Eigenvalue relation of the Heisenberg chain for the ground state

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- Introduction
- Heisenberg spin chain with the periodic boundary
  - Eigenvalues relation and their roots.
  - Thermodynamic limit.
- Heisenberg spin chain with general boundary terms
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  - Thermodynamic limit.
- Conclusion and Comments

Quantum integrable systems have many applications in

- String/ Gauge theories: AdS/CFT, Super-symmetric Yang-Mills theories...
- Statistical mechanics: The Ising model, the six-vertex models...
- Condensed Matter Physics: The super-symmetric $t - J$ Model, the Hubbard model...
- Mathematics: Quantum group, Representation theory, Algebraic Topology, ...

There are many methods to solve quantum integrable systems (The case of $T = 0$):

- The Coordinate Bethe Ansatz method (Bethe 1931)
- The Baxter's $T - Q$ relation method (Baxter 1970s)
- The Quantum Inverse Scattering (or Algebraic Bethe Ansatz) method (Faddeev's School 1979s) and its generalizations (such as the analytic Bethe Ansatz method (Reshetikhin 1983 and Mezincescu & Nepomechie 1992, the separation of variables method (Sklyanin 1985, the Lyon's group 2013), the q-Onsager algebra approach (Baseilhac 2006))
- The off-diagonal Bethe Ansaz method (Wang's school 2013s)
- The modified algebraic Bethe Ansatz (Belliard et. al 2013s)

$\vdots$

The Hamiltonian of the closed Heisenberg chain is

$$H = \sum_{k=1}^{N} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = x, y, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar \frac{\partial}{\partial t} h_i = [H, h_i] = 0, \qquad i = 1, \ldots.$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i = trT(u) = A(u) + D(u).$$

Then

$$[t(u),\ t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u-\theta_N)\ldots R_{01}(u-\theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \begin{pmatrix} u+\eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u+\eta \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$, where $\eta = \sqrt{-1}$.

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\;T_0(u)\;T_{0'}(v)=T_{0'}(v)\;T_0(u)\,R_{0\,0'}(u-v).$$

This leads to

$$[t(u),t(v)]=0, \tag{2}$$

which ensures the integrability of the Heisenberg chain with periodic boundary condition.

Besides the YBE, the R-matrix has the following properties

$$\begin{aligned}
&\text{Initial condition}: \; R_{0,j}(0) = \eta P_{0,j},\\
&\text{Unitary relation}: \; R_{0,j}(u)R_{j,0}(-u) = \phi(u) \times \mathrm{id} \otimes \mathrm{id},\\
&\text{Crossing relation}: \; R_{0,j}(u) = -\sigma_0^y R_{0,j}^{t_0}(-u-\eta)\sigma_0^y,\\
&\text{PT-symmetry}: \; R_{0,j}(u) = R_{j,0}(u) = R_{0,j}^{t_0\, t_j}(u),\\
&Z_2\text{-symmetry}: \; \sigma_0^\alpha \sigma_j^\alpha R_{0,j}(u) = R_{0,j}(u)\sigma_0^\alpha \sigma_j^\alpha, \; \text{for } \alpha = x, y, z,\\
&\text{Fusion condition}: \; R_{0,j}(\pm\eta) = \eta(\pm 1 + P_{0,j}) = \pm 2\eta P_{0,j}^{(\pm)},
\end{aligned}$$

where $\phi(u) = \eta^2 - u^2$.

By using the fusion technique (Kulish et al 1981, Kirillov et al, 1986), one can derive the relation

$$t(u)\,t(u-\eta) = a(u)\,d(u-\eta) \times \mathrm{id} + d(u)\,\mathbb{W}(u), \quad d(u) = \prod_{j=1}^{N}(u-\theta_j) = a(u-\eta), \tag{3}$$

where $\mathbb{W}(u)$ is a descendent operator can be given in terms of the fused R-matrix

$$\mathbb{W}(u) = tr_0\left(R_{0\,N}^{(1,\frac{1}{2})}(u-\theta_N)\cdots R_{0\,1}^{(1,\frac{1}{2})}(u-\theta_1)\right).$$

Here the fused R-matrix $R^{(1,\frac{1}{2})}(u)$ is given by

$$R^{(1,\frac{1}{2})}(u) = \left(\begin{array}{cc|cc|cc} u+\eta & & & & & \\ & u-\eta & \sqrt{2}\eta & & & \\ \hline & \sqrt{2}\eta & u & & & \\ & & & u & \sqrt{2}\eta & \\ \hline & & & \sqrt{2}\eta & u-\eta & \\ & & & & & u+\eta \end{array}\right).$$

The transfer matrices $t(u)$ and $\mathbb{W}(u)$ commutate with each other,

$$[t(u),\, t(v)] = [\mathbb{W}(u),\, \mathbb{W}(v)] = [t(u),\, \mathbb{W}(v)] = 0. \tag{4}$$

Moreover, from the definitions we know that they are the operator-valued polynomial of u with degree N. Acting the operators on a common eigenstate $|\Psi\rangle$

$$t(u)\,|\Psi\rangle = \Lambda(u)\,|\Psi\rangle, \quad \mathbb{W}(u)\,|\Psi\rangle = W(u)\,|\Psi\rangle,$$

we have the very relation between $\Lambda(u)$ and $W(u)$, called it as the $t-W$ relation,

$$\Lambda(u)\,\Lambda(u-\eta) = a(u)\,d(u-\eta) + d(u)W(u), \tag{5}$$

where the polynomials $\Lambda(u)$ and $W(u)$ with the degree N have decompositions

$$\Lambda(u) = 2\prod_{j=1}^{N}(u - z_j + \frac{\eta}{2}), \quad W(u) = 3\prod_{j=1}^{N}(u - w_j).$$

The eigenvalues of the Hamiltonian can be expressed in terms of the zero roots $\{z_j\}$ as

$$E = -2\eta \times \sum_{j=1}^{N} \frac{1}{z_j - \frac{\eta}{2}} - N.$$

Taking $u = \theta_j$ for the $t - W$ relation (5), we have

$$\Lambda(\theta_j)\Lambda(\theta_j - \eta) = a(\theta_j)\, d(\theta_j - \eta), \quad j = 1, \cdots, N. \tag{6}$$

The relations allow us that the eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ can be parameterized by some parameters $\{\lambda_1, \cdots, \lambda_M | M = 0, \cdots, N\}$ as follows (see also the conventional Bethe ansatz methods):

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \quad Q(u) = \prod_{j=1}^{M}(u - \lambda_j),$$

the parameters $\{\lambda_i\}$ should satisfy Bethe ansatz equations,

$$\prod_{k\neq j}^{M}\frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta} = \prod_{l=1}^{N}\frac{\lambda_j - \theta_l + \eta}{\lambda_j - \theta_l}, \qquad j = 1, \ldots, M.$$

$$\text{BAEs} \;\Rightarrow\; \Lambda(u) \;\Rightarrow\; W(u)$$

Taking $\{u = z_j - \frac{\eta}{2}\}$, $\{u = w_j\}$ and $\{\theta_j = 0\}$, we have

$$(z_j + \frac{\eta}{2})^N (z_j - \frac{3}{2}\eta)^N = -(z_j - \frac{\eta}{2})^N W(z_j - \frac{\eta}{2}), \quad j = 1, \cdots, N, \tag{7}$$

$$\Lambda(w_j)\Lambda(w_j - \eta) = (w_j + \eta)^N (w_j - \eta)^N, \quad j = 1, \cdots, N. \tag{8}$$

The above equations allow one to determine the polynomials $\Lambda(u)$ and $W(u)$. Moreover, one can show that

$$\{z_j^*\} = \{z_j\}, \quad \{w_j^*\} = \{w_j\}.$$

The eigenvalue can be given in terms of a homogeneous $T - Q$ relation

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{9}$$

$$W(u) = a(u)\frac{Q(u-2\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)Q(u-2\eta)}{Q(u)Q(u-\eta)} + d(u-\eta)\frac{Q(u+\eta)}{Q(u-\eta)},$$

where the roots of $Q(u)$ satisfy the Bethe ansatz equations (BAEs)

$$\frac{a(\lambda_j)}{d(\lambda_j)} = -\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}, \quad j = 1, \cdots, M. \tag{10}$$

BAEs $\Rightarrow$ TBA

Alternatively, we may consider the root patterns of $\{z_j\}$ and $\{w_j\}$ for some particular states such as the ground state.

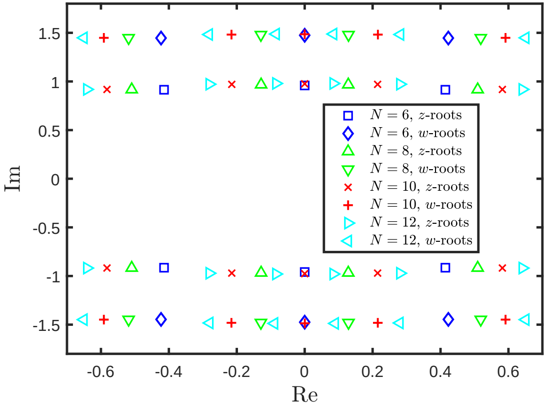

Fig. 1. Patterns of zero roots at the ground state with $N = 6, 8, 10, 12$. The data are obtained by using the exact numerical diagonalization with $\{\theta_j = 0\}$.

For the ground state, we have

- All the z-roots form conjugate pairs as $\{u_j^{(2)} \pm \eta | j = 1, \cdots, N/2\}$ with real $u_j^{(2)}$.
- All the w-roots form conjugate pairs as $\{\bar{u}_j^{(2)} \pm \frac{3\eta}{2} | j = 1, \cdots, N/2\}$ with real $\bar{u}_j^{(2)}$.

the corresponding eigenvalues $\Lambda_g(u)$ and $W_g(u)$ can be given as

$$\Lambda_g(u) = 2\prod_{j=1}^{N/2}(u - u_j^{(2)} - \frac{\eta}{2})(u - u_j^{(2)} + \frac{3\eta}{2}),$$

$$W_g(u) = 3\prod_{j=1}^{N/2}(u - \bar{u}_j^{(2)} - \frac{3}{2}\eta)(u - \bar{u}_j^{(2)} + \frac{3}{2}\eta).$$

In the thermodynamic limit $N \to \infty$, $u_j^{(2)}$ and $\bar{u}_j^{(2)}$ become dense on the real line

$$\Lambda_g(u) = e^{N[\lambda_g^{(0)}(u) + \frac{1}{N}\lambda_g^{(1)}(u) + O(\frac{1}{N^2})]},$$

$$W_g(u) = e^{N[w_g^{(0)}(u) + \frac{1}{N}w_g^{(1)}(u) + O(\frac{1}{N^2})]},$$

and form the densities of $u_j^{(2)}$ and $\bar{u}_j^{(2)}$:

$$\frac{\partial}{\partial u}\lambda_g^{(\beta)}(u) = \int_{-\infty}^{\infty}\Big(\frac{1}{u-\lambda-\frac{\eta}{2}}+\frac{1}{u-\lambda+\frac{3\eta}{2}}\Big)\rho_\Lambda^{(\beta)}(\lambda)\mathrm{d}\lambda, \quad \lambda_g^{(\beta)}(0), \quad \beta=0,1,$$

$$\frac{\partial}{\partial u}w_g^{(\beta)}(u) = \int_{-\infty}^{\infty}\Big(\frac{1}{u-\lambda-\frac{3\eta}{2}}+\frac{1}{u-\lambda+\frac{3\eta}{2}}\Big)\rho_w^{(\beta)}(\lambda)\mathrm{d}\lambda, \quad \lambda_g^{(\beta)}(0), \quad \beta=0,1.$$

The relation (6) implies that

$$\frac{\partial}{\partial u}[\lambda_g^{(0)}(u)+\lambda_g^{(0)}(u-\eta)] = \frac{1}{u+\eta}+\frac{1}{u-\eta}, \qquad \lambda_g^{(0)}(0)=0,$$

$$\frac{\partial}{\partial u}[\lambda_g^{(1)}(u)+\lambda_g^{(1)}(u-\eta)] = 0, \qquad \lambda_g^{(1)}(0)=0.$$

Finally, we obtain

$$\rho_\Lambda^{(0)}(\lambda)=\frac{1}{2\cosh(\pi\lambda)}, \qquad \rho_\Lambda^{(1)}(\lambda)=0.$$

The density of the z-roots allow us rederive

- The ground energy E_g

$$E_g = -2Ni\int_{-\infty}^{\infty}\left(\frac{1}{\lambda+\frac{i}{2}}+\frac{1}{\lambda-\frac{3i}{2}}\right)\left(\rho_w^{(0)}(\lambda)+\rho_w^{(1)}(\lambda)\right)\mathrm{d}\lambda - N = (1-4\ln 2)N.$$

- The eigenvalues of the transfer matrix for the ground state

$$\Lambda_g(u) = \left(\frac{2\Gamma(1+\frac{iu}{2})\Gamma(\frac{3}{2}-\frac{iu}{2})}{\Gamma(\frac{1}{2}+\frac{iu}{2})\Gamma(1-\frac{iu}{2})}\right)^N e^{O(\frac{1}{N})}.$$

⋮

Substituting $z_j = u_j^{(2)} + \frac{\eta}{2}$ and $z_j = u_j^{(2)} + \frac{\eta}{2}$ into (7) resectively, we obtain

$$(u_j^{(2)} + \frac{3\eta}{2})^N\,(u_j^{(2)} - \frac{\eta}{2})^N = -(u_j^{(2)} + \frac{\eta}{2})^N\, W_g(u_j^{(2)} + \frac{\eta}{2}), \quad j = 1, \cdots, \frac{N}{2},$$

$$(u_j^{(2)} + \frac{\eta}{2})^N\,(u_j^{(2)} - \frac{3\eta}{2})^N = -(u_j^{(2)} - \frac{\eta}{2})^N W_g(u_j^{(2)} - \frac{\eta}{2}), \quad j = 1, \cdots, \frac{N}{2},$$

which implies

$$(u_j^{(2)} + \frac{3\eta}{2})^N(u_j^{(2)} - \frac{3\eta}{2})^N = W_g(u_j^{(2)} + \frac{\eta}{2})W_g(u_j^{(2)} - \frac{\eta}{2}), \quad j = 1, \cdots, \frac{N}{2}.$$

Namely,

$$\frac{1}{N}\frac{\partial}{\partial u}\ln[W_g(u + \frac{\eta}{2})W_g(u - \frac{\eta}{2})] = \frac{1}{u + \frac{3\eta}{2}} + \frac{1}{u - \frac{3\eta}{2}} + O(\frac{1}{N^2}),$$

$$\frac{1}{N}\ln W_g(0) = \frac{1}{N}\ln 3 + O(\frac{1}{N^2}).$$

As a consequence, we have

$$\rho_w^{(0)}(\lambda) = \frac{1}{2\cosh(\pi\lambda)} = \rho_\Lambda^{(0)}(\lambda), \qquad \rho_w^{(1)}(\lambda) = 0 = \rho_\Lambda^{(1)}(\lambda),$$

which leads to

$$W_g(u) = 3\left(\frac{(u+\eta)(u-\eta)}{u}\right)^N \left(\tanh\frac{\pi u}{2}\right)^N e^{O(\frac{1}{N})}. \tag{11}$$

The $t - W$ relation (6) becomes

$$\Lambda_g(u)\Lambda_g(u-\eta) = (u+\eta)^N(u-\eta)^N\left[1 + 3\left(\tanh\frac{\pi u}{2}\right)^N e^{O(\frac{1}{N})}\right]. \tag{12}$$

This gives rise to the inverse relation

$$\Lambda_g(u)\Lambda_g(u-\eta) = (u+\eta)^N(u-\eta)^N\left[1 + e^{-\delta N}\right], \quad \text{for a postive } \delta.$$

The Hamiltonian of the Heisenberg chain with unparallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{\eta}{p}\sigma_1^z + \frac{\eta}{q}(\sigma_N^z + \xi \sigma_N^x). \quad (13)$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = tr(K^+(u)\mathcal{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix},$$

with the boundary parameters

$$p^* = -p, \quad q^* = -q, \quad \xi^* = \xi.$$

The Hamiltonian can be given in terms of the transfer matrix

$$H = \eta \frac{\partial}{\partial u} \ln t(u)|_{u=0,\{\theta_j\}=0} - N.$$

Following the similar fusion technique, we can derive the relation of the transfer matrices

$$t(u)\,t(u-\eta)=\frac{\Delta(u)\times \mathrm{id}}{(u+\frac{\eta}{2})(u-\frac{\eta}{2})}+\frac{u^2\,\bar{d}(u)}{(u+\frac{\eta}{2})(u-\frac{\eta}{2})}\mathcal{W}(u),\quad \Delta(u)=a(u)d(u-\eta).$$

where

$$a(u)=(u+\eta)(u+p)(\sqrt{1+\xi^2}\,u+q)\prod_{j=1}^{N}(u-\theta_j+\eta)(u+\theta_j+\eta),$$

$$d(u)=u(u-p+\eta)(\sqrt{1+\xi^2}(u+\eta)-q)\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j),$$

$$\bar{d}(u)=\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j).$$

The associated transfer matrices $t(u)$ and $\mathcal{W}(u)$ commutate with each other,

$$[t(u),\,t(v)]=[\mathcal{W}(u),\,\mathcal{W}(v))]=[t(u),\,\mathcal{W}(v)]=0.$$

Denote the corresponding eigenvalues of the transfer matrices by $\bar{\Lambda}(u)$ and $\bar{W}(u)$, we have

$$\Delta(u) - (u+\frac{\eta}{2})\,(u-\frac{\eta}{2})\bar{\Lambda}(u)\bar{\Lambda}(u-\eta) = u^2\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j)\,\bar{W}(u), \tag{14}$$

where $\bar{\Lambda}(u)$ (or $\bar{W}(u)$) is a polynomial of u with degree $2N+2$ (or $2N+4$):

$$\begin{aligned}\bar{\Lambda}(u) &= 2\prod_{j=1}^{N+1}(u-z_j+\frac{\eta}{2})(u+z_j+\frac{\eta}{2}), \quad \bar{\Lambda}(-u-\eta)=\bar{\Lambda}(u),\\ \bar{W}(u) &= (\xi^2-3)\prod_{k=1}^{N+2}(u-w_k)(u+w_k), \quad \bar{W}(-u)=\bar{W}(u).\end{aligned}$$

The roots satisfy the equations:

$$\Delta(z_j-\frac{\eta}{2}) = (z_j-\frac{\eta}{2})^{2N+2}\,\bar{W}(z_j-\frac{\eta}{2}), \quad j=1,\cdots,N+1, \tag{15}$$

$$\Delta(w_k) = (w_k+\frac{\eta}{2})(w_k-\frac{\eta}{2})\bar{\Lambda}(w_k)\bar{\Lambda}(w_k-\eta), \quad k=1,\cdots,N+2. \tag{16}$$

Let us consider the root patterns of $\{z_j\}$ and $\{w_j\}$ for the ground state.

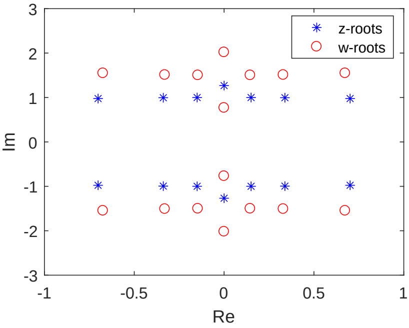

Fig. 2. The patterns of z-roots and w-roots in complex plane at the ground state with $N = 6, \eta = i, p = -1.2i, \bar{q} = 0.8i, \xi = 1$. The data are obtained by the exact numerical diagonalization. The blue asterisks indicate z-roots $\{z_j + \frac{\eta}{2}\}$ and red circles represent the w-roots $\{w_j\}$ with the inhomogeneous parameters $\{\theta_j = 0\}$.

- The z-roots form conjugate pairs as $\{\pm z_1\eta, u_j^{(2)}\pm\eta|j=1,\cdots,N\}$.
- The w-roots form conjugate pairs as $\{\pm\chi_1\eta,\pm\chi_2\eta, w_j^{(2)}\pm\frac{3\eta}{2}|j=1,\cdots,N\}$.

the corresponding eigenvalues $\bar{\Lambda}_g(u)$ and $\bar{W}_g(u)$ can be given as

$$\begin{aligned}
\bar{\Lambda}_g(u) &= 2\big(u-(z_1-\frac{1}{2})\eta\big)\big(u+(z_1+\frac{1}{2})\eta\big)\prod_{j=1}^{N/2}(u-u_j^{(2)}-\frac{\eta}{2})(u+u_j^{(2)}-\frac{\eta}{2})\\
&\quad\times(u-u_j^{(2)}+\frac{3\eta}{2})(u+u_j^{(2)}+\frac{3\eta}{2})\\
&\approx 2\big(u-(z_1-\frac{1}{2})\eta\big)\big(u+(z_1+\frac{1}{2})\eta\big)e^{2N(\bar{\lambda}_g^{(0)}(u)+\frac{1}{2N}\bar{\lambda}_g^{(1)}(u)+O(\frac{1}{N^2}))},
\end{aligned}$$

$$\begin{aligned}
\bar{W}_g(u) &= (\xi^2-3)(u-\chi_1\eta)(u+\chi_1\eta)(u-\chi_2\eta)(u+\chi_2\eta)\prod_{j=1}^{N/2}(u-w_j^{(2)}-\frac{3}{2}\eta)(u+w_j^{(2)}-\frac{3}{2}\eta)\\
&\quad\times(u-w_j^{(2)}+\frac{3}{2}\eta)(u+w_j^{(2)}+\frac{3}{2}\eta)\\
&\approx (\xi^2-3)(u-\chi_1\eta)(u+\chi_1\eta)(u-\chi_2\eta)(u+\chi_2\eta)e^{2N(\bar{\omega}_g^{(0)}(u)+\frac{1}{2N}\bar{\omega}_g^{(1)}(u)+O(\frac{1}{N^2}))},\\
\bar{\Lambda}_g(0) &\approx 2(z_1-\frac{1}{2})(z_1+\frac{1}{2})e^{2N(\bar{\lambda}_g^{(0)}(u)+\frac{1}{2N}\bar{\lambda}_g^{(1)}(u)+O(\frac{1}{N^2}))}=2\,p\,q\equiv 2\,p\,\bar{q}\sqrt{1+\xi^2},
\end{aligned}$$

$$
\begin{aligned}
\bar{\Lambda}_g(u) &= \frac{8\sqrt{1+\xi^2}}{u+\frac{\eta}{2}} \frac{\cosh(\frac{\pi u}{2}-\frac{i\pi}{4})}{\sinh(\frac{\pi u}{2}-\frac{i\pi}{4})} \frac{\Gamma(1+\frac{iu}{2})\Gamma(\frac{3}{2}-\frac{iu}{2})}{\Gamma(\frac{1}{2}+\frac{iu}{2})\Gamma(1-\frac{iu}{2})} \frac{\Gamma(\frac{p+1}{2}+\frac{iu}{2})\Gamma(\frac{p+2}{2}-\frac{iu}{2})}{\Gamma(\frac{p}{2}+\frac{iu}{2})\Gamma(\frac{p+1}{2}-\frac{iu}{2})} \\
&\quad \times \frac{\Gamma(\frac{\bar{q}+1}{2}+\frac{iu}{2})\Gamma(\frac{\bar{q}+2}{2}-\frac{iu}{2})}{\Gamma(\frac{\bar{q}}{2}+\frac{iu}{2})\Gamma(\frac{\bar{q}+1}{2}-\frac{iu}{2})} \left(\frac{2\Gamma(1+\frac{iu}{2})\Gamma(\frac{3}{2}-\frac{iu}{2})}{\Gamma(\frac{1}{2}+\frac{iu}{2})\Gamma(1-\frac{iu}{2})}\right)^{2N} e^{O(\frac{1}{N})}, \qquad (17) \\
\bar{W}_g(u) &= (\xi^2-3)(u-p\eta)(u+p\eta)(u-\bar{q}\eta)(u+\bar{q}\eta)\tanh^2\frac{\pi u}{2} \\
&\quad \times \frac{(u+\eta)^{2N+1}(u-\eta)^{2N+1}}{u^{2N+2}} \left(\tanh\frac{\pi u}{2}\right)^{2N} e^{O(\frac{1}{N})},
\end{aligned}
$$

which leads to the relation

$$
\begin{aligned}
(u+\frac{\eta}{2})(u-\frac{\eta}{2})\bar{\Lambda}_g(u)\bar{\Lambda}_g(u-\eta) &= (1+\xi^2)(u-p\eta)(u+p\eta)(u-\bar{q}\eta)(u+\bar{q}\eta) \\
&\quad \times (u-\eta)^{2N+1}(u+\eta)^{2N+1}\left\{1-\frac{(\xi^2-3)}{1+\xi^2}\left(\tanh\frac{\pi u}{2}\right)^{2N+2} e^{O(\frac{1}{N})}\right\}. \qquad (18)
\end{aligned}
$$

VI. Conclusion and comments

So far, we have used an unified method to solve the eigenvalue of the ground state for quantum integrable spin chain with/without $U(1)$-symmetry:

- The spin-$\frac{1}{2}$ Heisenberg chain with periodic boundary fields.
- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields.
- The open spin chains with general boundary condition associated with the other algebras.
- The super-symmetric t-J model with unparallel boundary fields.
- The Hubbard model with unparallel boundary fields.

Thanks for your attentions