]\right| \Longleftrightarrow Det[\phi_i(x_j)]$$

VOLUME 82, NUMBER 12 PHYSICAL REVIEW LETTERS 22 MARCH 1999

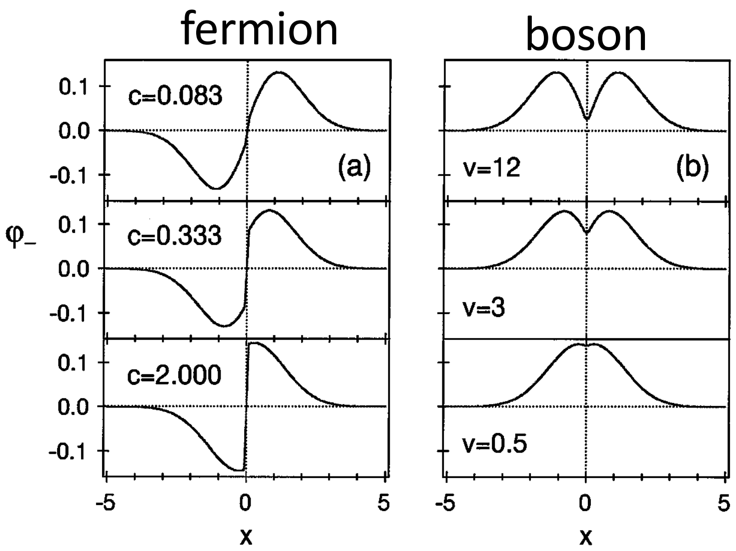
Fermion-Boson Duality of One-Dimensional Quantum Particles with Generalized Contact Interactions

Taksu Cheon[1,2] and T. Shigehara[3]
[1]*Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan*
[2]*Theory Division, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan*
[3]*Department of Information and Computer Sciences, Saitama University, Urawa, Saitama 338-8570, Japan*

(Received 12 June 1998)

For a system of spinless one-dimensional fermions, the nonvanishing short-range limit of a two-body interaction is shown to induce the wave-function discontinuity. We prove the equivalence of this fermionic system and the bosonic particle system with a two-body δ-function interaction with the reversed role of strong and weak couplings. [S0031-9007(99)08775-X]

Bose-Fermi mapping/duality



$$v = \frac{1}{c}$$

Non-interacting fermions = hard-core bosons

Resonant fermions = non-interacting bosons

Cheon, Shigehara, PRL 82, 2536 (1999)

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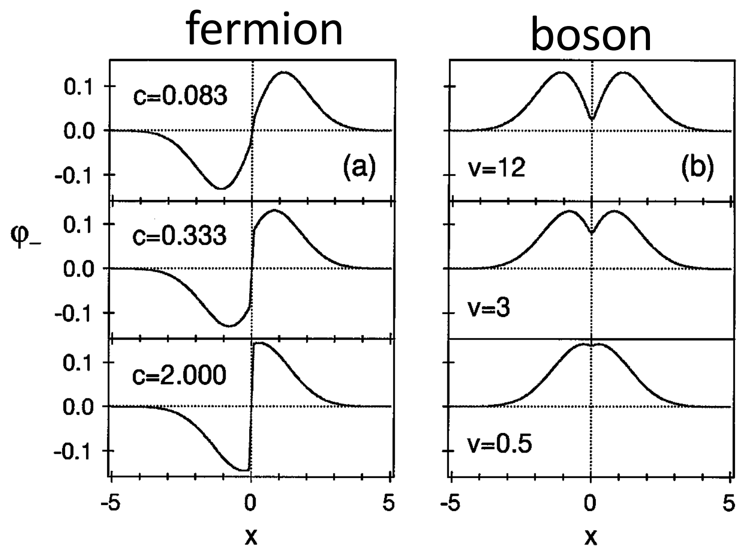
Bose-Fermi mapping/duality



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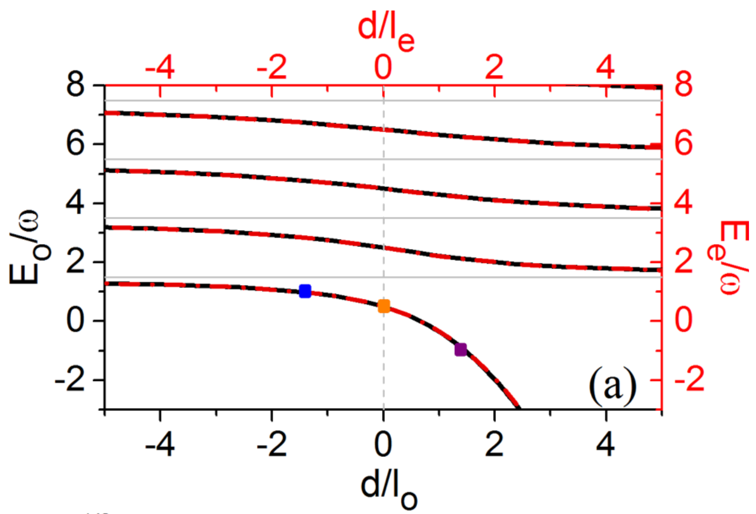
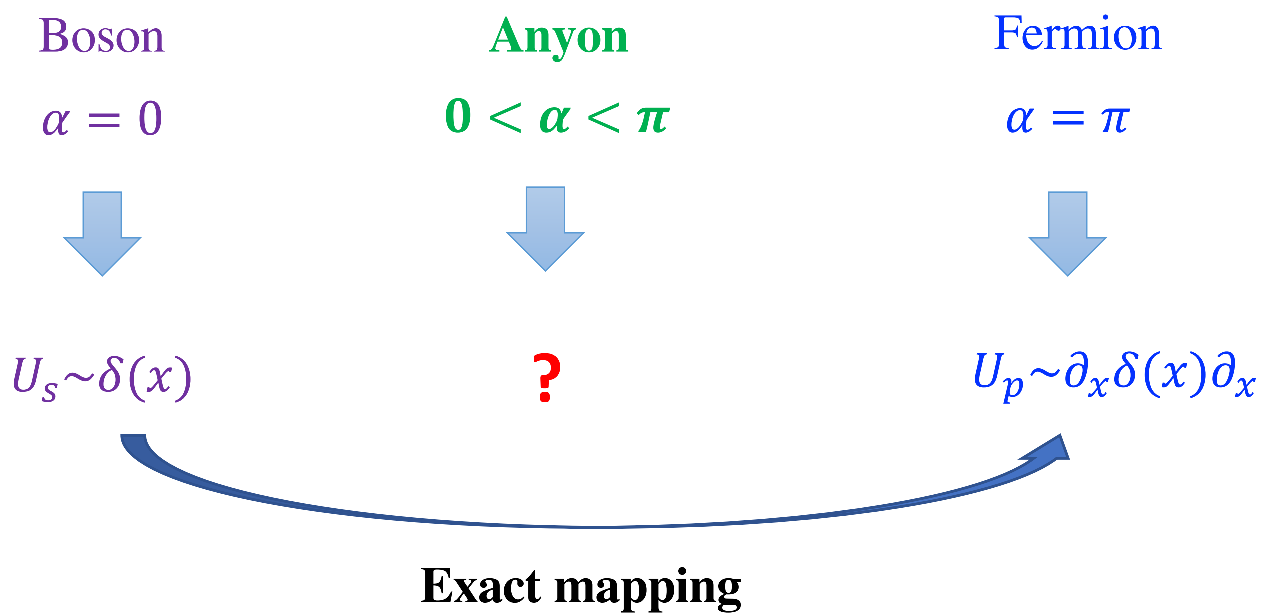


BF duality ⟺ **the same scattering length**

P-wave RG equation $$\frac{1}{U} = \frac{m}{2l_o} - \frac{1}{L}\sum_k \frac{k^2}{2\epsilon_k}$$

Cui, PRA 94, 043636 (2016)



How about anyons --- interaction? exact mapping?

1D anyons --- exact solutions:

Kundu, PRL 83, 1275 (1999)

$$H_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \boxed{g_{1D}\sum_{1\le i<j\le N}\delta(x_i - x_j)}$$

assume anyons experience the same contact potential as bosons **(?)**

Permutation relations:

$$\tilde{\psi}^\dagger(x_1)\tilde{\psi}^\dagger(x_2) = e^{i\kappa\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}^\dagger(x_1)$$

$$\tilde{\psi}(x_1)\tilde{\psi}^\dagger(x_2) = e^{-i\kappa\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}(x_1) + \delta(x_1 - x_2)$$

Pseudo-boson **(?)**

$\epsilon(x-y) = \pm 1$ for $x>y,\ x<y$ and $=0$ for $x=y$

Under PBC

$$\mathrm{e}^{\,\mathrm{i}k_jL} = -e^{\mathrm{i}\kappa(N-1)}\prod_{l=1}^{N}\frac{k_j - k_l + \mathrm{i}c'}{k_j - k_l - \mathrm{i}c'}$$

$$\boxed{c' = c/\cos(\kappa/2)}$$

modified coupling

$$H_N = -\sum_k^N \partial_{x_k}^2 + \sum_{<k,l>}\delta(x_k - x_l)\left(c + i\kappa(\partial_{x_k} + \partial_{x_l})\right) + \gamma_1 \sum_{<j,k,l>}\delta(x_j - x_k)\delta(x_l - x_k) + \gamma_2\sum_{<k,l>}\left(\delta(x_k - x_l)\right)^2$$

Bosons with double-delta potential

1D anyons --- lattice version:

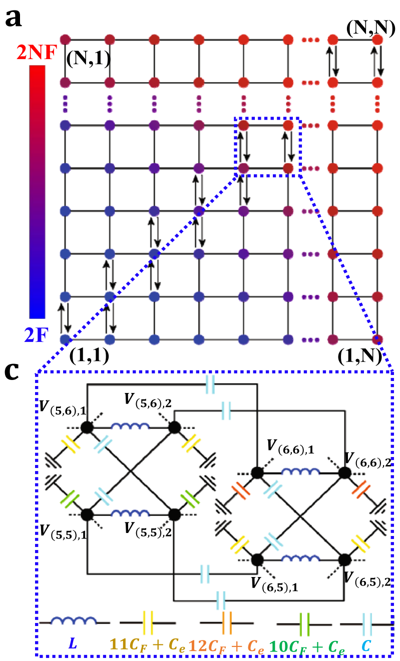
$$\hat{H}^a = -J\sum_{j=1}^{L}(\hat{a}_j^\dagger \hat{a}_{j+1} + \text{h.c.}) + \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j(\hat{n}_j - 1)$$

Anyon Hubbard model

$$\hat{a}_j = \hat{b}_j \exp\left(\mathrm{i}\theta \sum_{i=1}^{j-1}\hat{n}_i\right)$$ (pseudo-boson)

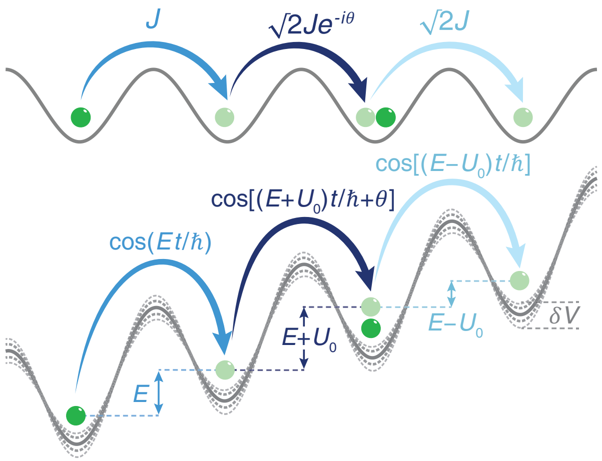
$$\hat{H}^b = -J\sum_{j=1}^{L}(\hat{b}_j^\dagger \hat{b}_{j+1} e^{\mathrm{i}\theta\hat{n}_j} + \text{h.c.}) + \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j(\hat{n}_j - 1)$$

Bose Hubbard with density-dependent hopping



Electric circuit

Nat. Comm. 13, 2392 (2022)



Ultracold bosons in tilted lattice

2306.01737

Questions:

- How to describe the interaction of anyons?
- Any mapping between anyon and boson/fermion?
- Realize anyon without lattice?

Our work:

- General description of short-range interaction
- **Boson-Anyon-Fermion mapping**
- Anyon construction in a spin-1/2 Fermi gas

Haitian Wang, Yu Chen, XC, arxiv: 2410.21632

Short-range interaction ←→ Asymptotic behavior of two-body wavefunction

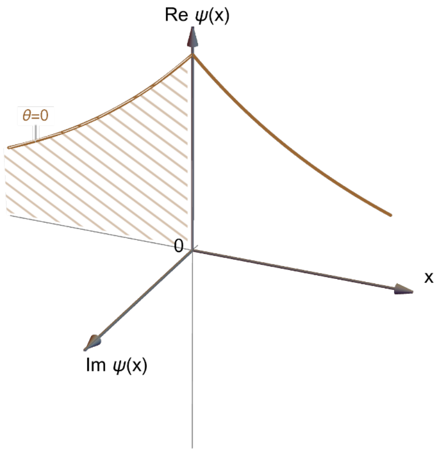
statistics

$$\psi(x \to 0^+) \propto (x - l); \qquad (l\text{: } scattering\ length)$$

$$\psi(x \to 0^-) \propto e^{i\alpha}(-x - l)$$

Boson

$\alpha = 0$

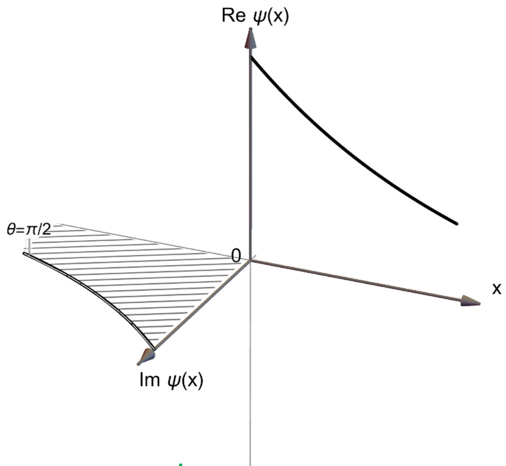


$\psi(0^+) = \psi(0^-)$

$U_s \sim \delta(x)$

Anyon

$\mathbf{0 < \alpha < \pi}$



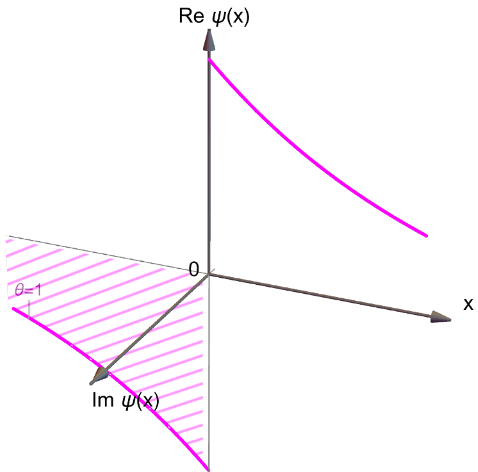
$\psi(0^+) \neq \psi(0^-)$

$\psi'(0^+) \neq \psi'(0^-)$

~~U_s~~, ~~U_p~~

Fermion

$\alpha = \pi$

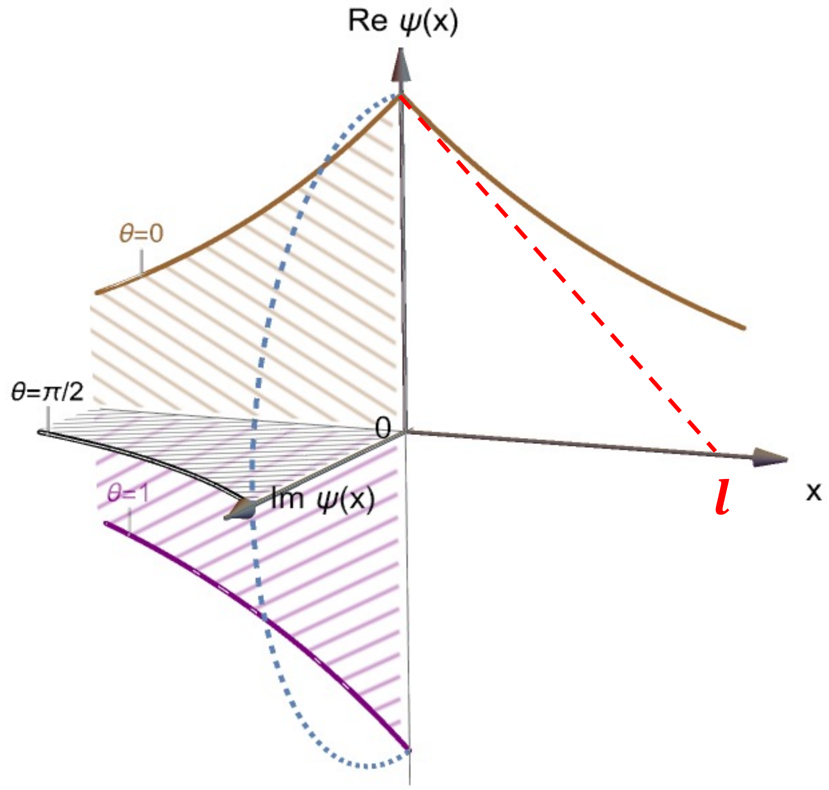


$\psi'(0^+) = \psi'(0^-)$

$U_p \sim \partial_x \delta(x) \partial_x$

A unified description of interaction effect for *all* statistics:

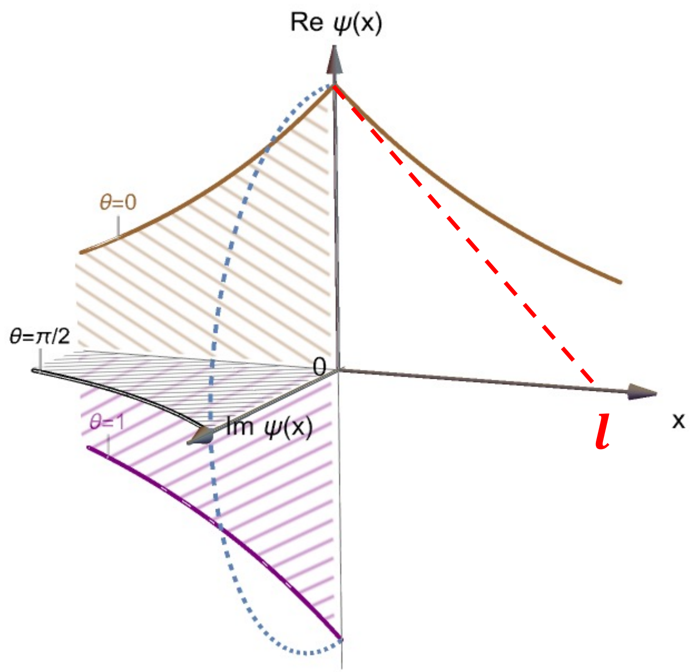
$$\lim_{x \equiv x_j - x_i \to 0^+} \left(\frac{1}{l} + \partial_x \right) \Psi(x_1, x_2, ...x_N) = 0$$



- Boundary condition at one side $(x \to 0^+)$ is enough to describe interaction effect
- Short-range behavior at the other side $(x \to 0^-)$ is determined by exchange statistics

Boson-Anyon-Fermion mapping:

Short-range interacting 1D systems with different statistics (boson, anyon or fermion) can be mapped to each other in both energy and real-space wavefunction as long as they are associated with the same scattering length ***l***.



$$\lim_{x\equiv x_j-x_i\to 0^+}\left(\frac{1}{l}+\partial_x\right)\Psi(x_1,x_2,...x_N)=0$$

Proof: given the same l for all α-systems,

- For $x_1 < x_2 < \cdots < x_N$, all α-systems share the same ψ and E:

  at $x_i \neq x_{i+1}$, governed by the same H_0;
  at $x_{i+1} - x_i \to 0^+$, governed by the same BC

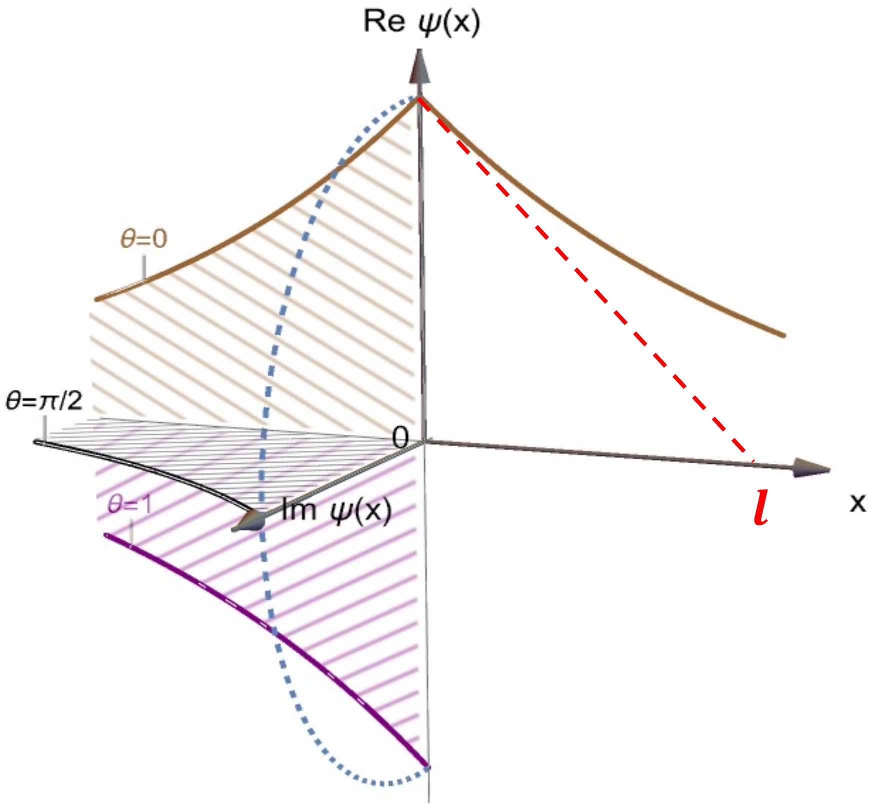
- For other regions, ψ is given by (exchange statistics):

$$\psi(x_1 \ldots x_j \ldots x_i \ldots x_N) = e^{i\alpha w}\, \psi(x_1 \ldots x_i \ldots x_j \ldots x_N)$$

$$w = \sum_{k=i+1}^{j} \mathrm{sgn}(x_k - x_i) - \sum_{k=i+1}^{j-1} \mathrm{sgn}(x_k - x_j)$$

*** ***l*** is the unique parameter to characterize interaction effect! ***

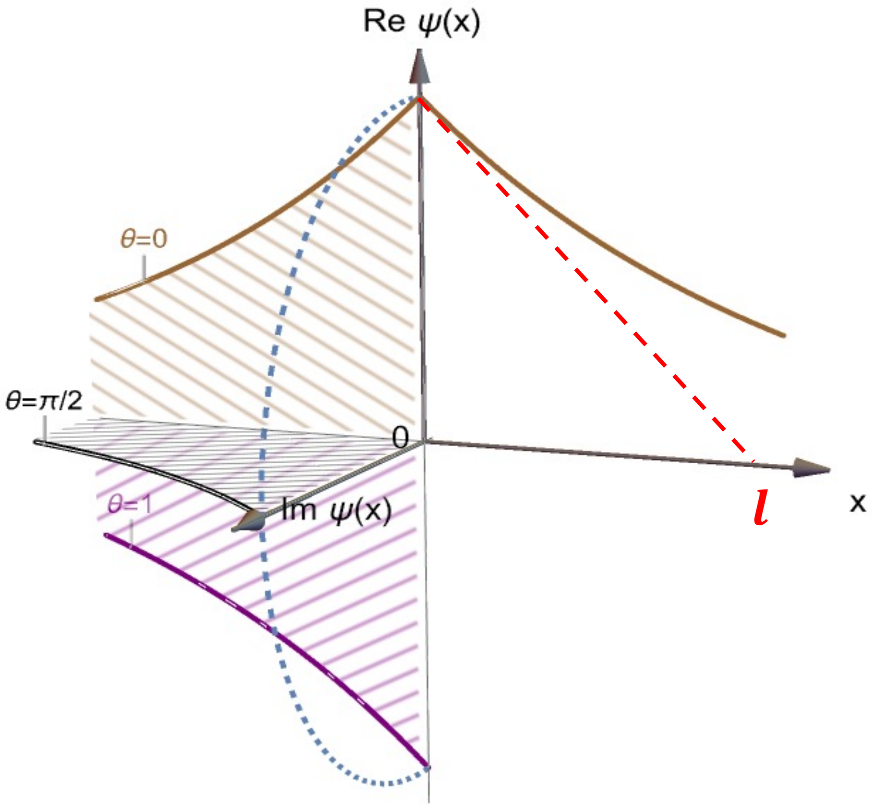
Boson-Anyon-Fermion mapping → **Anyon construction**



Given the same l for two-body systems,

- Boson: $\psi_s(x)$
- Fermion: $\psi_p(x) = \mathrm{sgn(x)}\,\psi_s(x)$

Boson-Anyon-Fermion mapping → Anyon construction



Given the same l for two-body systems,

- Boson: $\psi_s(x)$
- Fermion: $\psi_p(x) = \mathrm{sgn(x)}\, \psi_s(x)$
- Anyon:

$$\psi_{any}(x) = \psi_s(x) - \mathrm{i}\tan\frac{\alpha}{2}\psi_p(x)$$

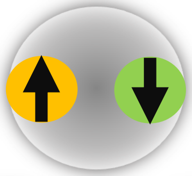
Anyonic state can be constructed simply from linear superposition of spatially symmetric (s-wave) and antisymmetric (p-wave) states!

- Where to find s+p?
- How to achieve superposition?

Constructing **Anyonic molecule and superfluidity** in a spin-1/2 Fermi gas:

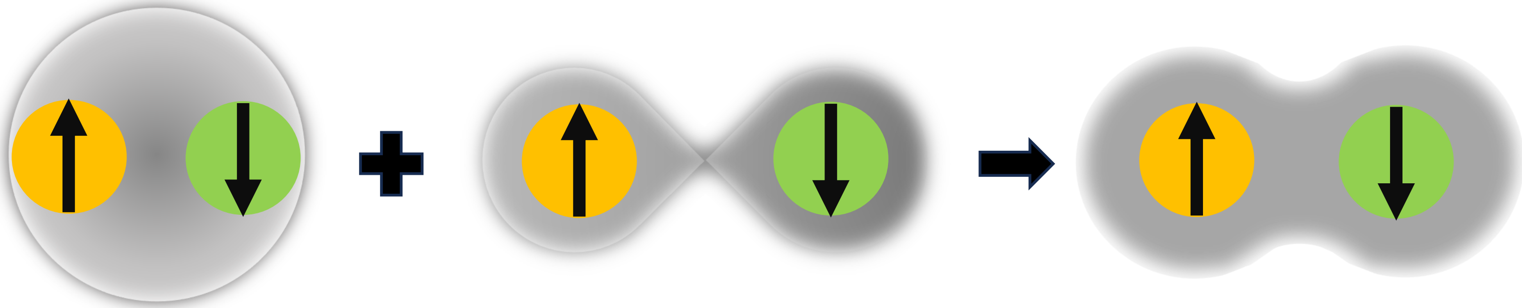
- Two types of bound states (molecules):

$$\Psi_s(x_1, x_2) = \phi_s | \uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2 \rangle$$



$$\Psi_p(x_1, x_2) = \phi_p | \uparrow_1 \downarrow_2 + \downarrow_1 \uparrow_2 \rangle$$

- A symmetry breaking field to couple s- and p-wave molecules:



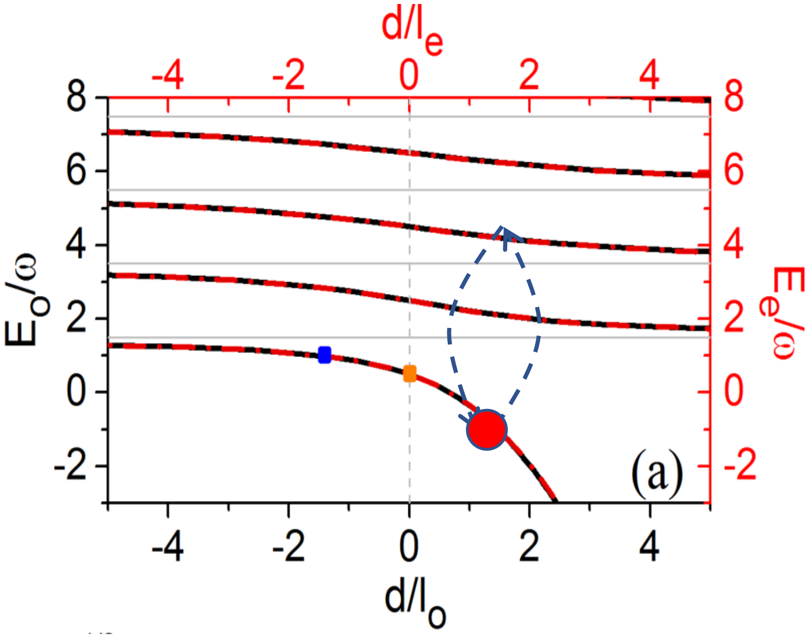
$\alpha = 0$ $\alpha = \pi$ $0 < \alpha < \pi$

Add a weak spin-orbit coupling:

$$V_{\rm soc} = \Omega \sum_i \left[e^{-iqx_i} \sigma_i^+ + e^{iqx_i} \sigma_i^- \right]$$

$$H_{\rm eff} = v \mathcal{M} v^T \qquad v = (\Psi_s, \Psi_p)$$

$$\mathcal{M} = -\frac{\Omega^2}{\omega_{ho}} \begin{pmatrix} A & iC \\ -iC & B \end{pmatrix}$$

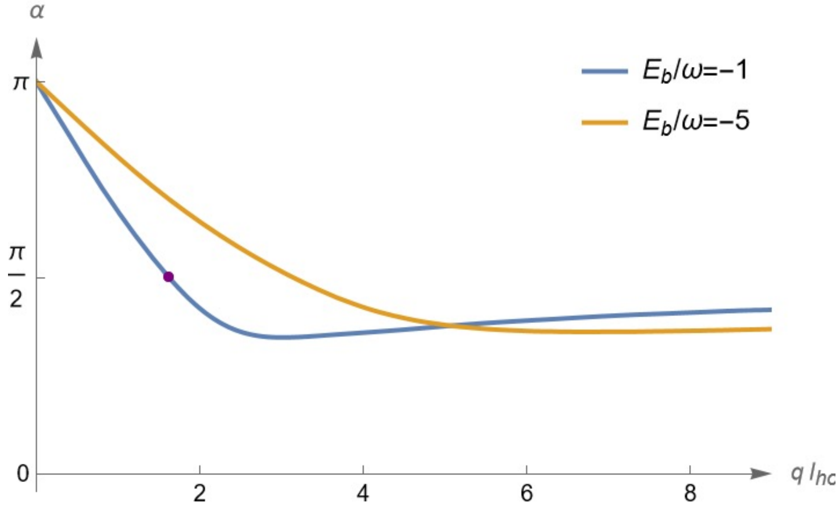


$$\Psi_g = (\phi_s + i\beta\phi_p)|\uparrow_1\downarrow_2\rangle - (\phi_s - i\beta\phi_p)|\downarrow_1\uparrow_2\rangle$$

anyonic molecule

$$\beta = \frac{1}{C}\left(\frac{A-B}{2} - \sqrt{\left(\frac{A-B}{2}\right)^2 + C^2} \right) = \tan\frac{\alpha}{2}$$

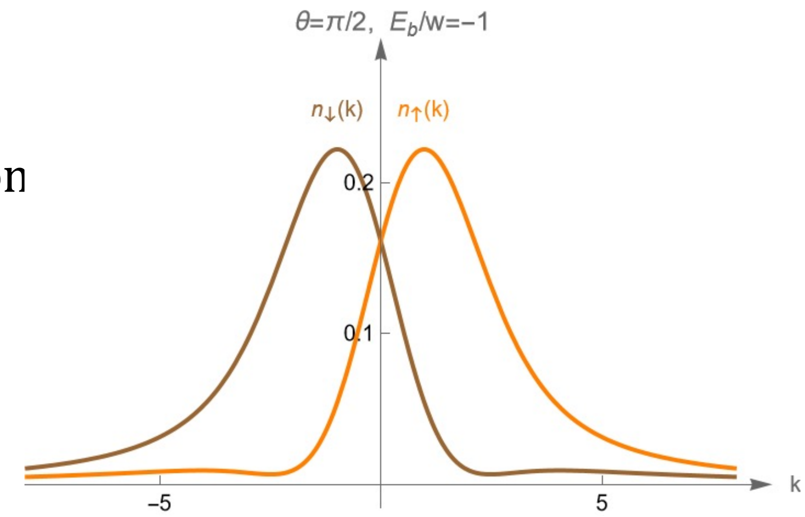
Anyonic molecule:



- $\boldsymbol{\alpha}$: tunable by SOC and harmonic trap

- **Asymmetric** momentum distribution

$$n_\sigma(k) \neq n_\sigma(-k)$$



From anyonic molecule to anyonic superfluidity:

$$d^\dagger = \sum_k \left[\phi_s(k) + i\beta\phi_p(k)\right] c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}$$

condense

$$\prod_k \left(1 + \lambda\left[\phi_s(k) + i\beta\phi_p(k)\right] c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right)$$

anyonic SF

$$n_\sigma(k) \sim \left(\frac{\kappa - 8\epsilon_\sigma \tan(\alpha/2)k}{k^2+\kappa^2}\right)^2 \qquad (\epsilon_\uparrow = 1,\ \epsilon_\downarrow = -1)$$

At high k,

$$n_\uparrow(\mathrm{k}) = \frac{C_2}{k^2} - \cot\frac{\alpha}{2}\frac{C_3}{k^3} + \cdots$$

$$n_\uparrow(\mathrm{k}) = \frac{C_2}{k^2} + \cot\frac{\alpha}{2}\frac{C_3}{k^3} + \cdots$$

Chiral k^{-3} tail

Anyon characteristics

Possible experimental setup:

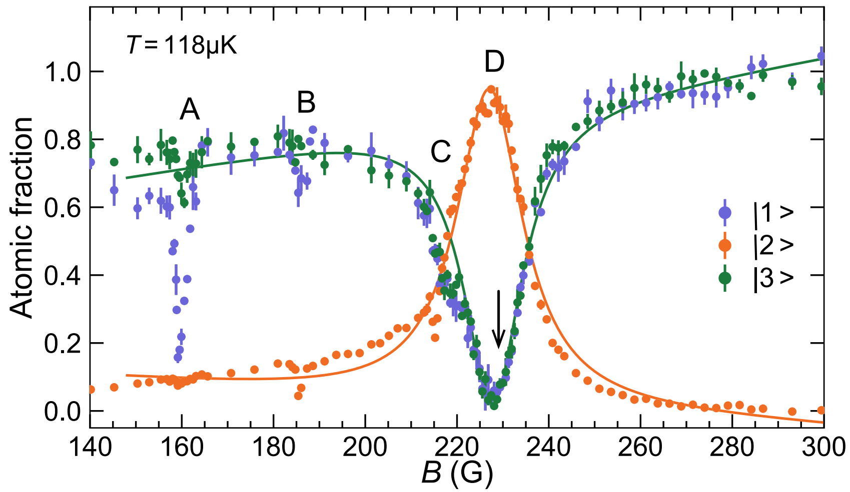
^{6}Li

$$|1\rangle \equiv |f = 1/2, m_f = 1/2\rangle$$

$$|3\rangle \equiv |f = 3/2, m_f = -3/2\rangle$$

- Large s-wave scattering length $a_s \sim 1000 a_0$
- Wide p-wave resonance near 225G

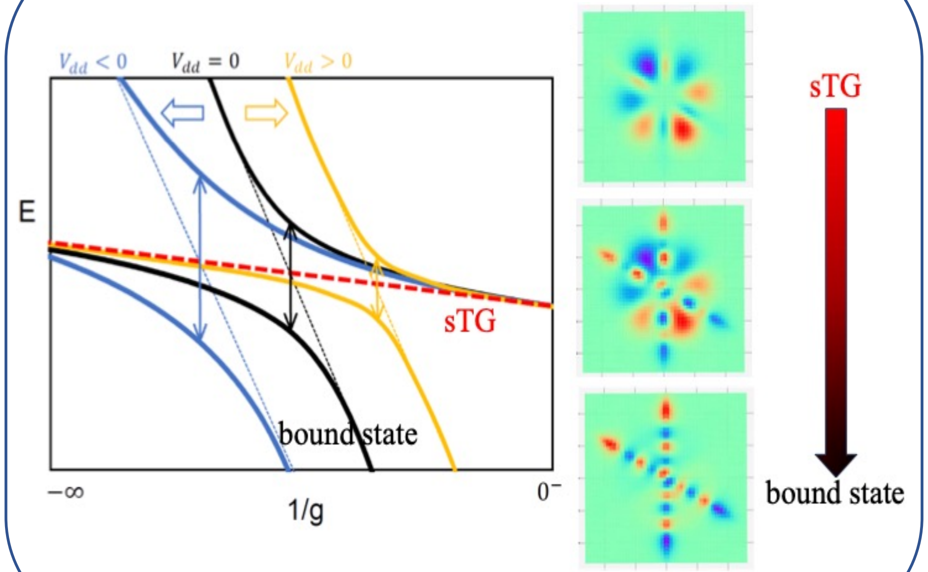
q1D $\rightarrow$ $l_s = l_p$



<u>P-wave resonance</u>: … Jiaming Li and Le Luo, 2406.01248

Summary

- Dipolar-enhanced stability of sTG: physical mechanism
- Statistics & interaction in 1D:
    - Boson-Anyon-Fermion mapping
    - Anyon construction



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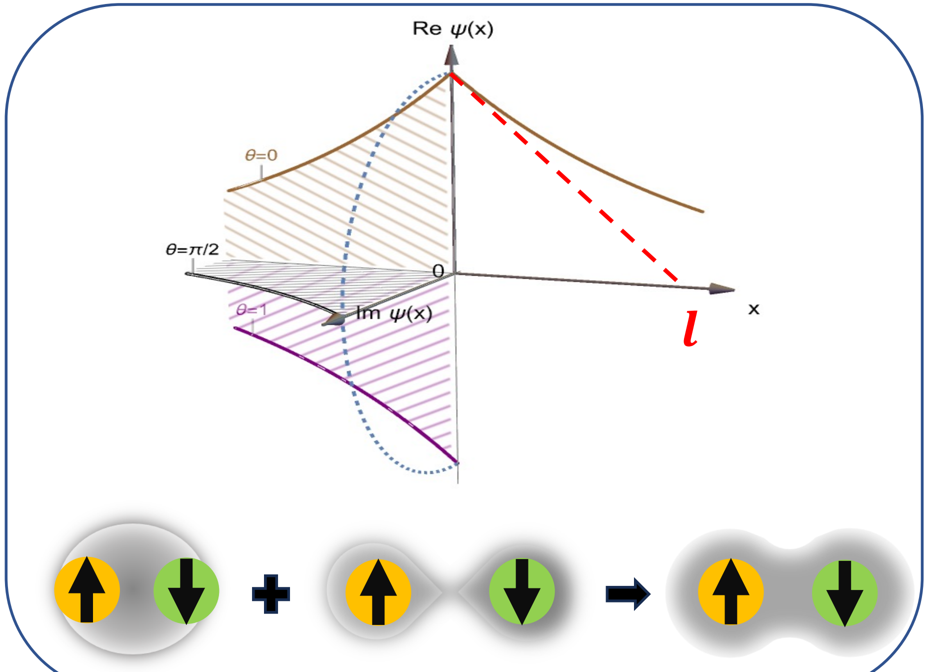

arxiv: 2410.21632

THANKS FOR YOUR ATTENSION