Exact g-function
without strings

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Based on the works

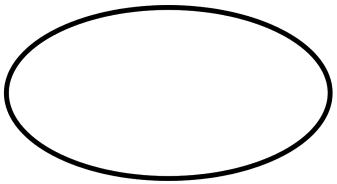
Y. Jiang and Y. He, *in progress*

I. Introduction

Boundary entropy / g-function

Boundary entropy / g-function

Closed boundary



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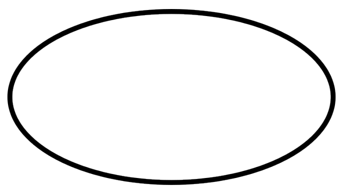

$$L \gg 1$$

Open boundary

Boundary entropy / g-function

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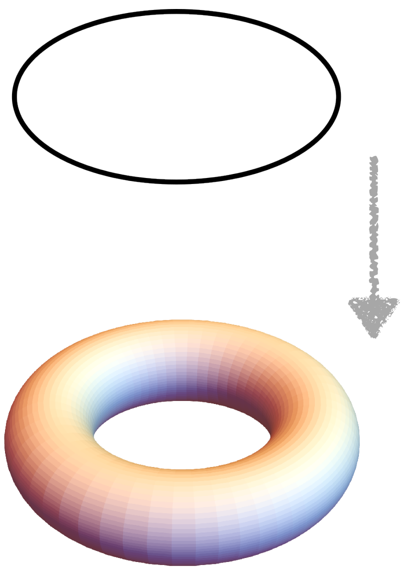

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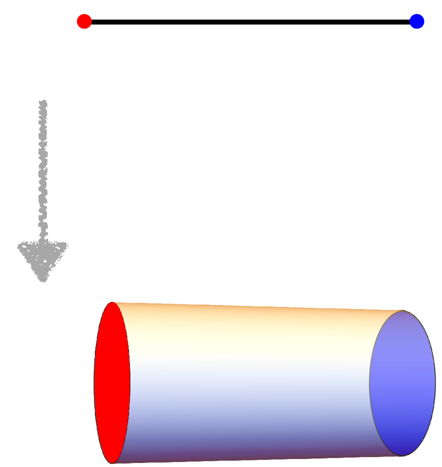

$L \gg 1$

Finite temperature

$T = 1/\beta$

Boundary entropy / g-function

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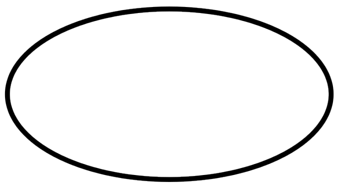

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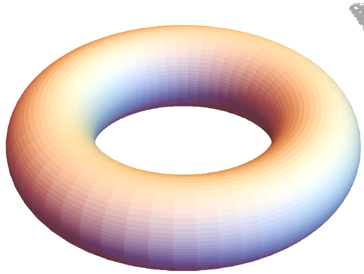
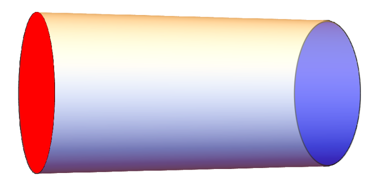




$$Z = \mathrm{Tr}\, e^{-\beta H} = e^{-\beta F}$$

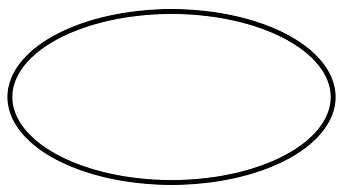
$$F = E - TS = E - S/\beta$$

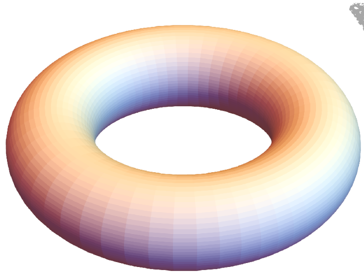
$$S = (1 - \beta \partial_\beta) \ln Z$$

Entropy

Boundary entropy / g-function

Closed boundary





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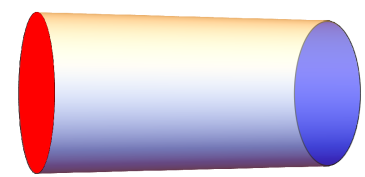

$$Z_{ab} = \mathrm{Tr}\, e^{-\beta H_{ab}} = e^{-\beta F_{ab}}$$

$$F_{ab} = F_{\mathrm{bulk}} + f_a + f_b$$

Boundary entropy / g-function

Closed boundary

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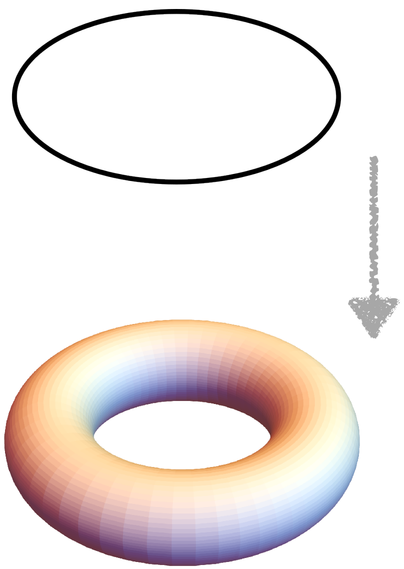

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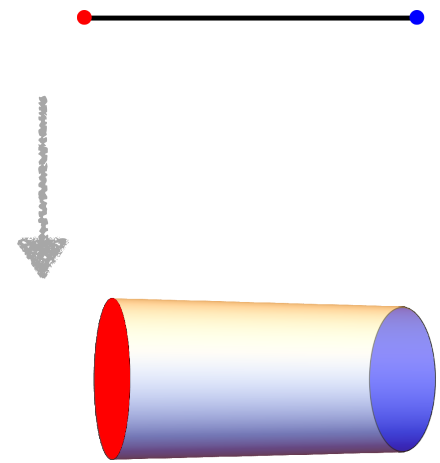

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Boundary entropy / g-function

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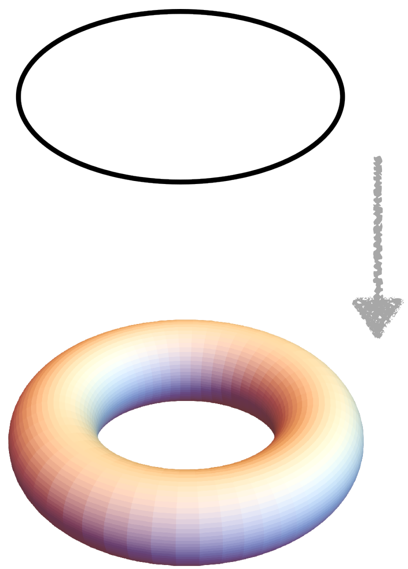

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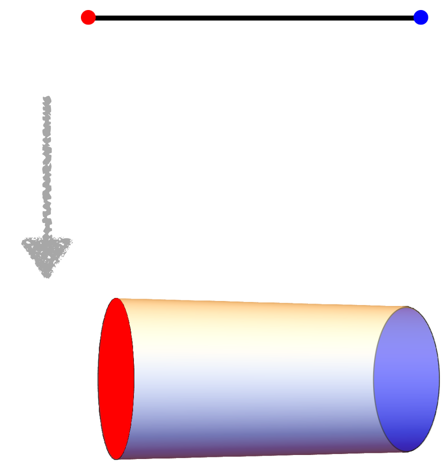

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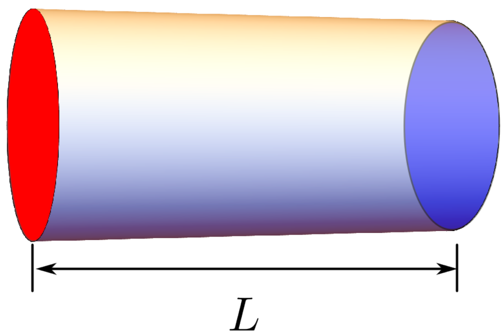
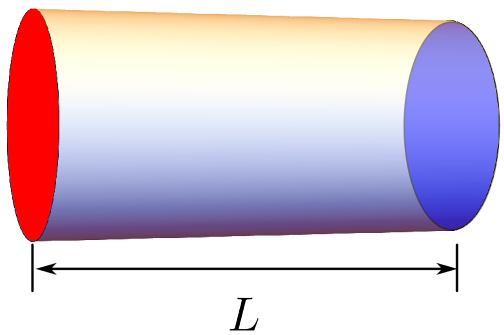


Open channel

$$Z_{ab}(\beta, L) = \mathrm{Tr}\, e^{-\beta H_{ab}(L)}$$

Boundary entropy / g-function



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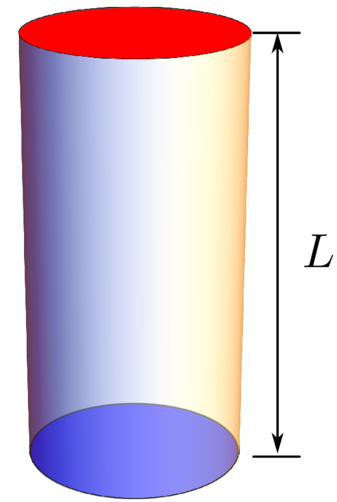

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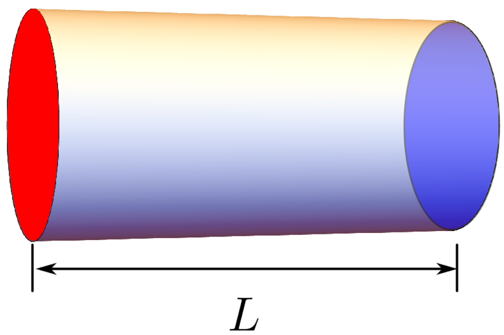
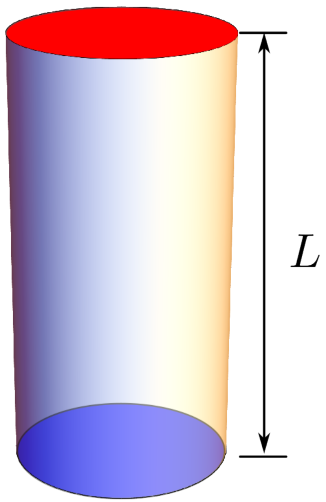
$$Z_{ab}(\beta, L) = \langle B_a | e^{-L H(\beta)} | B_b \rangle$$

Boundary entropy / g-function





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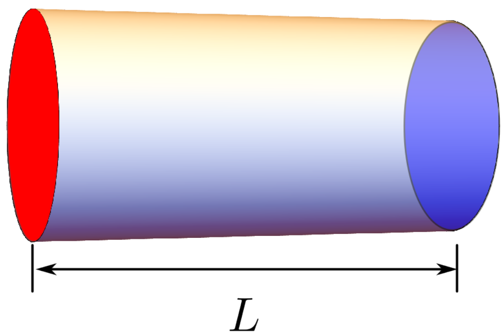
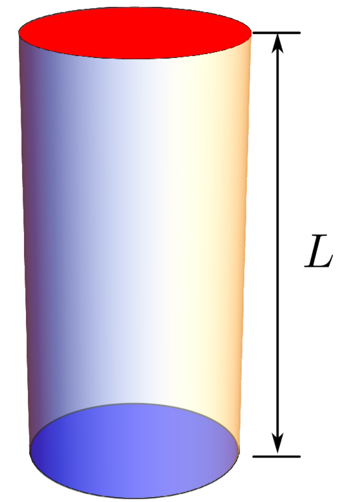
Boundary states

Boundary entropy / g-function





Open channel

Closed channel

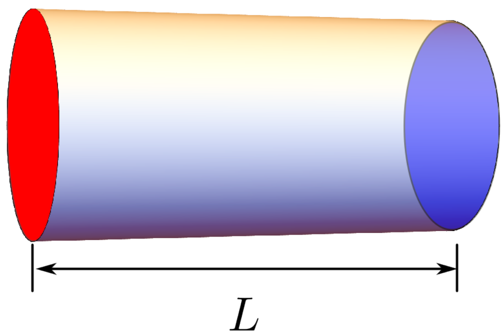
$$Z_{ab}(\beta, L) = \mathrm{Tr}\, e^{-\beta H_{ab}(L)}$$

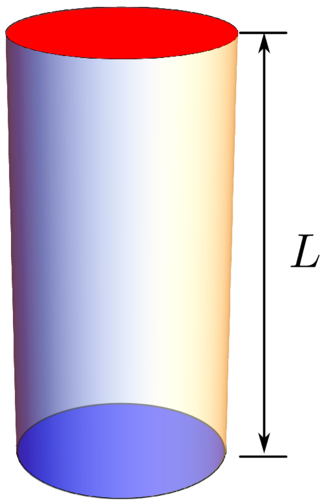
$$Z_{ab}(\beta, L) = \langle B_a | e^{-LH(\beta)} | B_b \rangle$$

For $L \gg 1$

$$Z_{ab}(\beta, L) \approx \langle B_a | 0 \rangle \langle 0 | B_b \rangle e^{-LE_0(\beta)}$$

Boundary entropy / g-function





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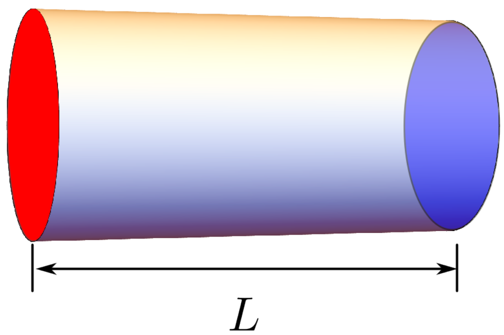

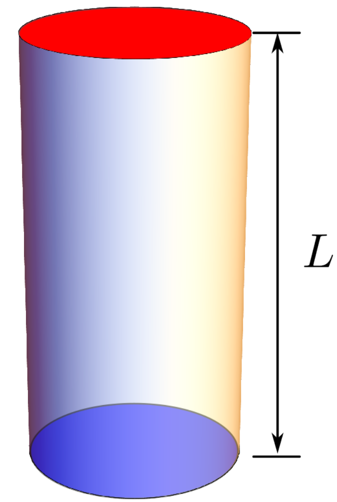

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The g-function

Boundary entropy / g-function

Physical meaning

- Introduced by Affleck and Ludwig in Kondo problem [Affleck, Ludwig 1991]
- Measures the universal ground state degeneracy
- Related to tension of D-brane in string theory

[Harvey, Kachru, Moore, Silverstein 1999]

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Computation in CFT

- For CFT, **boundary states** are constructed by Cardy [Cardy 1989]

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Computation in CFT

- g-function given by overlap of Cardy state and the ground state

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Computation in CFT

- Example of Ising model

$$|(+)\rangle = \frac{1}{\sqrt{2}}|0\rangle\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle\rangle + \frac{1}{\sqrt[4]{2}}|\sigma\rangle\rangle$$

$$|(-)\rangle = \frac{1}{\sqrt{2}}|0\rangle\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle\rangle - \frac{1}{\sqrt[4]{2}}|\sigma\rangle\rangle$$

$$|(f)\rangle = |0\rangle\rangle - |\varepsilon\rangle\rangle$$

Ishibashi states

One for each conformal family

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Modular S-matrix

Modular property of characters

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$$|(-)\rangle\rangle = \frac{1}{\sqrt{2}}|0\rangle\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle\rangle - \frac{1}{\sqrt[4]{2}}|\sigma\rangle\rangle$$ → **Fixed** spin down

$$|(f)\rangle\rangle = |0\rangle\rangle - |\varepsilon\rangle\rangle$$ → **Free** boundary

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$$\ln g_{(+)} = \ln(1/\sqrt{2})$$

$$\ln g_{(-)} = \ln(1/\sqrt{2})$$

$$\ln g_{(f)} = 0$$

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Note that

$$\ln g_{(f)} > \ln g_{(\pm)}$$

Is it a coincidence ?

Boundary entropy / g-function

The *g*-theorem

For fixed bulk critical theory, *g*-function ***decreases monotonically*** along the boundary RG flow

- *g*-function is the analog of Zamolodchikov's *c*-function
- Conjectured by Affleck and Ludwig in 1991 [Affleck, Ludwig 1991]
- Proved by Friedan and Konechny in 2003 [Friedan, Konechny 2003]
- Revisited from quantum information point of view

[Casini, Salazar Landea, Torroba 2016]

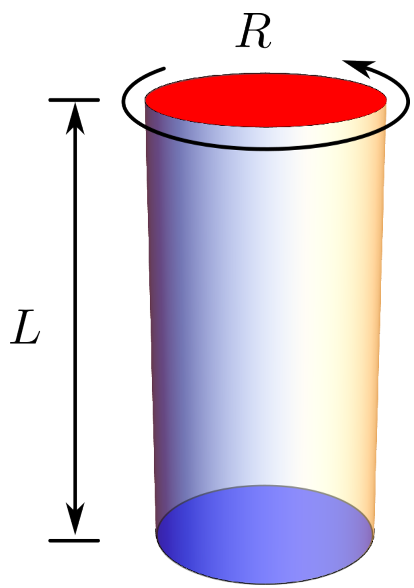
II. g-function in IQFT

Diagonal scattering theories

Off-critical g-function

Bulk massive QFT

The bulk theory is not a CFT, but a massive QFT, can still define g-function



$$\langle B|0\rangle \quad \sim \quad e^{-R\varepsilon}\, g(mR)$$

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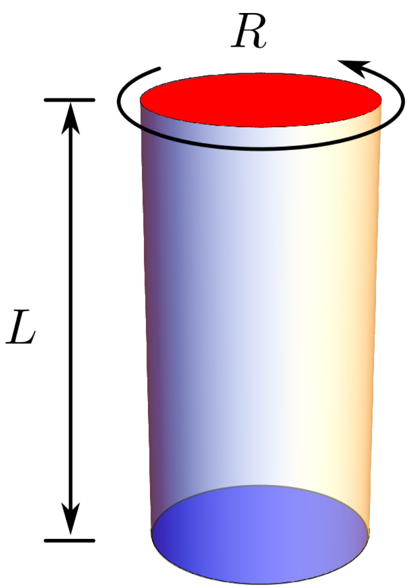

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boundary state

vacuum

Off-critical g-function

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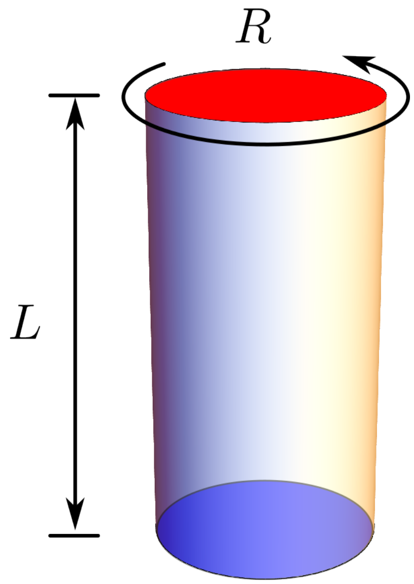

$$\langle B|0\rangle \quad \sim \quad e^{-R\varepsilon}\, g(mR)$$

$e^{-R\varepsilon}$: extensive part

$g(mR)$: g-function

Off-critical g-function

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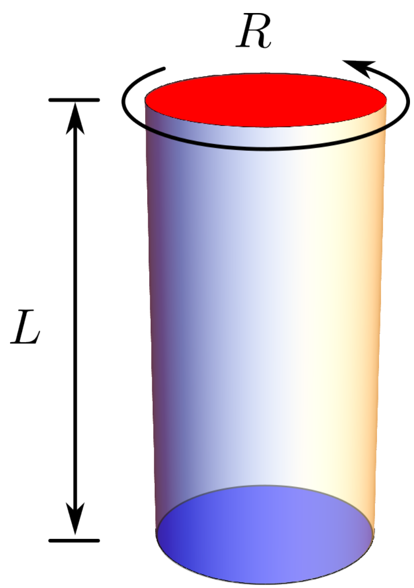

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Hard to compute for **generic QFT** and **generic boundary** condition

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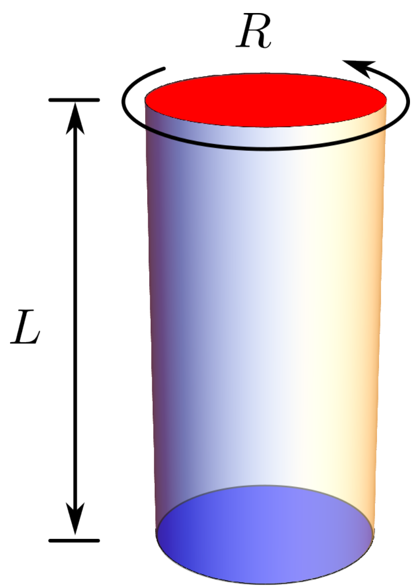

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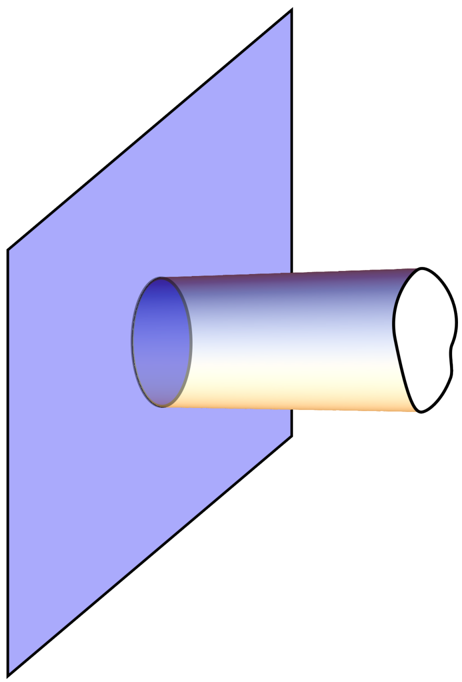
Exact computation for

- ***Integrable*** QFT : $[Q_m, Q_n] = 0$
- ***Integrable*** boundary : $Q_{2n+1}\,|B\rangle = 0$

Off-critical g-function

AdS/CFT correspondence

In planar $\mathcal{N} = 4$ super-Yang-Mills theory, some **OPE coefficients** can be given by the **worldsheet g-function**



D-brane emit a closed string

Amplitude = $\langle B | \psi \rangle$

- Giant-graviton 1pt

$$\langle \mathcal{D}_1(x_1)\mathcal{D}_2(x_2)\mathcal{O}(x_3)\rangle$$

- Wilson-loop 1pt

$$\langle W(\mathcal{C})\mathcal{O}(x)\rangle$$

- Defect 1pt

$$\langle \mathcal{O}(x)\rangle_{\text{dCFT}}$$

Spacetime dependence fixed
OPE coefficient given by *g*-function

Off-critical g-function

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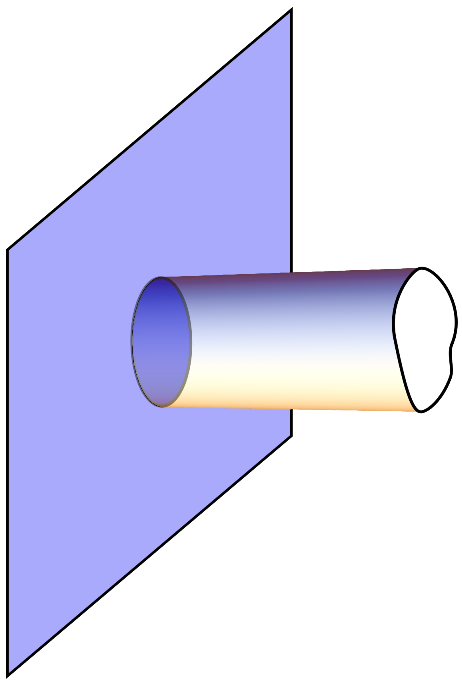

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- Defect 1pt

$$\langle \mathcal{O}(x) \rangle_{\text{dCFT}}$$

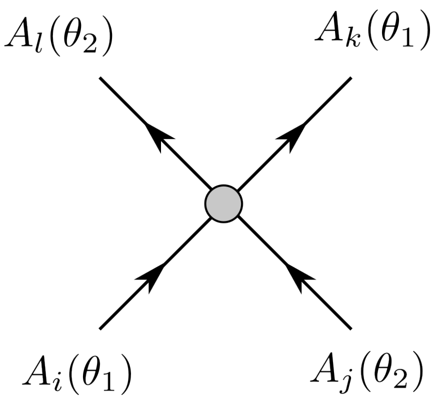
So far the only cases where **finite size corrections** can be computed exactly

IQFT with boundaries

Integrable QFTs

Bulk scattering $S_{ij}^{kl}(\theta)$

$S_{ij}^{kl}(\theta_1 - \theta_2) \quad =$



diagonal scattering

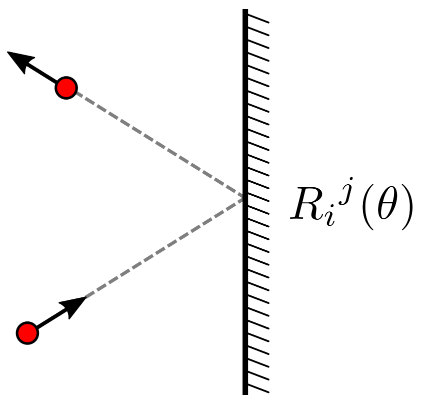
$$S_{ij}^{kl}(\theta) = \delta_i^k \delta_j^l \, S_{ij}(\theta)$$

Integrable boundary conditions

Boundary S-matrix $R_i^{\,j}(\theta)$

diagonal boundary

$$R_i^{\,j}(\theta) = \delta_i^j \, S_i(\theta)$$



Compute g-function

Cluster expansion

From the identity

$$\sum_{n=0}^{\infty} e^{-RE_{aa}^{(n)}(L)} = \sum_{n=0}^{\infty} \langle B_a|n\rangle\langle n|B_a\rangle e^{-LE_n(R)}$$

For $L \gg R \gg 1$

$$2\ln g_a \sim L(E_0(R) - \mathcal{E}m^2R) + \ln\left(1 + \sum_{n=1}^{\infty} e^{-R(E_{aa}^{(n)}(L) - E_{aa}^{(0)}(L))}\right)$$

- Estimate energy levels by Bethe ansatz
- Take into account 1-particle, 2-particles,… n-particle contributions
- Find a pattern and postulate general expressions

[Dorey, Fioravanti, Rim, Tateo 2004]

Compute g-function

Cluster expansion

For **diagonal scattering** theory, N particle species

$$2\ln g_a = \frac{1}{2}\sum_{j=1}^{N}\int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta)\right)\ln\left(1 + e^{-\varepsilon_j(\theta)}\right)$$

$$+\sum_{n=1}^{\infty}\sum_{j_1,\cdots,j_n=1}^{N}\frac{1}{n}\int_{\mathbb{R}}\frac{\mathrm{d}\theta_1}{1+e^{\varepsilon_{j_1}(\theta_1)}}\cdots\frac{\mathrm{d}\theta_n}{1+e^{\varepsilon_{j_n}(\theta_n)}}$$

$$\times\left(\phi_{j_1 j_2}(\theta_1+\theta_2)\phi_{j_2 j_3}(\theta_2-\theta_3)\cdots\phi_{j_n j_1}(\theta_n-\theta_1)\right)$$

Compute g-function

Cluster expansion

For **diagonal scattering** theory, N particle species

Finite sum, boundary dependent

$$2\ln g_a = \boxed{\frac{1}{2}\sum_{j=1}^{N}\int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta)\right)\ln\left(1 + e^{-\varepsilon_j(\theta)}\right)}$$

$$+ \sum_{n=1}^{\infty}\sum_{j_1,\cdots,j_n=1}^{N}\frac{1}{n}\int_{\mathbb{R}}\frac{\mathrm{d}\theta_1}{1+e^{\varepsilon_{j_1}(\theta_1)}}\cdots\frac{\mathrm{d}\theta_n}{1+e^{\varepsilon_{j_n}(\theta_n)}}$$

$$\times\left(\phi_{j_1j_2}(\theta_1+\theta_2)\phi_{j_2j_3}(\theta_2-\theta_3)\cdots\phi_{j_nj_1}(\theta_n-\theta_1)\right)$$

Compute g-function

Cluster expansion

For **diagonal scattering** theory, N particle species

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$$\times\left(\phi_{j_1j_2}(\theta_1+\theta_2)\phi_{j_2j_3}(\theta_2-\theta_3)\cdots\phi_{j_nj_1}(\theta_n-\theta_1)\right)$$

$$\phi_a^{(j)}(\theta) = -\frac{\mathrm{i}}{\pi}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_a^{(j)}(\theta)$$

Boundary S-matrix

$$\phi_{ij}(\theta) = -\frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_{ij}(\theta)$$

Bulk S-matrix

Compute g-function

Cluster expansion

For **diagonal scattering** theory, N particle species

Finite sum, boundary dependent

$$2\ln g_a = \boxed{\frac{1}{2}\sum_{j=1}^{N}\int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta)\right)\ln\left(1+e^{-\varepsilon_j(\theta)}\right)}$$

$$+\sum_{n=1}^{\infty}\sum_{j_1,\cdots,j_n=1}^{N}\frac{1}{n}\int_{\mathbb{R}}\frac{\mathrm{d}\theta_1}{1+e^{\varepsilon_{j_1}(\theta_1)}}\cdots\frac{\mathrm{d}\theta_n}{1+e^{\varepsilon_{j_n}(\theta_n)}}$$

$$\times\left(\phi_{j_1j_2}(\theta_1+\theta_2)\phi_{j_2j_3}(\theta_2-\theta_3)\cdots\phi_{j_nj_1}(\theta_n-\theta_1)\right)$$

$\varepsilon_j(\theta)$ pseudo-energy, solution of TBA

$$\varepsilon_j(\theta) = m_j R\cosh\theta - \sum_{k=1}^{N}\int \mathrm{d}\theta'\phi_{jk}(\theta-\theta')\ln\left(1+e^{-\varepsilon_k(\theta)}\right)$$

Compute g-function

Cluster expansion

For **diagonal scattering** theory, N particle species

$$2\ln g_a = \frac{1}{2}\sum_{j=1}^{N}\int_{\mathbb{R}} d\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta)\right)\ln\left(1+e^{-\varepsilon_j(\theta)}\right)$$

Infinite sum,
Boundary
independent

$$+\sum_{n=1}^{\infty}\sum_{j_1,\cdots,j_n=1}^{N}\frac{1}{n}\int_{\mathbb{R}}\frac{d\theta_1}{1+e^{\varepsilon_{j_1}(\theta_1)}}\cdots\frac{d\theta_n}{1+e^{\varepsilon_{j_n}(\theta_n)}}$$
$$\times\left(\phi_{j_1j_2}(\theta_1+\theta_2)\phi_{j_2j_3}(\theta_2-\theta_3)\cdots\phi_{j_nj_1}(\theta_n-\theta_1)\right)$$

The infinite sum can be rewritten in terms of
ratio of determinants

Compute g-function

TBA approach

First try: LeClair, Mussardo, Saleur, Sorik (LMSS) 1995

$$2\ln g_a = \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta) \right) \ln\left(1 + e^{-\varepsilon_j(\theta)}\right)$$

Compute g-function

TBA approach

First try: LeClair, Mussardo, Saleur, Sorik (LMSS) 1995

$$2\ln g_a = \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta) \right) \ln \left(1 + e^{-\varepsilon_j(\theta)} \right)$$

Problem: A boundary independent piece is missing !

[Dorey, Runkel, Tateo, Watts 1999]

Compute g-function

TBA approach

First try: LeClair, Mussardo, Saleur, Sorik (LMSS) 1995

$$2\ln g_a = \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta) \right) \ln\left(1 + e^{-\varepsilon_j(\theta)}\right)$$

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Second try: Woynarovich 2004

Take into account Gaussian fluctuation around the saddle-point

$$\text{LMSS result} + \ln \frac{1}{\det(1-\hat{K}_1)}$$

Compute g-function

TBA approach

First try: LeClair, Mussardo, Saleur, Sorik (LMSS) 1995

$$2\ln g_a = \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\theta \left(\phi_a^{(j)}(\theta) - \delta(\theta) - 2\phi_{jj}(2\theta)\right) \ln\left(1 + e^{-\varepsilon_j(\theta)}\right)$$

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[Dorey, Runkel, Tateo, Watts 1999]

Second try: Woynarovich 2004

Take into account Gaussian fluctuation around the saddle-point

$$\text{LMSS result} + \ln \frac{1}{\det(1 - \hat{K}_1)}$$

Problem: Gives universal $\mathcal{O}(1)$ contribution, but not correct

Compute g-function

TBA approach

Third try: Pozsgay 2010

Functional measure for the partition function need to be corrected !

$$\text{LMSS result} + \ln \frac{\det(1-\hat{K}_2)}{\det(1-\hat{K}_1)}$$

Compute g-function

TBA approach

Third try: Pozsgay 2010

Functional measure for the partition function need to be corrected !

$$\text{LMSS result} + \ln \frac{\det(1-\hat{K}_2)}{\det(1-\hat{K}_1)}$$

$$\ln \frac{\det(1-\hat{K}_2)}{\det(1-\hat{K}_1)} = \sum_{n=1}^{\infty} \sum_{j_1,\cdots,j_n=1}^{N} \frac{1}{n} \int_{\mathbb{R}} \frac{d\theta_1}{1+e^{\varepsilon_{j_1}(\theta_1)}} \cdots \frac{d\theta_n}{1+e^{\varepsilon_{j_n}(\theta_n)}}$$
$$\times \left(\phi_{j_1 j_2}(\theta_1+\theta_2)\phi_{j_2 j_3}(\theta_2-\theta_3)\cdots\phi_{j_n j_1}(\theta_n-\theta_1)\right)$$

The proposal of LMSS is thus proven.

Compute g-function

General lesson

$O(1)$ contribution of saddle point fluctuations to the free energy of Bethe Ansatz systems

F. Woynarovich*
Institute for Solid State Physics and Optics
Hungarian Academy of Sciences
1525 Budapest 114, Pf 49.

In addition to the technical problems the calculation of non macroscopic corrections to the macroscopic free energy rises some conceptional questions too. The Yang and Yang method has been developed to pick up the leading contribution only, thus in calculating further terms one has to see, that this refinement is meaningful, the method is accurate enough to calculate the next to leading contributions too. This involves two kinds of problems. The first is if it is possible at all to define an accurate enough free energy density in terms of the momentum

- Exact g-function function is **more delicate** than free energy
- Need to carefully take into account $\mathcal{O}(1)$ contributions

Compute g-function

Problem with ***non-diagonal*** scattering

- The above results only apply to diagonal scattering theories

  - Theory diagonalized by ***nested*** Bethe ansatz
  - Nee to introduce ***auxiliary roots,*** or ***Bethe strings***
  - The densities of auxiliary roots also enter TBA

- These densities must satisfy ***additional constraints***
- Such constraints are not taken into account in usual TBA
- Ignoring them lead to ***divergences*** in computing g-functions

[Woynarovich 2004]

[Kostov, Serban, Vu 2019]

III. Lattice approach

sine-Gordon theory

Sine-Gordon Theory

The sine-Gordon theory

$$\mathcal{A} = \int_{-\infty}^{+\infty} \mathrm{d}y \int_{x_-}^{x_+} \mathrm{d}x\, \mathcal{L}(\phi) + \int_{-\infty}^{+\infty} \mathcal{B}_\pm(\phi) \mathrm{d}y$$

The bulk action

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 + \mu_{\mathrm{bulk}} \cos(\beta\phi)$$

The boundary conditions

$$\mathcal{B}_\pm(\phi) = \mu_\pm \cos\left(\frac{\beta}{2}(\phi - \phi_0^\pm)\right)\Bigg|_{x=x_\pm}$$

- The boundary condition is shown to be **integrable**
- The bootstrap description has been worked out

[Ghoshal, Zamolodchikov 1993]

Sine-Gordon Theory

The sine-Gordon theory

$$\mathcal{A} = \int_{-\infty}^{+\infty} \mathrm{d}y \int_{x_-}^{x_+} \mathrm{d}x\, \mathcal{L}(\phi) + \int_{-\infty}^{+\infty} \mathcal{B}_{\pm}(\phi)\mathrm{d}y$$

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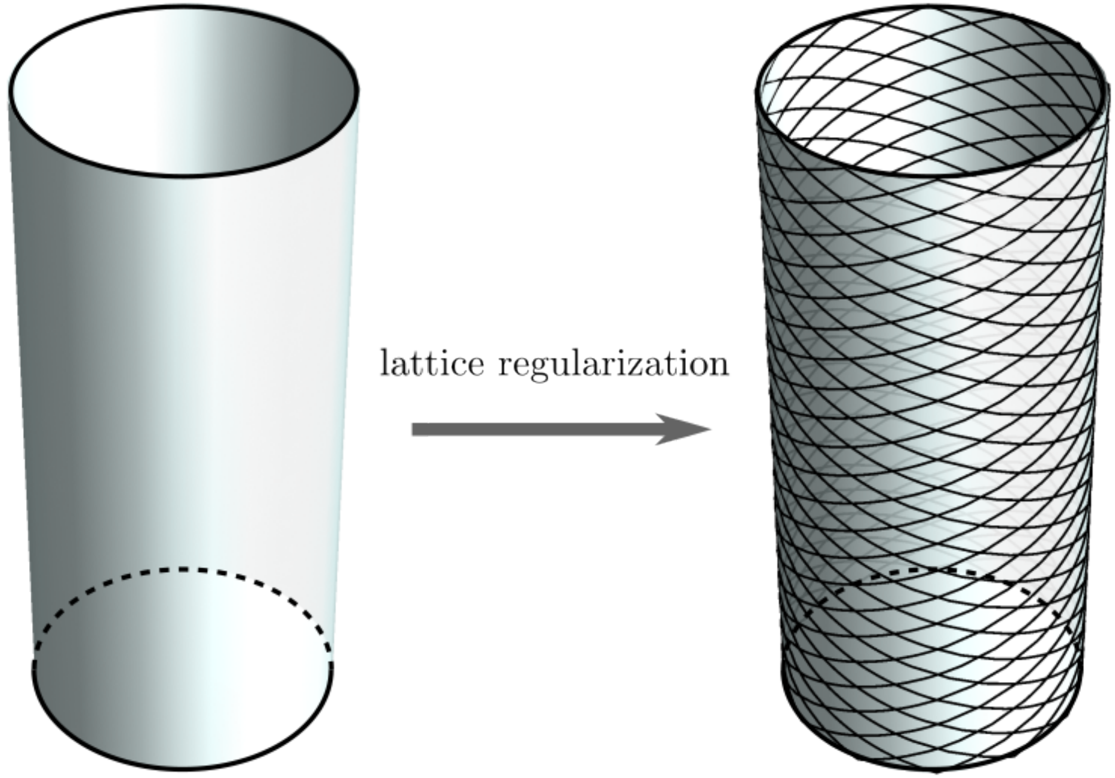
The boundary conditions

$$\mathcal{B}_{\pm}(\phi) = \mu_{\pm} \cos\left(\frac{\beta}{2}(\phi - \phi_0^{\pm})\right)\Big|_{x=x_{\pm}}$$

Challenge Compute *g*-function for sG model with GZ boundary condition

So far TBA does not work !

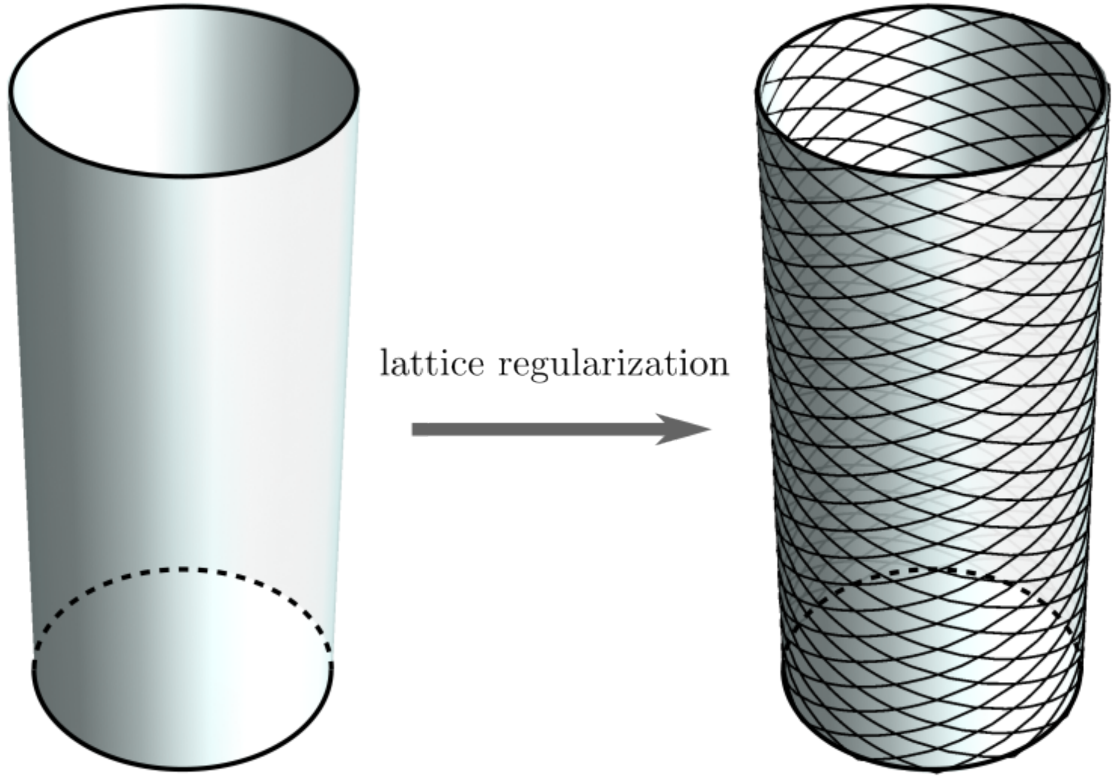
[Ghoshal, Zamolodchikov 1993]



[Destri, de Vega 1992]

For spectral problem, an alternative method to TBA is

lattice discretization

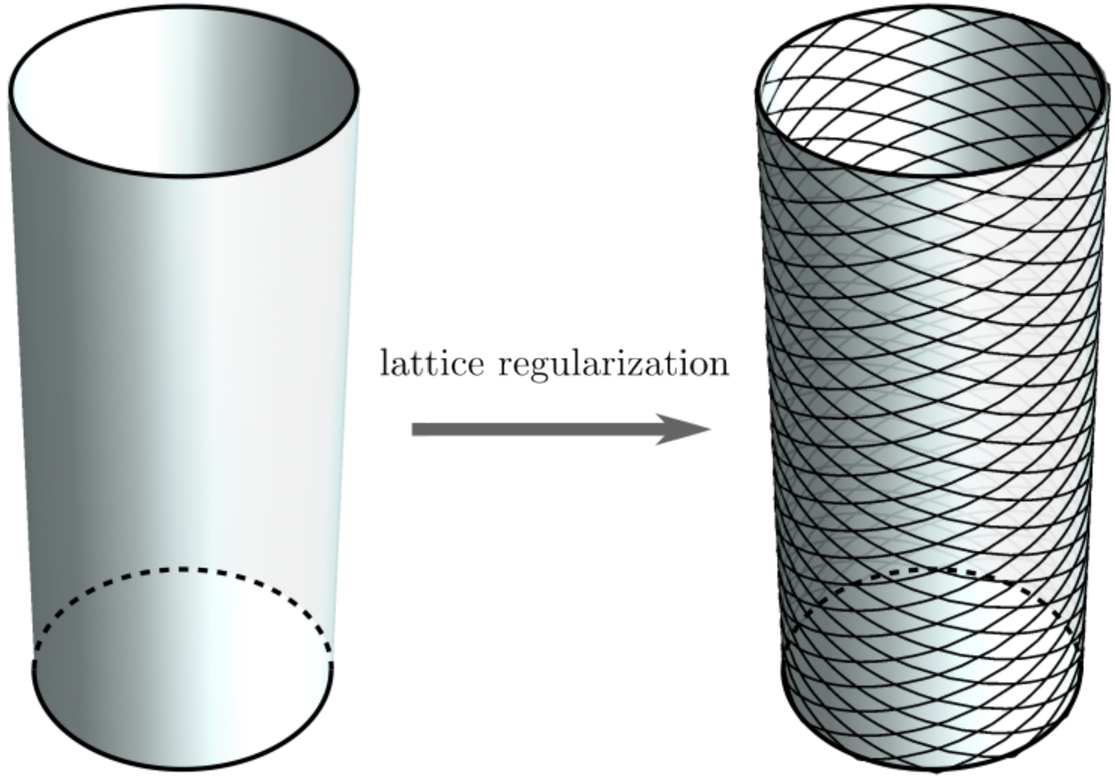


[Destri, de Vega 1992]

For spectral problem, an alternative method to TBA is

lattice discretization

Advantage *No need to introduce Bethe strings*



[Destri, de Vega 1992]

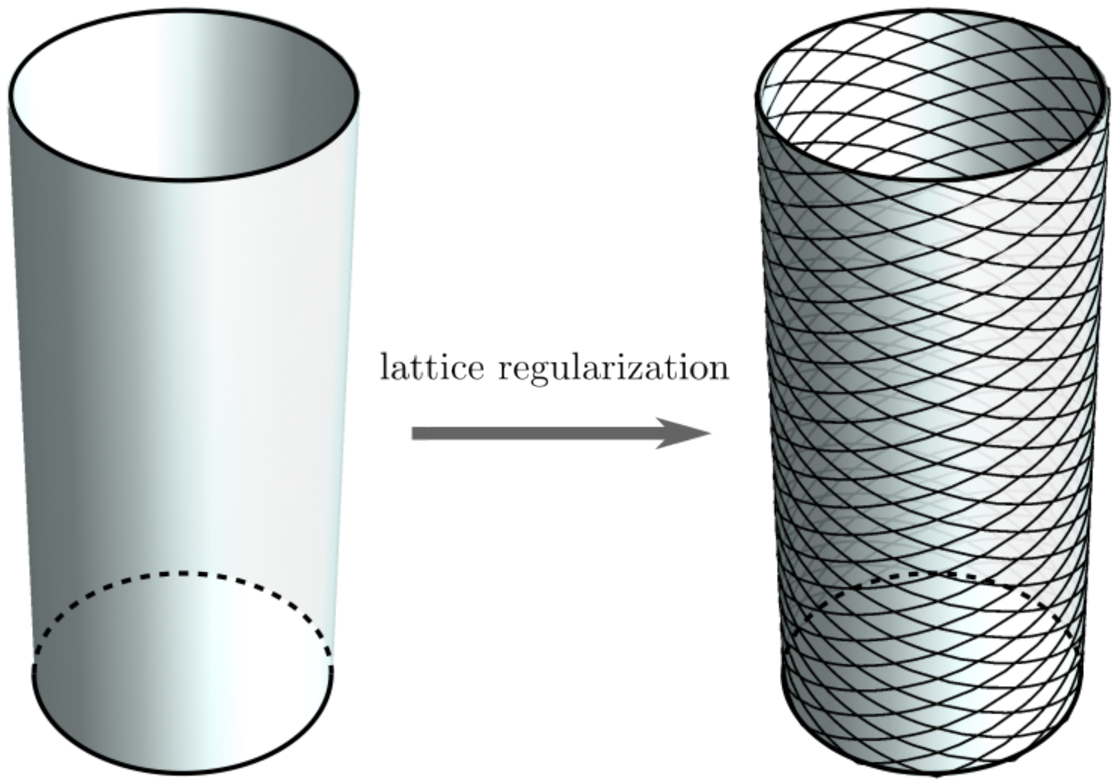
New Thermodynamic Bethe Ansatz Equations without Strings

C. Destri[(1),(a)] and H. J. de Vega[(2),(b)]

[(1)]*Dipartimento di Fisica, Università di Parma, and Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Parma, Parma, Italy*

[(2)]*Laboratoire de Physique Théorique et Hautes Energies, Université de Paris VI, Paris, France*

(Received 24 March 1992)

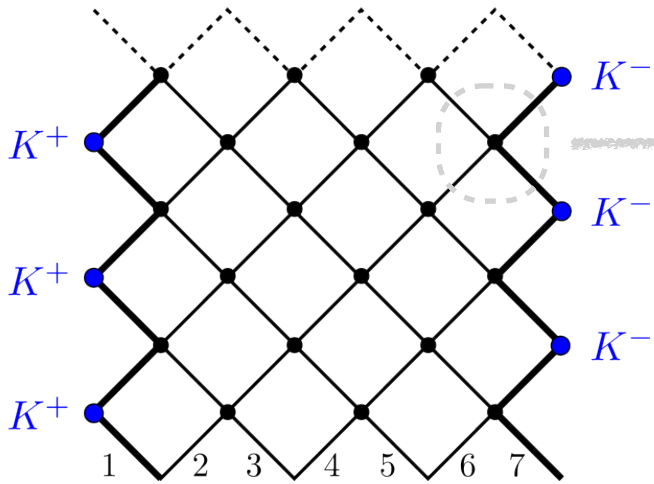


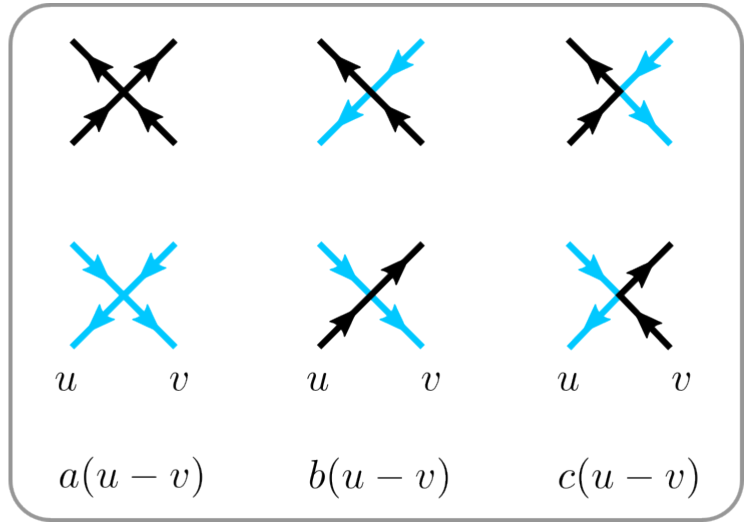
[Destri, de Vega 1992]

For spectral problem, an alternative method to TBA is
lattice discretization

Idea: Lattice overlap → *g*-function

The lattice model





Integrability

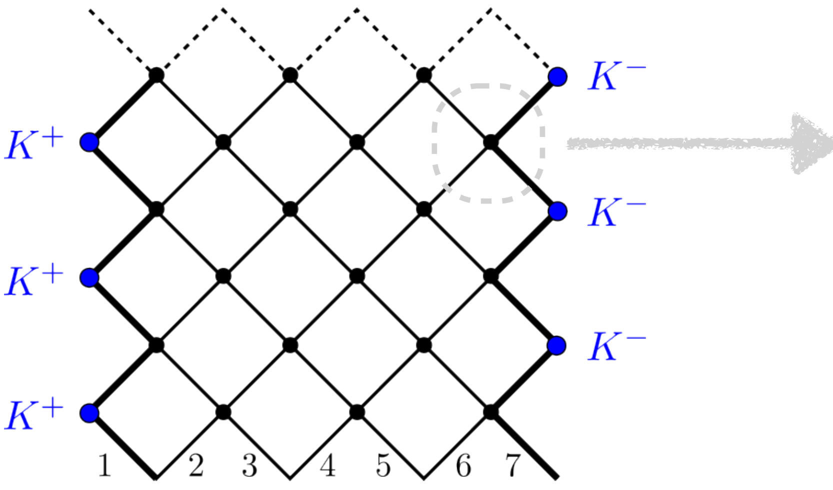
Boltzmann weights

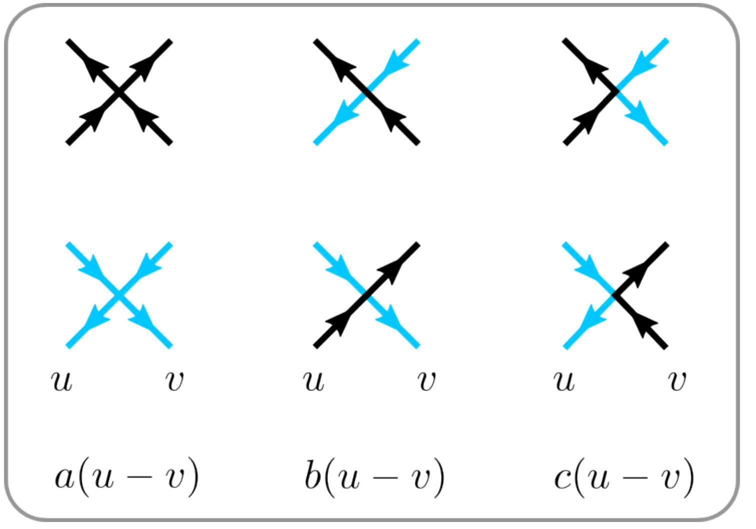
$$R_{jk}(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}$$

Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

The lattice model





Integrability

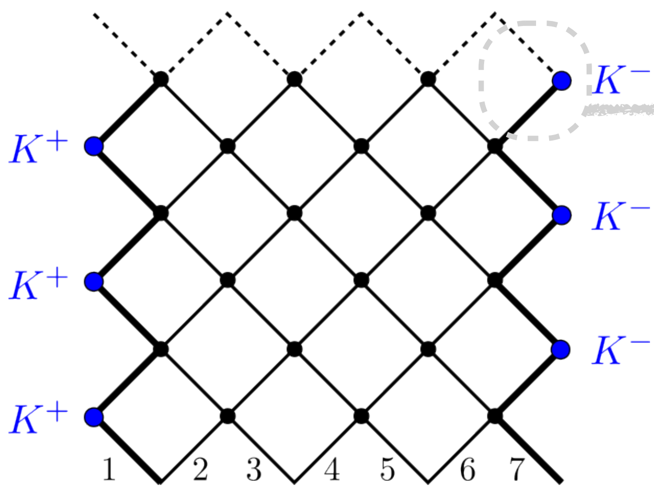
Boltzmann weights

$$a(u) = \sinh(u + \mathrm{i}\gamma)$$
$$b(u) = \sinh u$$
$$c(u) = \mathrm{i}\sin\gamma$$

Parameter relations

$$\frac{8\pi}{\beta^2} - 1 = \frac{1}{\pi/\gamma - 1}$$

The lattice model



$$K^+(u) = K(u + \mathrm{i}\gamma), \quad K^-(u) = K(u)$$

$$K(u) = \begin{pmatrix} K_{11}(u) & K_{12}(u) \\ K_{21}(u) & K_{22}(u) \end{pmatrix}$$

Boundary K-matrix

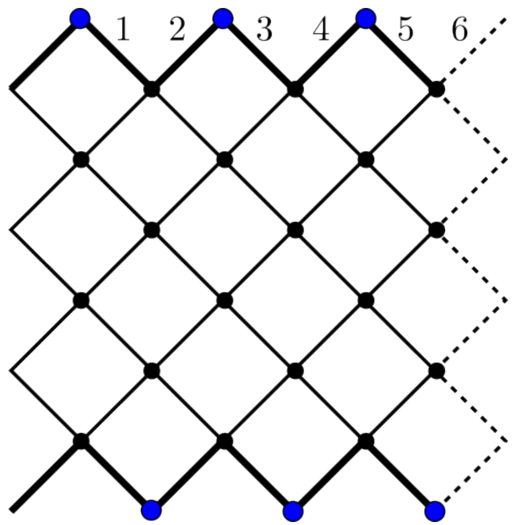
$$K_{11}(u) = 2\mathrm{i}(\sin a \cosh b \cosh u + \cos a \sinh b \sinh u)$$
$$K_{12}(u) = K_{21}(u) = \sinh 2u$$
$$K_{22}(u) = 2\mathrm{i}(\sin a \cosh b \cosh u - \cos a \sinh b \sinh u)\,.$$

Boundary YBE

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v) = K_2(v)R_{12}(u+v)K_1(u)R_{12}(u-v)$$

The lattice model

Partition function in closed channel



$$Z_{M,N}(u) = \frac{\langle \Phi_0^+ | U^\dagger \mathbf{T}_R(u)^M | \Phi_0^- \rangle}{(\mathrm{i} \sin \gamma)^{2MN}}$$

Due to integrability, $Z_{M,N}(u)$ can be computed by Bethe ansatz

Two-site states

$$|\Phi_0^-\rangle = |\psi^-\rangle \otimes |\psi^-\rangle \otimes \cdots \otimes |\psi^-\rangle$$

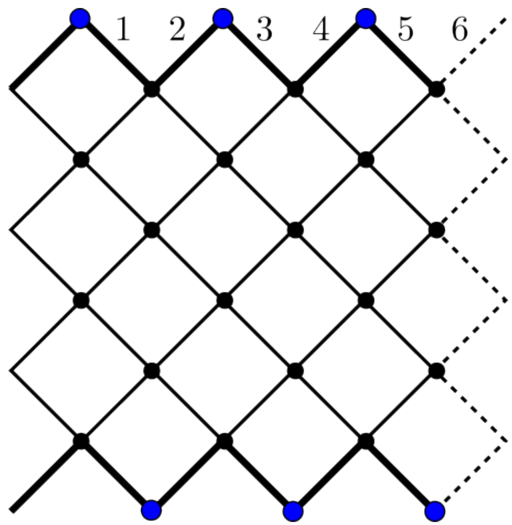
$$\langle \Phi_0^+| = \langle \psi^+| \otimes \langle \psi^+| \otimes \cdots \otimes \langle \psi^+|$$

$$|\psi^-\rangle = \left(\sigma^x \check{K}^-(\tfrac{u}{2})\right)_{ij} (-1)^{i-1} |i\rangle \otimes |j\rangle$$

$$\langle \psi^+| = \left(\sigma^x \check{K}^+(\tfrac{-u}{2})\right)_{ij} (-1)^{j-1} \langle i| \otimes \langle j|$$

The lattice model

Partition function in closed channel

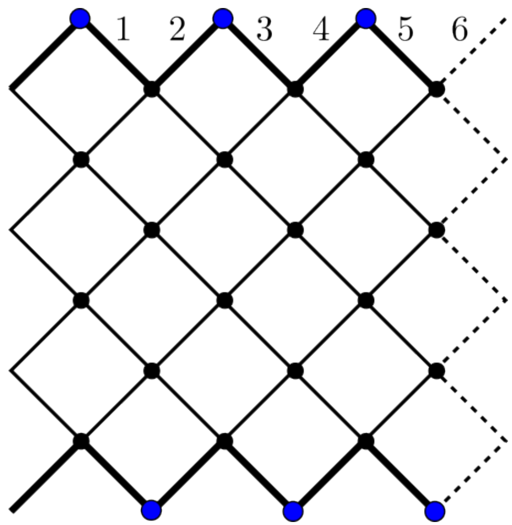


Bethe ansatz

$$Z_{M,N}(u) = \frac{1}{(\mathrm{i}\sin\gamma)^{2MN}} \sum_{\mathrm{sol}} \tau_R(u|\mathbf{u})^M W(u|\mathbf{u})$$

The lattice model

Partition function in closed channel



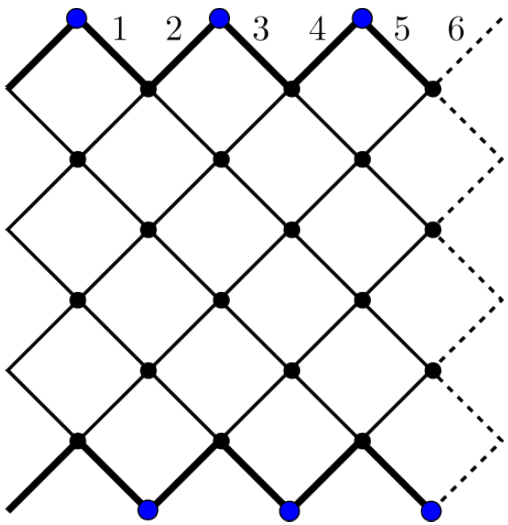
Bethe ansatz

$$Z_{M,N}(u) = \frac{1}{(\mathrm{i}\sin\gamma)^{2MN}} \sum_{\mathrm{sol}} \tau_R(u|\mathbf{u})^M W(u|\mathbf{u})$$

Sum over solution of BAE

The lattice model

Partition function in closed channel



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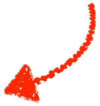
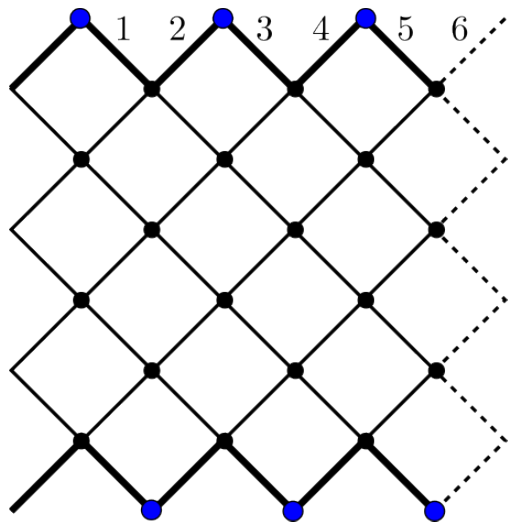


But for ground state, only one solution is needed

The lattice model

Partition function in closed channel



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Sum over solution of BAE

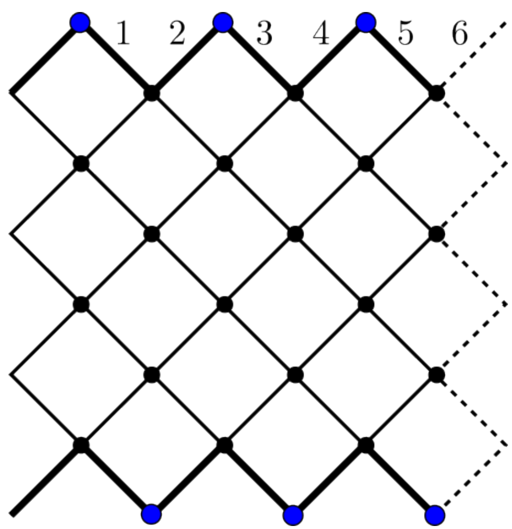
Eigenvalue of transfer matrix

$$\mathbf{T}_R(u)|\mathbf{u}_K\rangle = \tau_R(u|\mathbf{u}_K)|\mathbf{u}_K\rangle$$

$$\tau_R(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2}-\mathrm{i}\gamma)}{Q(-\frac{u}{2})}\frac{Q(\frac{u}{2}+\mathrm{i}\gamma)}{Q(\frac{u}{2})}$$

The lattice model

Partition function in closed channel



Bethe ansatz

$$Z_{M,N}(u) = \frac{1}{(\mathrm{i}\sin\gamma)^{2MN}} \sum_{\mathrm{sol}} \tau_R(u|\mathbf{u})^M W(u|\mathbf{u})$$

Sum over solution of BAE

$$Q(u) = \prod_{k=1}^{N} \sinh(u - u_k)$$

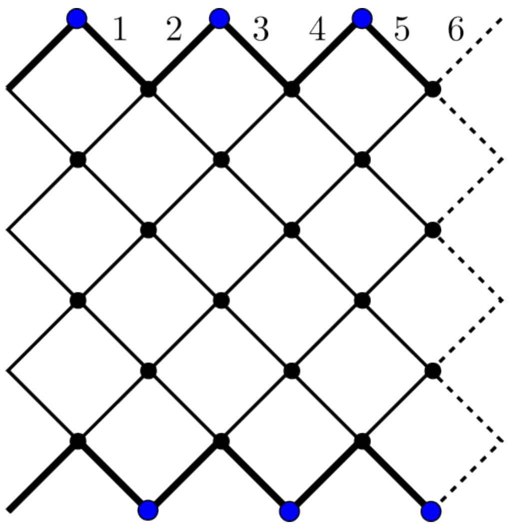
Eigenvalue of transfer matrix

$$\mathbf{T}_R(u)|\mathbf{u}_K\rangle = \tau_R(u|\mathbf{u}_K)|\mathbf{u}_K\rangle$$

$$\tau_R(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \frac{Q(\frac{u}{2} + \mathrm{i}\gamma)}{Q(\frac{u}{2})}$$

The lattice model

Partition function in closed channel



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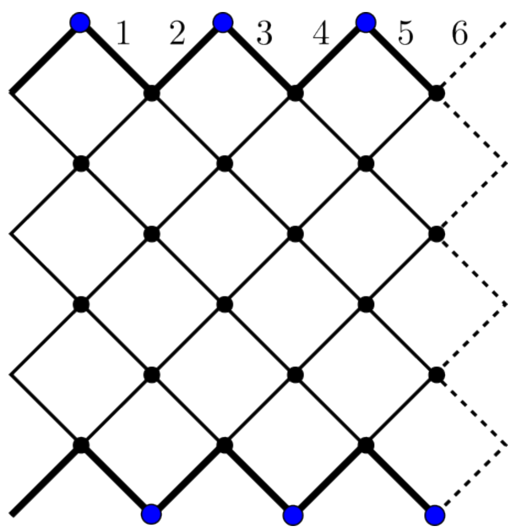
Sum over solution of BAE

The lattice model

Partition function in closed channel



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Sum over solution of BAE

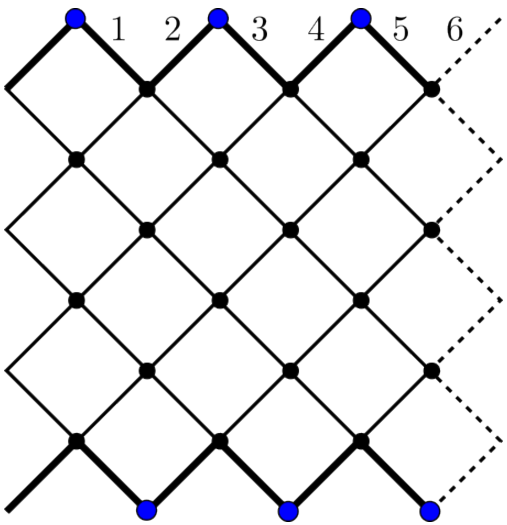
Overlap formula

$$W(u|\mathbf{u}_K) = \frac{\langle \Phi_0^+ | U^\dagger | \mathbf{u} \rangle \langle \mathbf{u} | \Phi_0^- \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle}$$

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \prod_{j=1}^{K} f(u_j) \times \frac{\det G^+}{\det G^-}$$

The lattice model

Partition function in closed channel



Bethe ansatz

$$Z_{M,N}(u) = \frac{1}{(\mathrm{i}\sin\gamma)^{2MN}} \sum_{\mathrm{sol}} \tau_R(u|\mathbf{u})^M W(u|\mathbf{u})$$

Sum over solution of BAE

$$f(u) = \frac{4\sinh^2(u+\mathrm{i}a)\cosh^2(u+b)}{\sinh(2u+\mathrm{i}\gamma)\sinh(2u)}$$

Overlap formula

$$W(u|\mathbf{u}_K) = \frac{\langle\Phi_0^+|U^\dagger|\mathbf{u}\rangle\langle\mathbf{u}|\Phi_0^-\rangle}{\langle\mathbf{u}|\mathbf{u}\rangle}$$

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The continuum limit

Main proposal

g-function is encoded in the overlap $W(u\,|\,\mathbf{u})$ in the continuum limit

The continuum limit

Main proposal

g-function is encoded in the overlap $W(u\,|\,\mathbf{u})$ in the continuum limit

The continuum limit

Spectral parameter $u = -2\Theta - i\gamma$

Lattice site	$N \to \infty$,
lattice spacing	$\Delta \to 0$,
Rapidity cut-off	$\Theta \to \infty$

Such that

$$R = N\Delta, \qquad m = \frac{4}{\Delta}\exp\left(-\frac{\Theta\pi}{\gamma}\right)$$

is fixed and finite

The continuum limit

The Nonlinear Integral Equation (NLIE)

$$Z(u) = mR\sinh\left(\frac{\pi u}{\gamma}\right) + 2\mathrm{Im}\int_{-\infty}^{\infty} \mathrm{d}v\, G(u-v-\mathrm{i}\xi)\ln\left(1+e^{\mathrm{i}Z(v+\mathrm{i}\xi)}\right)$$

with the kernel

$$G(u) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{iku} \frac{\sinh\left((\frac{\pi}{2}-\gamma)k\right)}{2\sinh\left((\pi-\gamma)\frac{k}{2}\right)\cosh\left(\frac{\gamma k}{2}\right)}$$

- A non-linear integral equation for counting function $Z(u)$
- Describes the vacuum state of the sine-Gordon model
- One equation, much simpler than TBA (infinitely many equations)

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \prod_{j=1}^{K} f(u_j) \times \frac{\det G^+}{\det G^-}$$

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \boxed{\prod_{j=1}^{K} f(u_j)} \times \frac{\det G^+}{\det G^-}$$

The continuum limit

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$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \boxed{\prod_{j=1}^{K} f(u_j)} \times \frac{\det G^+}{\det G^-}$$

The scalar part

$$\ln \prod_{j=1}^{N} f(u_j) \mapsto -2\varepsilon_a R + \text{UV divergences} + 2 \ln g_{\text{pref}}$$

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \boxed{\prod_{j=1}^{K} f(u_j)} \times \frac{\det G^+}{\det G^-}$$

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Boundary energy

The continuum limit

The exact overlap

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The scalar part

$$\ln \prod_{j=1}^{N} f(u_j) \mapsto -2\varepsilon_a R + \text{UV divergences} + 2\ln g_{\text{pref}}$$

Boundary energy

$$\ln g_{\text{pref}} = -\mathrm{Im}\int_{-\infty}^{\infty} \tilde{f}(-v-\mathrm{i}\xi)\ln\left(1+e^{\mathrm{i}Z(v+\mathrm{i}\xi)}\right) + \texttt{discrete terms}$$

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2}-\mathrm{i}\gamma)}{Q(-\frac{u}{2})} \boxed{\prod_{j=1}^{K} f(u_j)} \times \frac{\det G^+}{\det G^-}$$

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Boundary energy

$$\ln g_{\text{pref}} = -\mathrm{Im} \int_{-\infty}^{\infty} \tilde{f}(-v-\mathrm{i}\xi) \ln\left(1+e^{\mathrm{i}Z(v+\mathrm{i}\xi)}\right) + \texttt{discrete terms}$$

$$\boxed{\tilde{f}(v) = f(v) - \int_{-\infty}^{\infty} f(x) G(-x+v) \mathrm{d}x}$$

depends on boundary parameters

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \prod_{j=1}^{K} f(u_j) \times \frac{\det G^+}{\det G^-}$$

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \prod_{j=1}^{K} f(u_j) \times \frac{\det G^+}{\det G^-}$$

The determinant part

$$\ln \frac{\det G^+}{\det G^-} \mapsto 2 \ln g_{\det} = \ln \frac{\det(1 - \hat{H}^+)}{\det(1 - \hat{H}^-)}$$

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2}-\mathrm{i}\gamma)}{Q(-\frac{u}{2})}\prod_{j=1}^{K} f(u_j) \times \frac{\det G^+}{\det G^-}$$

The determinant part

$$\ln \frac{\det G^+}{\det G^-} \mapsto 2\ln g_{\mathrm{det}} = \ln \frac{\det(1-\hat{H}^+)}{\det(1-\hat{H}^-)}$$

$$\hat{H}^{\pm}[y](x) = \int_\Gamma \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{\varphi(x-u)\pm\varphi(x+u)}{2(1+e^{-\mathrm{i}Z(u)})} y(u)$$

The continuum limit

The exact overlap

$$W(u|\mathbf{u}_K) \propto \frac{Q(-\frac{u}{2} - \mathrm{i}\gamma)}{Q(-\frac{u}{2})} \prod_{j=1}^{K} f(u_j) \times \boxed{\frac{\det G^+}{\det G^-}}$$

The determinant part

$$\ln \frac{\det G^+}{\det G^-} \mapsto 2 \ln g_{\det} = \ln \frac{\det(1 - \hat{H}^+)}{\det(1 - \hat{H}^-)}$$

$$\boxed{\hat{H}^{\pm}[y](x) = \int_{\Gamma} \frac{\mathrm{d}u}{2\pi \mathrm{i}} \frac{\varphi(x-u) \pm \varphi(x+u)}{2(1 + e^{-\mathrm{i}Z(u)})} y(u)}$$

$$\Gamma = \{\mathbb{R} + \mathrm{i}\xi\} \cup \{\mathbb{R} - \mathrm{i}\xi\}, \quad 0 < \xi < \frac{\gamma}{2}$$

$$\varphi(u) = -\frac{\mathrm{i}\sin(2\gamma)}{\sinh(u + \mathrm{i}\gamma)\sinh(u - \mathrm{i}\gamma)}$$

The g-function

The proposal for g-function

$$\ln|g|^2 = 2\ln g_{\text{pref}} + 2\ln g_{\text{det}}$$

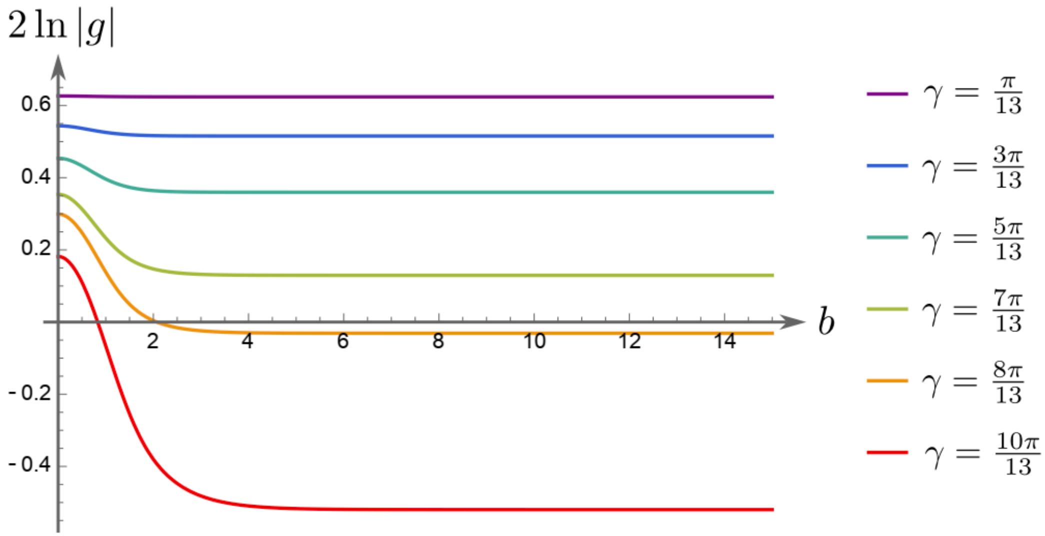
$$\ln g_{\text{pref}} = -\text{Im}\int_{-\infty}^{\infty} \tilde{f}(-v-\mathrm{i}\xi)\ln\left(1+e^{\mathrm{i}Z(v+\mathrm{i}\xi)}\right) + \texttt{discrete terms}$$

$$\ln g_{\text{det}} = \frac{1}{2}\ln\frac{\det(1-\hat{H}^{+})}{\det(1-\hat{H}^{-})}$$

- Solve NLIE and find the counting function $Z(u)$
- Plug in the formula of $\ln|g|^2$
- A well-defined procedure, leads to finite results

Results

The boundary flow



- From free boundary to fixed boundary with fixed R
- Consistent with g-theorem

Conclusions

Boundary entropy or ***g*-function** is an analog of the c-function, important for boundary systems

Off-critical *g*-functions are interesting, can be computed for diagonal scattering theories

For **non-diagonal scattering theories**, TBA leads to divergence, we propose **a lattice approach**

Lattice approach gives **finite results**, and is **simpler** both conceptually and for explicit computation

Outlook

- **Compare with TBA**

  Check results, fix TBA

- **Excited state**

  An extension of g-function, useful for AdS/CFT

- **Other theories**

  Consider theories with higher rank symmetries