Peierls Phenomenon via Bethe Ansatz

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Joint work with Valdemar Melin, Yuta Sekiguchi, P. W., and Konstantin Zarembo

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Cnoidal wave ⟷ Bethe Ansatz

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Cnoidal wave $\longleftrightarrow$ Bethe Ansatz

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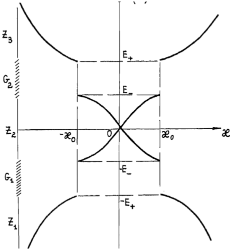
Near periodic swell in shallow water near Panama, 1938

Bethe Ansatz

$$Lp_i = 2\pi n_i + \sum_{j\neq i} \Phi(p_i, p_j)$$

Spectral curve

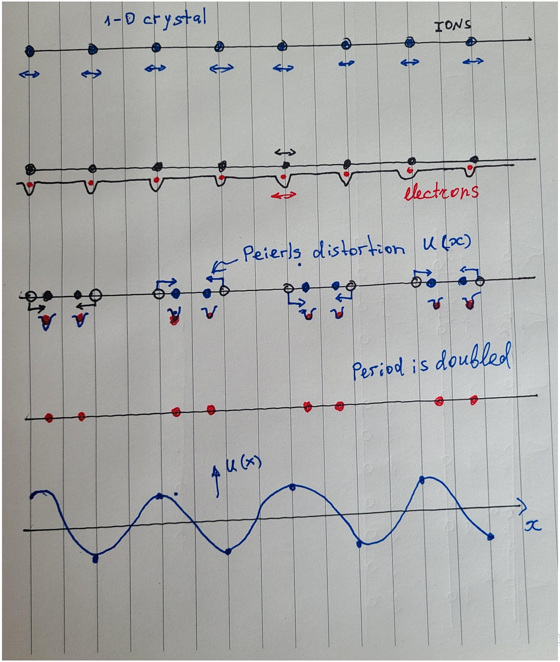


$$\left(-\frac{d^2}{dx^2} + q(x)\right)\Psi(x) = E\,\Psi(x)$$

Periodic solution:

$$q(x) \Rightarrow \text{ spectrum } E(p) \Rightarrow$$

$$\text{spectral curve } p^2 = f(E)$$


1-D crystal
IONS
electrons
Peierls distortion u(x)
Period is doubled
u(x)
x

Rudolf Peierls

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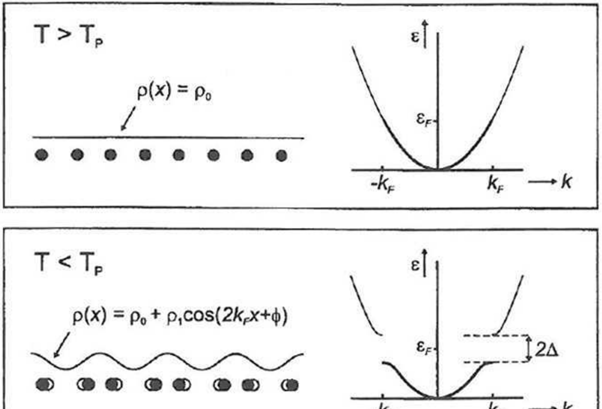
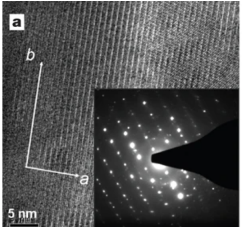
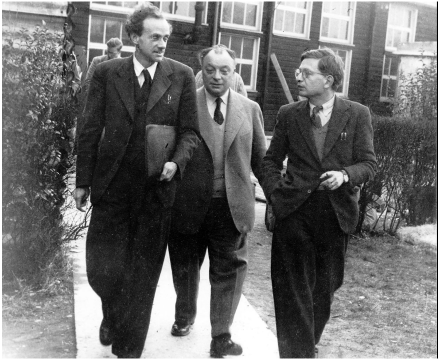


ELECTRONIC CRYSTAL

W. Little 1964



RUDOLF PEIERLS: 1907-1995



Peierls problem: Discrete version

1) Schrödinger equation: $c_n\psi_{n+1} + c_{n-1}\psi_{n-1} = \varepsilon\psi_n$

2) Find the spectrum as a functional of $C = \{c_1, c_n, \ldots, c_N\}$: $\varepsilon[C]$

3) Compute the energy: sum over all eigenvalues below μ: $E[C] = \sum_{\varepsilon<\mu} \varepsilon[C] + \sum_n c_n^2$

Peierls Problem: Find C which gives the minimum to the energy $\min_C E[C]$

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Krichever Solution: The extrema are given by the *finite-gap solutions* of the Toda chain.

The minimum is given by the *one-gap* solution.

Peierls problem and the Lax operator

1) The Schrödinger equation with a variable hopping:

$$L\psi = c_n \psi_{n+1} + c_{n-1}\psi_{n-1} = \varepsilon \psi_n$$ was identified with the Lax operator

2) Extrema of energy were identified with finite-gap periodic solutions

Peierls problem: Continuos version

(1+1) Dirac equation: $$\begin{cases} -i\big(\partial_x - \Delta(x)\big)\psi_+ = \varepsilon\psi_- \\ -i\big(\partial_x + \Delta(x)\big)\psi_- = \varepsilon\psi_+ \end{cases}$$

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Dirac Hamiltonian: $$H = \overline{\psi}\sigma_1\big(i\partial_x + \sigma_3\Delta\big)\psi + \frac{1}{2\lambda}\Delta^2$$

$$E = \operatorname*{Tr}_{H<\mu}(H)$$

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The minimum of energy is achieved if Δ is a periodic solution of mKdV

$$\Delta_t - 6\Delta^2\Delta_x + \Delta_{xxx} = 0, \quad \Delta = \text{function}(x - ct)\,.$$

Cnoidal wave

$$
\begin{aligned}
\text{mKdV:} \quad & \Delta_t - 6\Delta^2\Delta_x + \Delta_{xxx} = 0 \\
\text{Miura:} \quad & q^{\pm} = \Delta^2 + \Delta_x \\
\text{KdV:} \quad & q_t - 6qq_x + q_{xxx} = 0\,.
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$$

Cnoidal wave

mKdV:	$\Delta_t - 6\Delta^2\Delta_x + \Delta_{xxx} = 0$	Cnoidal wave:	$\Delta(x) = \Delta_0 k^{1/2}\,\mathrm{sn}(x\|k)$
Miura:	$q^\pm = \Delta^2 + \Delta_x$	# Particles=Period:	$N/N_0 = 2k^{1/2}K(k)$
KdV:	$q_t - 6qq_x + q_{xxx} = 0\,.$	Gap:	$\Delta_0 = \Lambda e^{-\pi/\lambda}$

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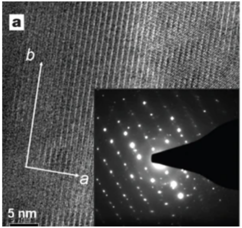
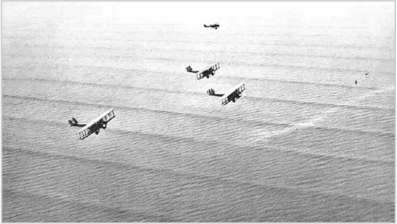
$$\text{\# Particles=Period:} \quad N/N_0 = 2k^{1/2}K(k)$$

$$\text{Gap:} \quad \Delta_0 = \Lambda e^{-\pi/\lambda}$$



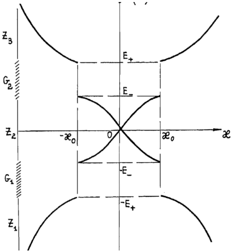


US Army bombers flying over near-periodic swell in shallow water, close to the Panama

Spectral Curve

$$-i\big(\partial_x \mp \Delta(x)\big)\psi_\pm = \varepsilon(p)\psi_\mp$$

$$dN(\varepsilon)/N_0 = dp$$



$$dp = \frac{|\varepsilon^2 - S|}{\sqrt{R}} d\varepsilon$$

$$R(\varepsilon) = (\varepsilon^2 - E_+^2)(\varepsilon^2 - E_-^2)$$

$$2S = -\overline{\Delta^2} + E_+^2 + E_-^2$$

Edges of the spectrum : $E_\pm = \frac{\Delta_0}{2}(k^{-1/2} \pm k^{1/2})$

Quantization of the spectral curve

Valdemar Melin, Yuta Sekiguchi, P. W., and Konstantin Zarembo

How to obtain periodic solutions of classical integrable equations from quantum integrable models?

QUANTIZATION OF THE SPECTRAL CURVE

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Quantum version of Peierls problem and the spectral curves

Quantum version: Gross-Neveu model

$$H = \overline{\psi}\sigma_2\left(i\partial_x + \sigma_3\Delta\right)\psi + \frac{1}{2\lambda}\Delta^2$$

Adiabatic approximation: Δ is determined by the extremum of $\mathrm{Tr}\,(H)$

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Quantum version: Δ itself is a quantum field.
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Large N as a semiclassical parameter: $\psi \to (\psi_1, \ldots \psi_N)$

$$H = \sum_{1\le k\le N} \overline{\psi}_k\sigma_2 i\partial_x\psi_k + \tfrac{\lambda}{2}\Big(\sum_{1\le k\le N} \overline{\psi}_k\psi_k\Big)^2$$

We recover the Peierls model in the limit of a large N

Lie group

$$H = \sum_{1\le k\le N} \overline{\psi}_k \sigma_2 i\partial_x \psi_k + \frac{\lambda}{2}\Big(\sum_{1\le k\le N} \overline{\psi}_k \psi_k \Big)^2$$

Integrable model controlled by its global symmetry $O(2N)$

Mass spectrum

Particle content: All fundamental representations.

Mass spectrum D_N:

$$\text{n-th tensor}: \quad m_n = m \frac{\sin \frac{\pi n}{2N-2}}{\sin \frac{\pi}{2N-2}}$$

$$\text{spinors:} \quad m_s = m_{\bar{s}} = \frac{m}{2 \sin \frac{\pi}{2N-2}},$$

Scattering matrices, the mass spectrum, the Bethe Ansatz are known for all simple Lie groups: E. Ogievetski, N. Reshetikhin, P. W.

The scattering matrix is factorized into a product of two-particle scattering

$$S_{ab}(\theta), \quad \theta = \theta_a - \theta_b, \quad p_a(\theta \to \infty) \sim m_a \sinh \theta$$

Quantum integrable systems: Scattering matrices and TBA

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Thermodynamic Bethe-Ansatz equations for the "spectral curve" $K_{ab} = \frac{1}{2\pi i}\frac{d}{d\theta}\log S_{ab}$

$$\int K_{ab}(\theta_a - \theta_b)dp_b = m_a \sinh\theta_a, \qquad \int K_{ab}(\theta_a - \theta_b)\varepsilon_b = \mu_a - m_a\cosh\theta_a$$

Sum over particle content (along the Dynkin diagram 

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Sum over particle content (along the Dynkin diagram 

Spectral curve $E =$ multivalued function (P) – spectral curve

$$P = \sum_a \int \sinh\theta\, dp_a, \qquad E - \mu N = \sum_a \int \cosh(\theta)\varepsilon_b d\theta$$

Scattering matrix D_N (in momentum space)

$$\hat{K}_{ab} = \begin{cases} \delta_{ab} + \frac{1}{2} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{(|a-b|-N+1)k}{2N-2} - \sinh \frac{(a+b-N+1)k}{2N-2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}}, & a, b \leq N-2 \\ \delta_{ab} + \frac{1}{4} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{(|N-1-b|-N+1)k}{2N-2} - \sinh \frac{b|k|}{2N-2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}}, & a > N-2, b \leq N-2 \\ \delta_{ab} + \frac{1}{4} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{(|N-1-a|-N+1)k}{2N-2} - \sinh \frac{a|k|}{2N-2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}}, & a \leq N-2, b > N-2 \\ \delta_{ab} - \frac{1}{4} e^{\frac{|k|}{2N-2}} \dfrac{\sinh \frac{k}{2}}{\sinh \frac{k}{2N-2} \cosh \frac{k}{2}} - \frac{1}{4} \dfrac{(-1)^{a+b} e^{\frac{|k|}{2N-2}}}{\cosh \frac{k}{2N-2}}, & a, b > N-2. \end{cases}$$

Karowski and Thun, 1981

Ground State

The vacuum is filled by spinors: 

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The TBA are reduced to:

$$\int_{-B}^{B}(K_{ss}+K_{s\bar{s}})(\theta-\theta')\,dp_s = m_s \sinh\theta, \quad 2\int_{-B}^{B}K_{as}(\theta-\theta')\,dp_s = m_a \cosh\theta, \quad \frac{N}{N_0} = \int_{-B}^{B}dp_s$$

$$K_{ss}+K_{s\bar{s}} = \frac{\tanh\frac{|k|}{2}}{2\left(1-e^{-\frac{|k|}{N-1}}\right)}, \qquad K_{as} = -\frac{\mathrm{e}^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}}$$

Singular large N limit

$$K_{ss} + K_{s\bar{s}} = \frac{\tanh \frac{k}{2}}{2\left(1 - e^{-\frac{|k|}{N-1}}\right)} \underset{N\to\infty}{\longrightarrow} -\frac{N}{\pi^2} \log\coth\frac{\theta}{2} \qquad K_{as} = -\frac{\mathrm{e}^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}} \underset{N\to\infty}{\longrightarrow} -\frac{1}{2\pi\cosh\theta}$$

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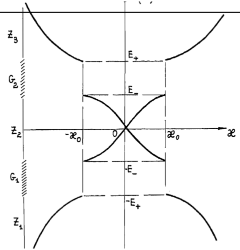
Integral equations degenerate to the Riemann-Hilbert problem

$$\int_{[-B,B]} \ln\coth\frac{\theta-\theta'}{2}\frac{dp_s}{\pi} = m\cosh\theta, \qquad \int_{[-B,B]} \ln\coth\frac{\theta-\theta'}{2}\varepsilon(\theta')\frac{d\theta'}{\pi} = m\cosh\theta - \frac{\mu}{2}$$



KdV spectral curve

$$dp = \frac{\varepsilon^2 + \overline{\Delta^2} - E_+^2 - E_-^2}{\sqrt{(\varepsilon^2 - E_+^2)(\varepsilon^2 - E_-^2)}}d\varepsilon$$

$$\frac{N}{N_0} = 2k^{1/2}K(k), \quad E_\pm = \frac{\Delta_0}{2}(k^{-1/2} \pm k^{1/2})$$

Comments

Relation between Lie algebras and integrable equations

- $A_N \Rightarrow NLS$
- $B_N, C_N, D_N \Rightarrow KdV$

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Quantum version of Krichever' algebro-geometric construct