Entanglement Swapping in Critical Quantum Spin Chains

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In Memoriam: Ian Affleck (2 July 1952 - 4 Oct 2024)

My memoir on Ian: https://bit.ly/3YfV7Gn

Special online issue for Ian Affleck: in preparation

Universal, finite temperature, crossover functions of the quantum transition in the Ising chain in a transverse field

Subir Sachdev
Department of Physics, P.O. Box 208120, Yale University, New Haven, CT 06520-8120
(September 13, 1995)

APPENDIX A: DERIVATION OF CROSSOVER FUNCTIONS OF THE ISING MODEL

It appears worthwhile to sketch a derivation from first principles using a consistent notation.

First, following Lieb, Schultz and Mattis [5], convert H_I into a free fermion Hamiltonian by the Jordan-Wigner transformation. Then, evaluate the equal-time spin correlator in terms of the free-fermion correlators. This yields an expression for the correlator in terms of a Toeplitz determinant [5,25]:

$$\langle \sigma_z(i)\sigma_z(i+n)\rangle = \begin{vmatrix} G_0 & G_{-1} & \cdots & G_{-n+1} \\ G_1 & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & G_0 & G_{-1} \\ G_{n-1} & & G_1 & G_0 \end{vmatrix} \tag{A1}$$

where

$$G_p = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ip\phi} \tilde{G}(\phi) \tag{A2}$$

with

$$\tilde{G}(\phi) = \left(\frac{1 - ge^{i\phi}}{1 - ge^{-i\phi}}\right)^{1/2} \tanh\left[\frac{J}{k_B T}((1-ge^{i\phi})(1-ge^{-i\phi}))^{1/2}\right] \tag{A3}$$

We can now take the large n limit of (A1) by Szego's lemma [13] provided $g < g_c = 1$. For $g \geq g_c$ a more complicated analysis is necessary to evaluate (A1) directly [11]; we shall not need this here as we shall obtain results for $g \geq g_c$ by our method of analytic continuation. By Szego's lemma we obtain

MO:
correlation length using the thermal transfer matrix

1/(correlation length)
= lowest gap on a finite ring

alternative derivation of (part of) Subir's result

$$\xi_I^{-1} = \frac{k_B T}{\hbar c} f_I\left(\frac{mc^2}{k_B T}\right)$$

$$f_I(s) = \int_0^\infty \frac{dy}{\pi} \ln\coth\frac{\sqrt{y^2+s^2}}{2} + |s|\theta(-s)$$

Ian: we can study a defect line in critical Ising model!

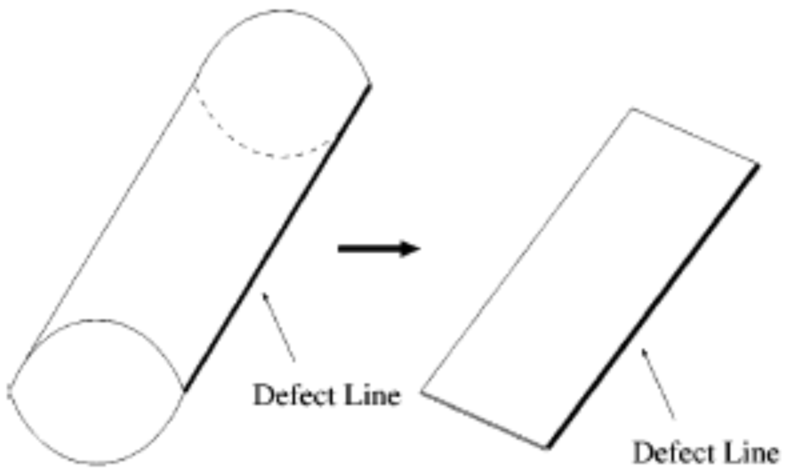
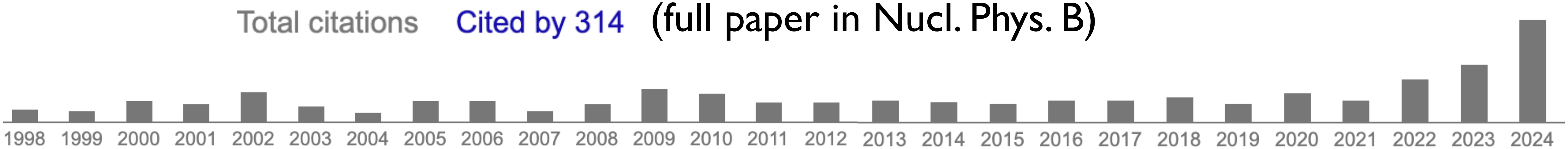


FIG. 1. The folding of the Ising model on a cylinder to a $c = 1$ theory on a strip. We fold at the defect line and also at the line on the opposite side. These lines correspond to the boundary in the folded system.

early example of “conformal defects”

Total citations Cited by 314 (full paper in Nucl. Phys. B)



VOLUME 77, NUMBER 13 PHYSICAL REVIEW LETTERS 23 SEPTEMBER 1996

Defect Lines in the Ising Model and Boundary States on Orbifolds

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(Received 24 June 1996)

Critical phenomena in the two-dimensional Ising model with a defect line are studied using boundary conformal field theory on the $c = 1$ orbifold. Novel features of the boundary states arising from the orbifold structure, including continuously varying boundary critical exponents, are elucidated. New features of the Ising defect problem are obtained including a novel universality class of defect lines and the universal boundary to bulk crossover of the spin correlation function. [S0031-9007(96)01226-4]

PACS numbers: 05.50.+q, 11.25.Hf, 61.72.Lk

arXiv:2406.12377

Entanglement swapping in critical quantum spin chains

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(Dated: June 19, 2024)

The transfer of quantum information between many-qubit states is a subject of fundamental importance in quantum science and technology. We consider entanglement swapping in critical quantum spin chains, where the entanglement between the two chains is induced solely by the Bell-state measurements. We employ a boundary conformal field theory (CFT) approach and describe the measurements as conformal boundary conditions in the replicated field theory. We show that the swapped entanglement exhibits a logarithmic scaling, whose coefficient takes a universal value determined by the scaling dimension of the boundary condition changing operator. We apply our framework to the critical spin-$\frac{1}{2}$ XXZ chain and determine the universal coefficient by the boundary CFT analysis. We also numerically verify these results by the tensor-network calculations. Possible experimental relevance to Rydberg atom arrays is briefly discussed.

with Masahiro Hoshino and Yuto Ashida (Physics UTokyo)

Entanglement Swapping

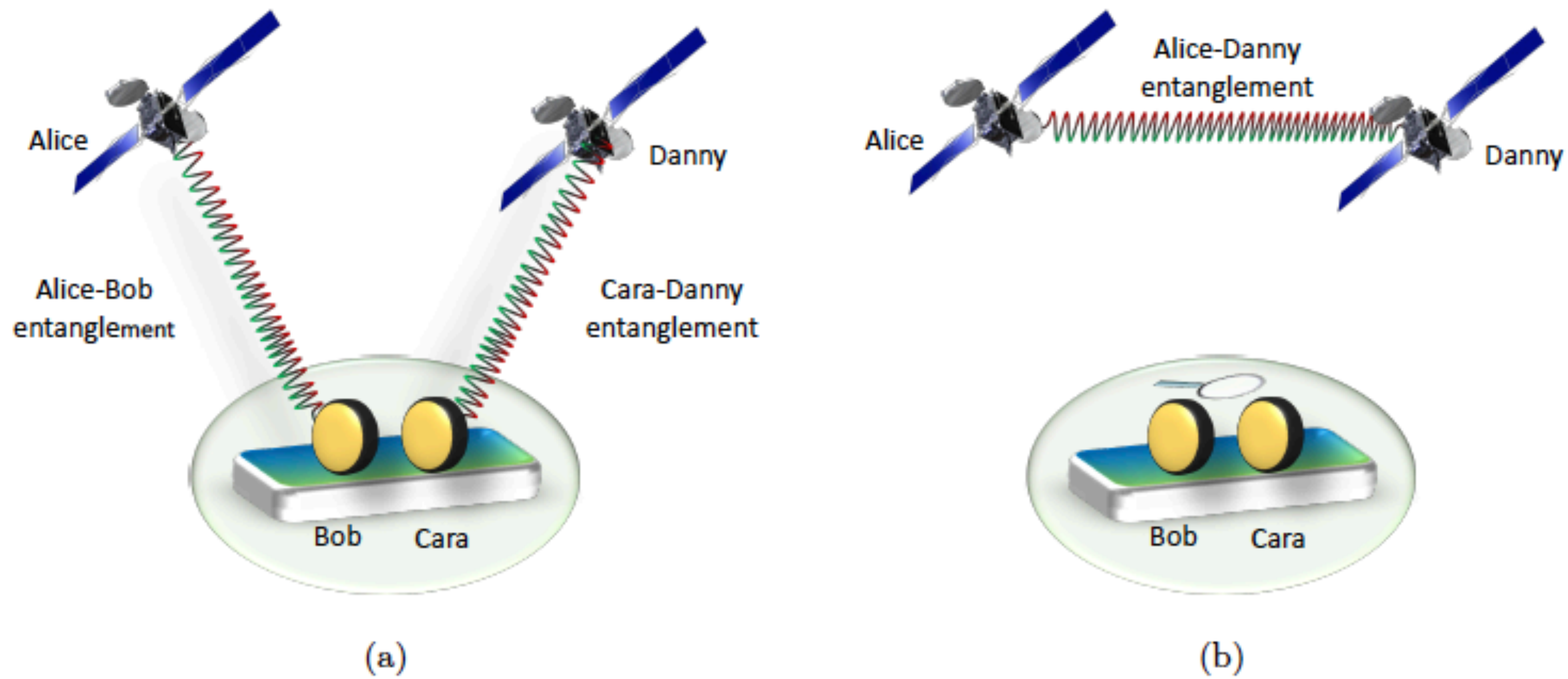


Figure 1: Entanglement swapping. Initially in Fig. (a) entangled pairs are shared between Alice and Bob, and between Cara and Danny. There is no entanglement between Alice and Danny. However, in Fig (b), the measurement on Bob and Cara's qubits project the entanglement between Alice and Danny.

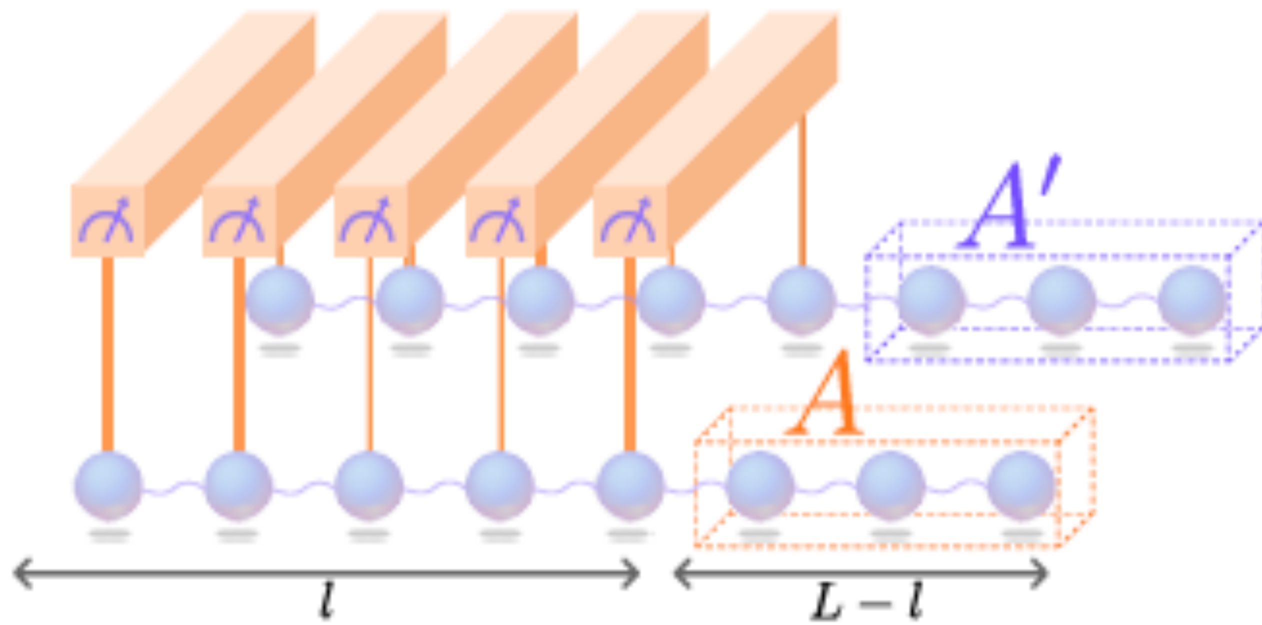
[taken from Zangi et al. arXiv:2212.03413]

Entanglement Swapping in Many-Body

Initial state: ground state of 2 independent S=1/2 XXZ chain

$\sim$ 2 independent Tomonaga-Luttinger Liquids (TLLs)

= 2 component free boson field theory in 1+1dim



Bell-pair measurement on a subsegment of the system

Entanglement is induced between the 2 chains in the unmeasured segment!

XXZ Chain

$$H = \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z), \quad \Delta \in (-1, 1],$$

TLL (free boson field theory)

$$\mathcal{L} = \frac{1}{2\pi K} (\partial_\mu \phi)^2$$

$$\sigma_j^z \simeq \frac{2}{\pi} \partial_x \phi + \frac{(-1)^j}{\pi \gamma} \cos(2\phi),$$

$$\sigma_j^x \pm i \sigma_j^y \simeq \frac{e^{\pm i\theta}}{\sqrt{\pi\gamma}} \left[(-1)^j + \cos(2\phi) \right],$$

Two chains: each chain is represented by a TLL

⇒ 2-component free boson field theory

Bell Pair Measurement

$$|\mathrm{Bell}^{b_1 b_2}\rangle \equiv \frac{1}{\sqrt{2}}\left[|0b_1\rangle + (-1)^{b_2}|1\bar{b}_1\rangle\right]$$

Projective measurement on 4 Bell pair state

→ one of 4 outcomes 00,01,10,11

e.g. "00" $$|\mathrm{Bell}^{00}\rangle \equiv \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

σ^z is same in both chains

total triplet, polarized as Sx=+1

$$\sigma_j^z \simeq \frac{2}{\pi}\partial_x\phi + \frac{(-1)^j}{\pi\gamma}\cos(2\phi),$$

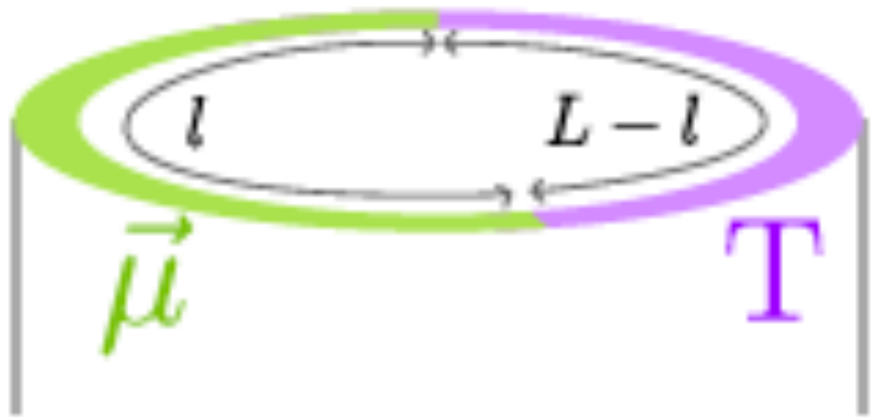
$$\sigma_j^x \pm i\sigma_j^y \simeq \frac{e^{\pm i\theta}}{\sqrt{\pi\gamma}}\left[(-1)^j + \cos(2\phi)\right],$$

(Locally) imposing

$$\phi_- \equiv \phi_1 - \phi_2 = 0,$$

$$\theta_+ \equiv \theta_1 + \theta_2 = 0.$$

(Forced) Uniform Outcome



Imaginary time evolution from τ=-∞

Measurement imposes the boundary condition at τ=0 in the measured segment

Introduce 2×*n* replicas (bra/ket) of the 2-component TLL (4*n*-component TLL in total)

Replica partition function: $Z_n \sim \mathrm{Tr}\rho^n$

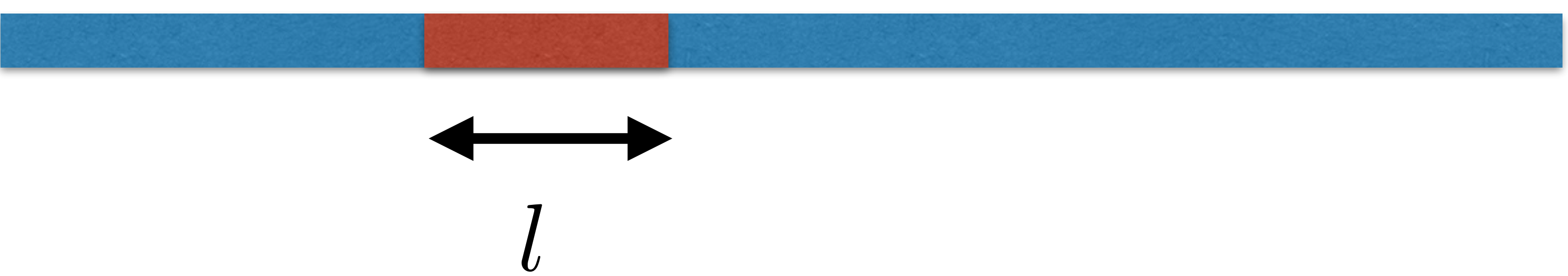
Measurement-imposed boundary condition on measured section

Different Dirichlet boundary condition *T* connecting bra and ket of neighboring replicas $\phi_{j+1}^{\mathrm{bra}} = \phi_j^{\mathrm{ket}}$ on unmeasured section

Entanglement Entropy in CFT

Warmup: the Entanglement Entropy between a finite segment and the rest of the system of a CFT in 1+1D

Holzhey-Larsen-Wilczek 1994/ Calabrese-Cardy 2004



$$S^E = \frac{c}{3} \log l + \text{const.}$$

Why?

ELSEVIER

Nuclear Physics B424 [FS] (1994) 443–467

NUCLEAR PHYSICS B [FS]

Geometric and renormalized entropy in conformal field theory

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Christoph F. E. Holzhey is the founding director of the ICI Berlin Institute for Cultural Inquiry, which he has directed since 2006. He read physics at Oxford (B.A. 1988) and received a PhD in Theoretical Physics from Princeton University in 1993 with a dissertation on the entropy and information loss of black holes. At Columbia University, he studied German Literature (MA 1994, MPhil 1996) and wrote a dissertation on paradoxical pleasures in aesthetics (PhD 2001). Returning to Germany, he was a postdoctoral research fellow at the Max Planck Institute for the History of Science in Berlin (2001-03) and at the Universität Siegen (2003-06).

Entanglement and BCFT

Renyi entanglement entropy

$$S_n^E = \frac{1}{1-n} \log\left(\mathrm{Tr}{\rho_A}^n\right) \qquad \rho_A = \mathrm{Tr}_{\bar{A}}\left(|\Psi_0\rangle\langle\Psi_0|\right)$$

(von Neumann) entanglement entropy: $S^E = \lim_{n\to 1} S_n^E$

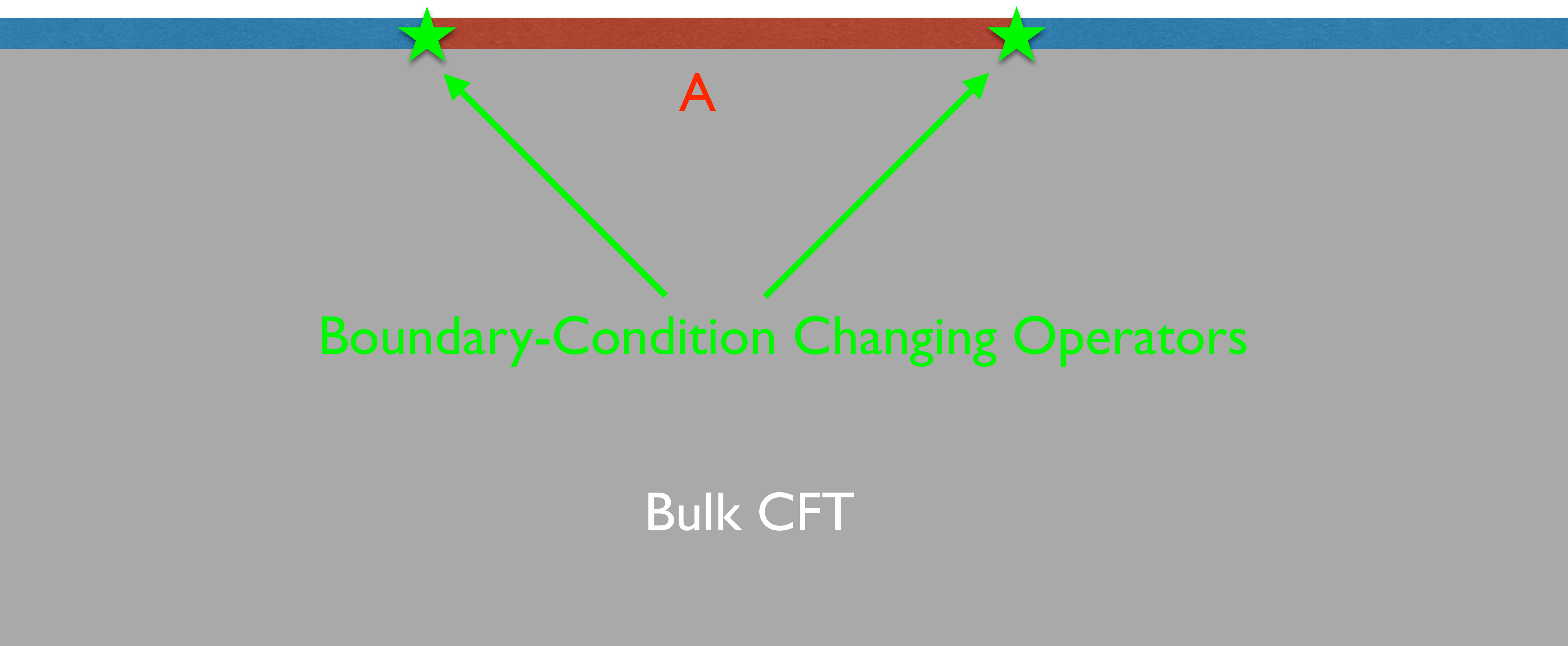
Replica trick

Introduce *n* replicas of the field (*n*: natural number)
"bra" and "ket" states for each field

Entanglement and BCFT

$$Z_n \sim \mathrm{Tr}({\rho_A}^n)$$

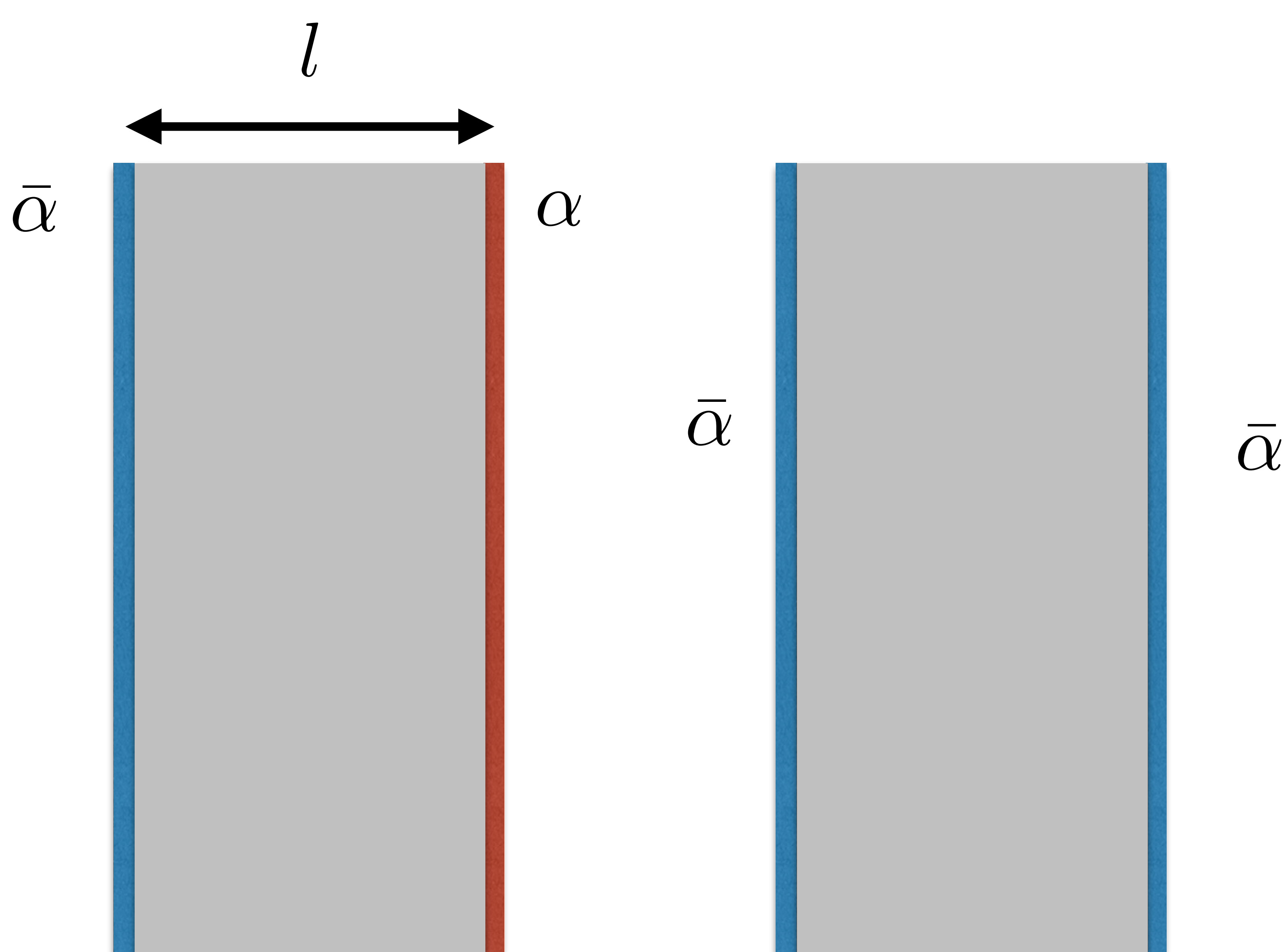
boundary condition $\bar{\alpha}$	boundary condition α	boundary condition $\bar{\alpha}$
$\phi_j^{\mathrm{bra}} = \phi_j^{\mathrm{ket}}$	$\phi_{j+1}^{\mathrm{bra}} = \phi_j^{\mathrm{ket}}$	$\phi_j^{\mathrm{bra}} = \phi_j^{\mathrm{ket}}$



Entanglement Entropy and BCFT

$$\frac{Z_n}{Z^n} \sim \frac{\mathrm{Tr}{\rho_A}^n}{(\mathrm{Tr}\rho)^n} \sim \langle \sigma(0)\sigma(l) \rangle \sim \left(\frac{1}{l}\right)^{2\Delta_\sigma}$$

Δ_σ : boundary scaling dimension of the BCCO



$$\frac{\pi \Delta_\sigma}{l} = E_0^{\alpha\bar{\alpha}} - E_0^{\bar{\alpha}\bar{\alpha}}$$

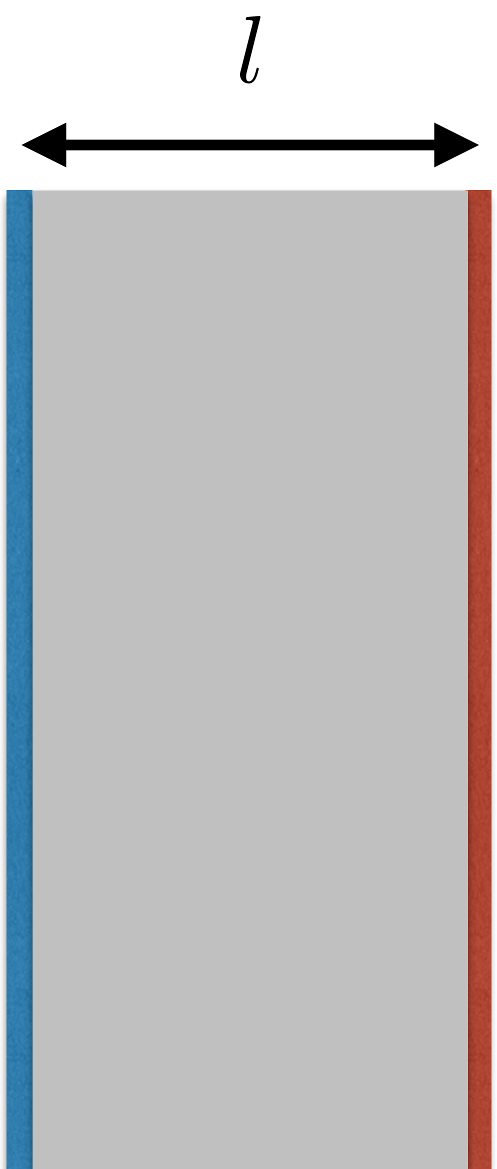
Entanglement Entropy and BCFT

$E_0^{\alpha\bar{\alpha}}$ g.s. energy of the CFT on a ring of circumference *2nl*

Blote-Cardy-Nightingale/ Affleck 1986

$$E_0^{\alpha\bar{\alpha}} \sim -\frac{\pi c}{6}\frac{1}{2nl}$$

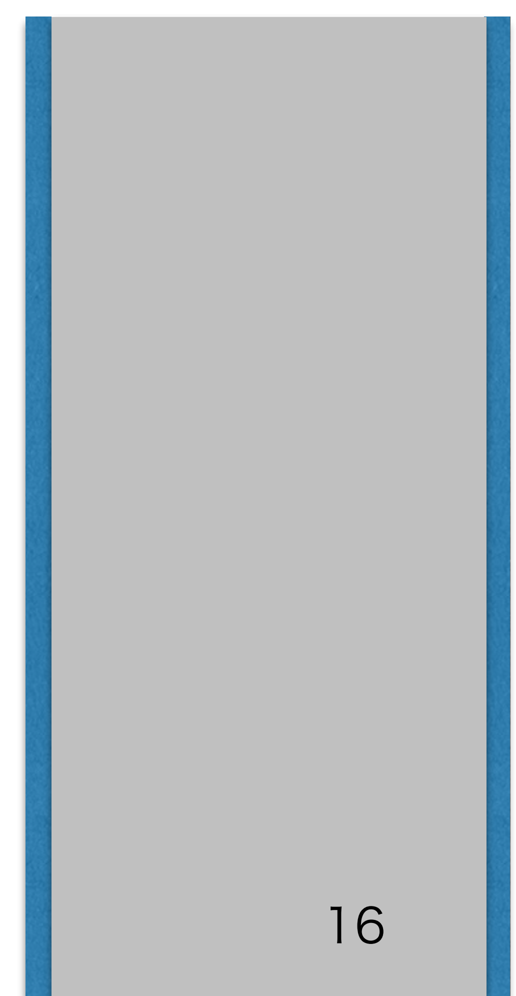
$$E_0^{\bar{\alpha}\bar{\alpha}} \sim -n\frac{\pi c}{6}\frac{1}{2l}$$



$$\frac{\pi \Delta_\sigma}{l} = E_0^{\alpha\bar{\alpha}} - E_0^{\bar{\alpha}\bar{\alpha}} \sim \frac{\pi c}{12 l}\left(n - \frac{1}{n}\right)$$

$\bar{\alpha}$ α

$\phi_j^{\mathrm{bra}} = \phi_j^{\mathrm{ket}}$ $\phi_{j+1}^{\mathrm{bra}} = \phi_j^{\mathrm{ket}}$



$\bar{\alpha}$ $\bar{\alpha}$

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Conformal Invariance, the Central Charge, and Universal Finite-Size Amplitudes at Criticality

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(Received 21 November 1985)

We show that for conformally invariant two-dimensional systems, the amplitude of the finite-size corrections to the free energy of an infinitely long strip of width L at criticality is linearly related to the conformal anomaly number c, for various boundary conditions. The result is confirmed by renormalization-group arguments and numerical calculations. It is also related to the magnitude of the Casimir effect in an interacting one-dimensional field theory, and to the low-temperature specific heat in quantum chains.

VOLUME 56, NUMBER 7 PHYSICAL REVIEW LETTERS 17 FEBRUARY 1986

Universal Term in the Free Energy at a Critical Point and the Conformal Anomaly

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(Received 6 December 1985)

We show that the leading finite-size correction to $\ln Z$ for a two-dimensional system at a conformally invariant critical point on a strip of length L, width β ($\beta << L$), is $(\pi/6)c(L/\beta)$, where c is the conformal anomaly. Equivalently, the leading low-temperature correction to the free energy of a one-dimensional quantum system is $-(\pi/6)cL(kT)^2/\hbar v$, where v is the effective "velocity of light." The latter formula is used to check recently derived critical theories of spin-s quantum chains against Bethe-*Ansatz* solutions.

Derivation of HLW/CC Formula

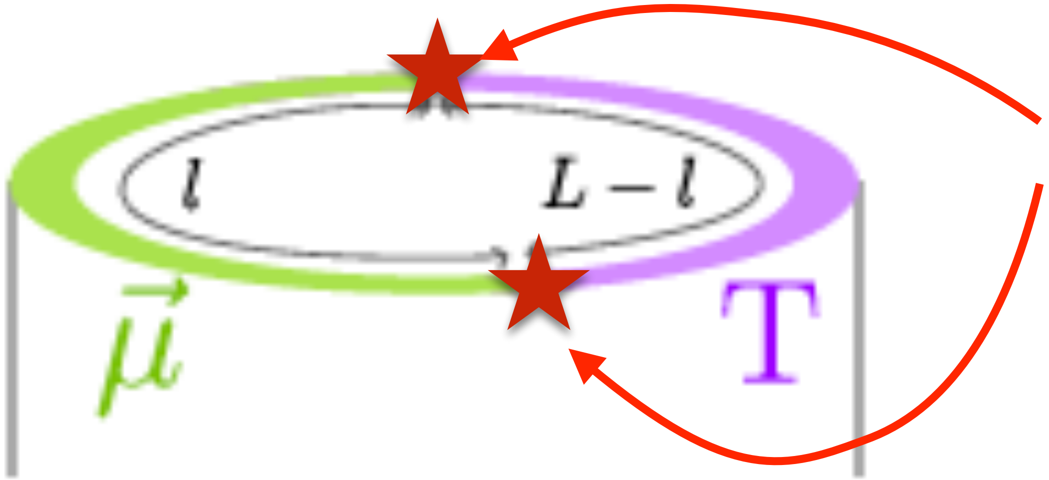
$$\frac{Z_n}{Z^n} \sim \frac{\mathrm{Tr}{\rho_A}^n}{(\mathrm{Tr}\rho)^n} \sim \langle \sigma(0)\sigma(l) \rangle \sim \left(\frac{1}{l}\right)^{2\Delta_\sigma}$$

$$\Delta_\sigma = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

$$S_n^E = \frac{1}{1-n}\log\frac{Z_n}{(Z)^n} \sim \frac{c}{6}\frac{1}{n-1}\left(n - \frac{1}{n}\right)\log l + \mathrm{const.}$$

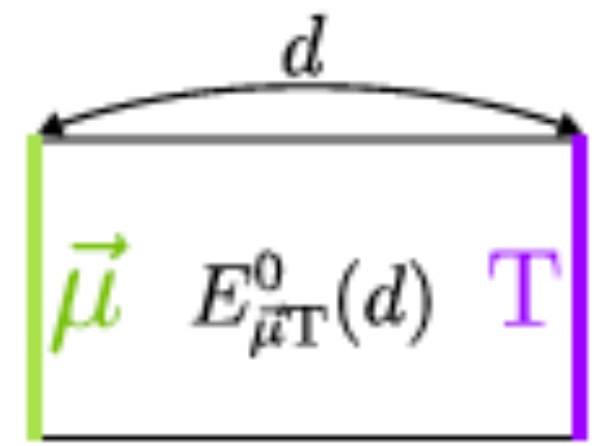
$$S^E = \lim_{n\to 1} S_n^E \sim \frac{c}{3}\log l + \mathrm{const.}$$

Conformal Mapping



Boundary condition changing operators

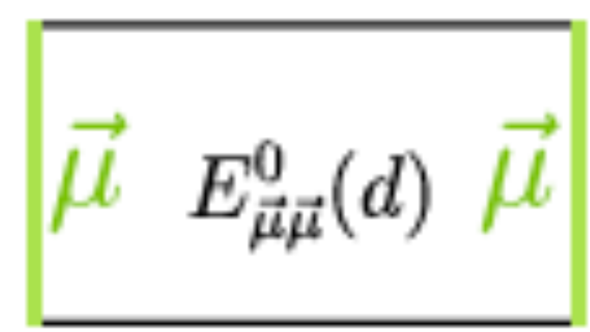
$\mathcal{B}_{T|\vec{\mu}}$



Scaling dimension $\Delta_{T|\vec{\mu}}$ of $\mathcal{B}_{T|\vec{\mu}}$

obtained by conformal mapping

$$\frac{\pi \Delta_{\mathrm{T}|\vec{\mu}}}{d} = E^0_{\vec{\mu}\mathrm{T}}(d) - E^0_{\vec{\mu}\vec{\mu}}(d) = \frac{\pi n}{6d} - \frac{\pi}{24dn},$$



$$\Delta_{T|\vec{\mu}} = \frac{n}{6} - \frac{1}{24n}$$

$$Z_n \sim \left(\frac{L}{\pi}\sin\frac{\pi}{L}\right)^{-2\Delta_{T|\vec{\mu}}}$$

Entanglement Entropy

$$\mathrm{Tr}\Big[(\rho_I^{\vec{\mu}})^n\Big] = \frac{Z_n(I)}{Z^n} = \frac{\langle \mathcal{B}_{\mathrm{T}|\vec{\mu}}(l)\mathcal{B}_{\mathrm{T}|\vec{\mu}}(0)\rangle_n}{\langle \mathcal{B}_{\mathrm{F}|\vec{\mu}}(l)\mathcal{B}_{\mathrm{F}|\vec{\mu}}(0)\rangle_1^n} \sim l^{-2\Delta_{\mathrm{T}|\vec{\mu}}+2n\Delta_{\mathrm{F}|\vec{\mu}}} = l^{-\frac{1}{12}\left(n-\frac{1}{n}\right)}.$$

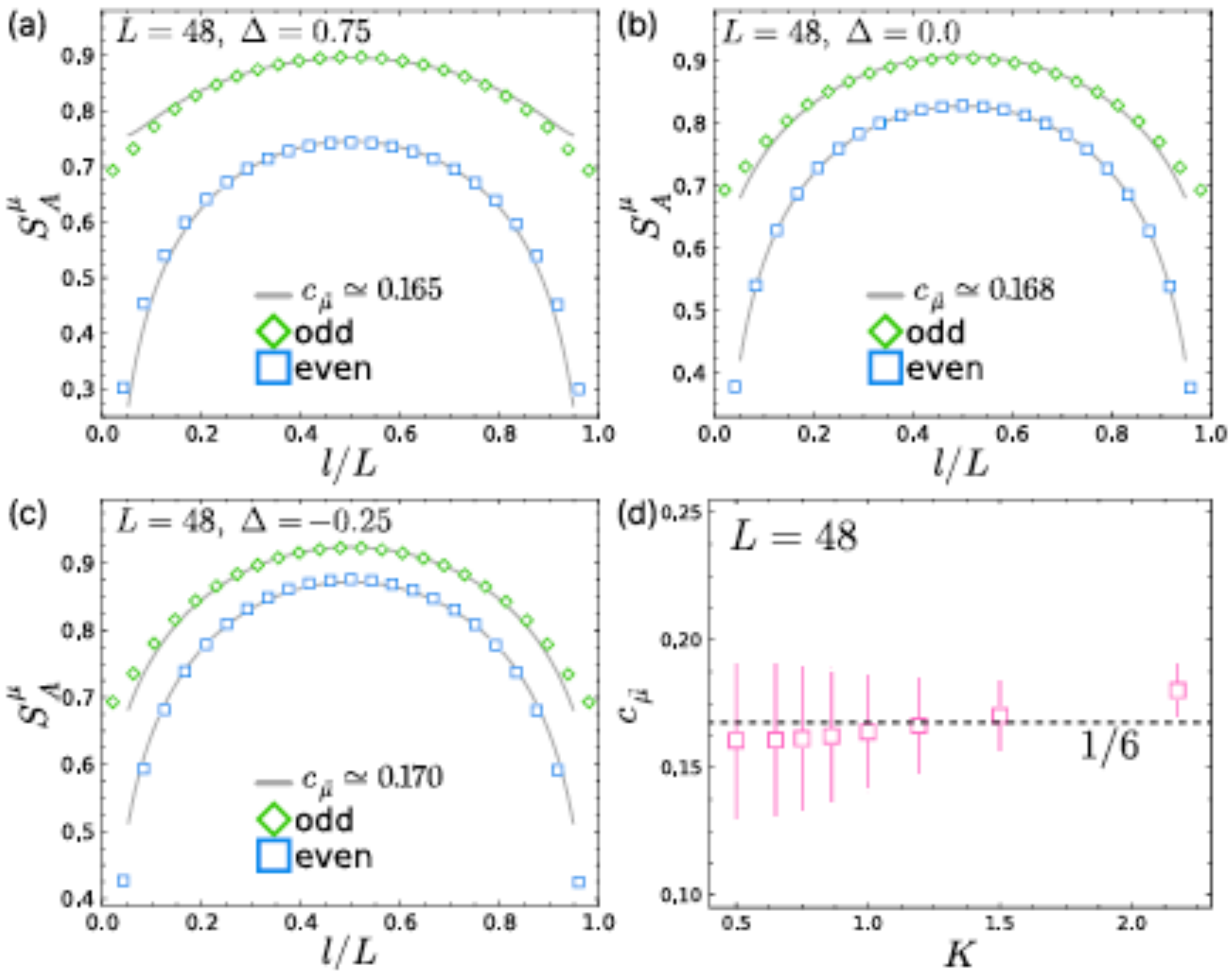
Renyi entanglement entropy

$$S_I^{(n)} = \frac{1}{1-n}\ln\frac{Z_n(I)}{Z^n} = \frac{1}{12}\left(1+\frac{1}{n}\right)\ln\left(\frac{L}{\pi}\sin\frac{\pi l}{L}\right) + \mathrm{const.}$$

Von Neumann entanglement entropy ($n\to 1$)

$$S_I = \frac{1}{6}\ln\left(\frac{L}{\pi}\sin\frac{\pi l}{L}\right) + \mathrm{const.}$$

Numerical Results



universal coefficient

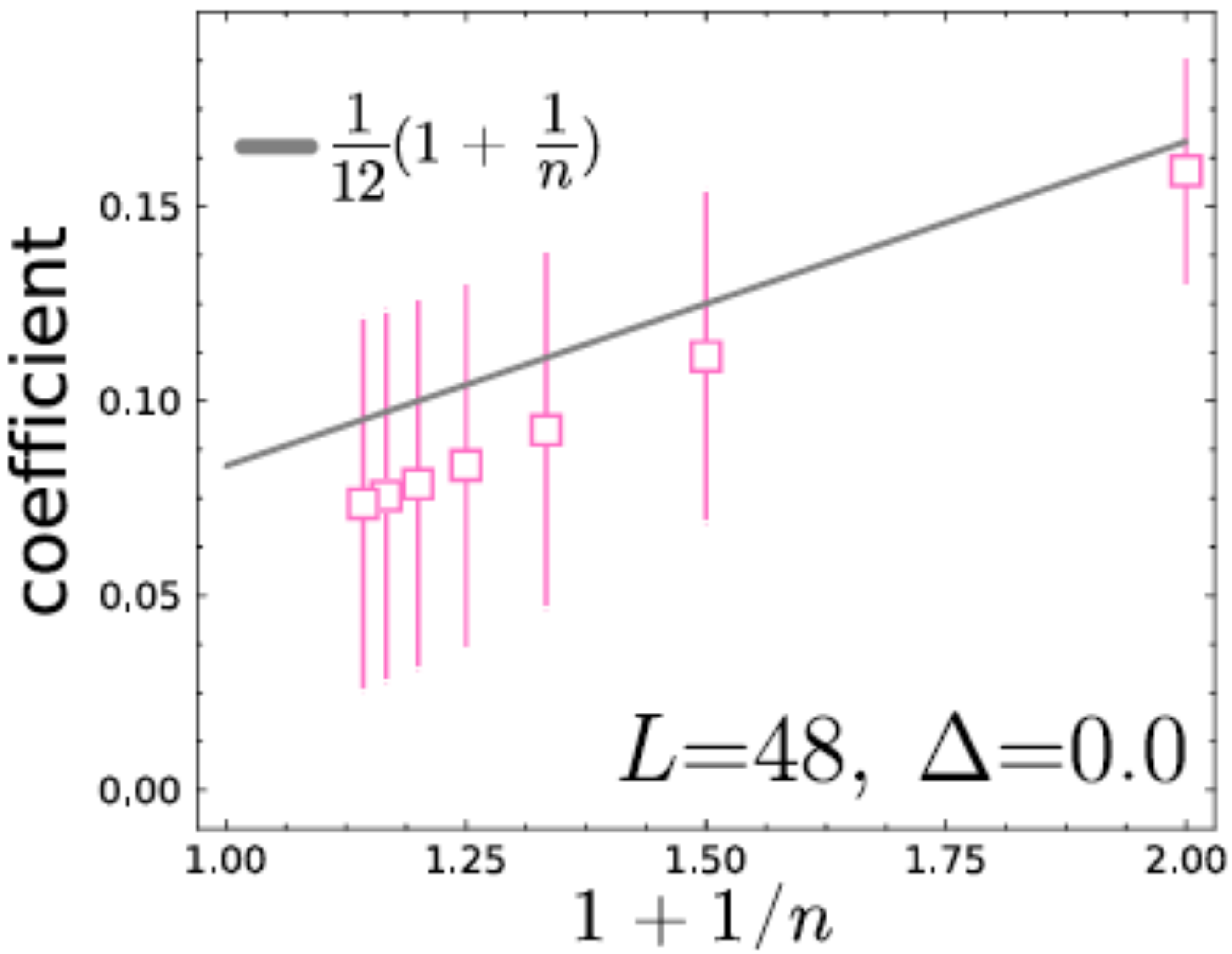
$$c_{\vec{\mu}} \sim \frac{1}{6}$$

independent of Luttinger parameter (i.e. XXZ anisotropy)

agree with CFT!

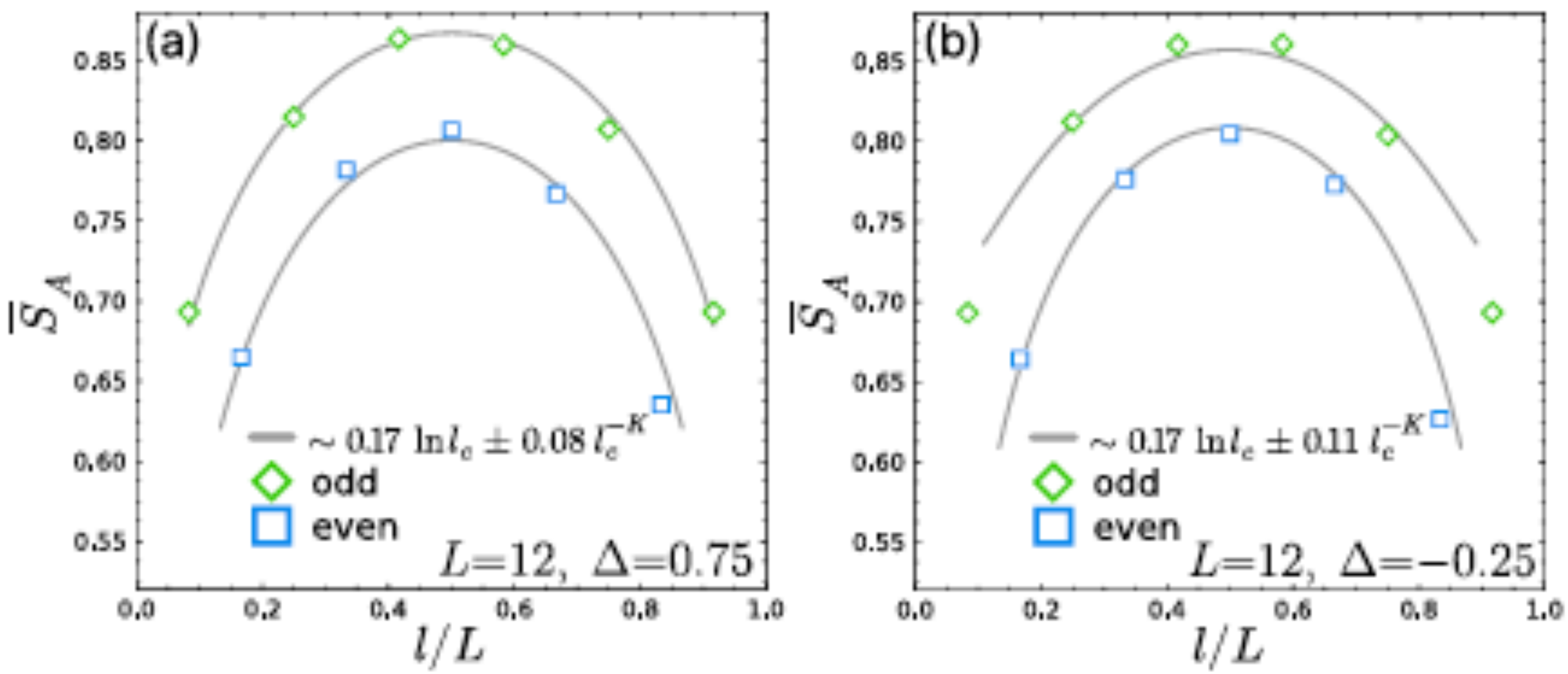
$$S_A^{\vec{\mu}} = c_{\vec{\mu}} \ln\left[\frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right] + c_1(c_2 + (-1)^l)\left[\frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right]^{-K}$$

n-Dependence of Renyi Entropy



$$S_I^{(n)} = \frac{1}{1-n} \ln \frac{Z_n(I)}{Z^n} = \frac{1}{12}\left(1+\frac{1}{n}\right) \ln\left(\frac{L}{\pi} \sin \frac{\pi l}{L}\right) + \text{const.}$$

Averaged Entanglement Entropy



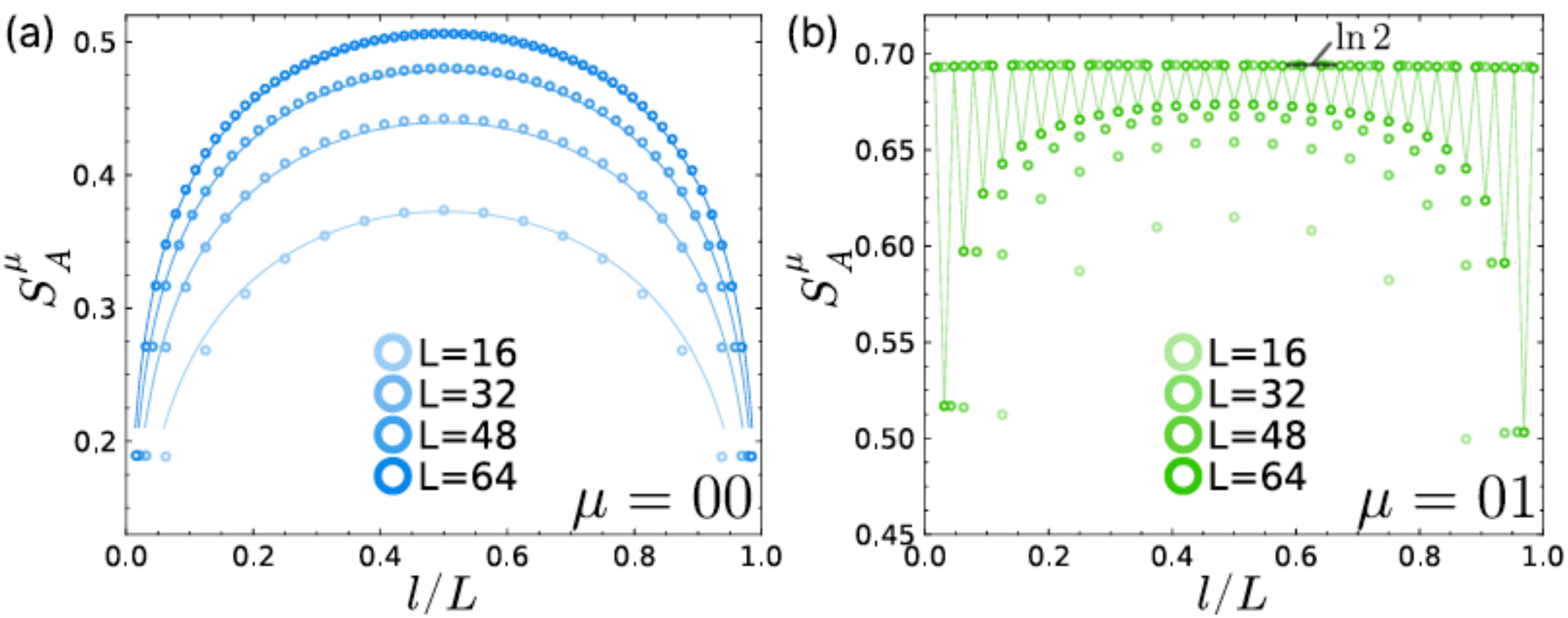
average over all possible outcomes of Bell-pair measurements

appears to follow the same universal behavior as in the case of the uniform outcome (no theoretical derivation at this point)

Mystery with Ising

same problem with 2 critical transverse-field Ising chains

$$H = -\sum_{j=1}^{L} (\sigma_j^z \sigma_{j+1}^z + \sigma_j^x).$$



conformal invariance only for some of the measurement outcomes??

Summary and Outlook

- Measurement-induced entanglement (“Entanglement Swapping”) between two S=1/2 XXZ chains
- Universal behavior of the entanglement entropy derived from boundary CFT
  (Replica partition function = correlation function of boundary condition changing operators)
- Generalization of the celebrated Holzhey-Larsen-Wilczek formula (twist operator → boundary condition changing op)
- Boundary CFT works! (cf. measurement-induced phase transitions) But Ising??
- How to compute averaged EE in CFT?